The number of full exceptional collections for extended Dynkin quivers

Takumi OTANI (Tsinghua University)

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Joint work with Yuuki SHIRAISHI (Osaka) and Atsushi TAKAHASHI (Osaka)

Full exceptional collection

Let $\mathcal{D}$ be a triangulated category.

Definition 1.
A full exceptional collection *in $\mathcal{D}$ is an ordered set of objects $(E_1, \ldots, E_n)$ s.t.*

1. $\mathrm{Hom}_{\mathcal{D}}(E_i, E_i) \cong \mathbb{C}$ *and* $\mathrm{Hom}_{\mathcal{D}}(E_i, E_i[p]) \cong 0$ *when* $p \neq 0$.
2. *If* $i > j$, *then* $\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[p]) \cong 0$ *for all* $p \in \mathbb{Z}$.
3. *The thick closure containing* $E_1, \ldots, E_n$ *in* $\mathcal{D}$ *is equivalent to* $\mathcal{D}$.

Denote by $\mathrm{FEC}(\mathcal{D})$ the set of isomorphism classes of full exceptional collections in $\mathcal{D}$.

There are two actions on $\mathrm{FEC}(\mathcal{D})$:

- $\mathbb{Z}^n$-action: $(p_1, \ldots, p_n) \cdot (E_1, \ldots, E_n) \coloneqq (E_1[p_1], \ldots, E_n[p_n])$
- $\mathrm{Aut}(\mathcal{D})$-action: $\Phi \cdot (E_1, \ldots, E_n) \coloneqq (\Phi(E_1), \ldots, \Phi(E_n))$

Dynkin case

Let $\vec{\Delta} = (\vec{\Delta}_0, \vec{\Delta}_1)$ be a Dynkin quiver and $\mathcal{D}^b(\vec{\Delta}) := \mathcal{D}^b \mathrm{mod}(\mathbb{C}\vec{\Delta})$. In this case, the maximal length n of a full exceptional collection is given by $|\vec{\Delta}_0|$.

Define $e(\mathcal{D}^b(\vec{\Delta})) \in \mathbb{Z}_{\geq 1} \cup \{+\infty\}$ by

$$e(\mathcal{D}^b(\vec{\Delta})) := \left|\mathrm{FEC}(\mathcal{D}^b(\vec{\Delta})) \,/\, \mathbb{Z}^n\right|.$$

Theorem 2 (Obaid–Nauman–Shammakh–Fakieh–Ringel, Deligne).

$$e(\mathcal{D}^b(\vec{\Delta})) = \frac{n!}{d_1 \cdots d_n} h^n,$$

where h is the Coxeter number associated with $\vec{\Delta}$ and $d_1, \ldots, d_n$ are degrees of the Weyl group of $\vec{\Delta}$, i.e.,

$\vec{\Delta}$	h	$d_1, \ldots, d_n$
$\vec{A}_n$	$n+1$	$n+1, n, \ldots, 3, 2$
$\vec{D}_n$	$2n-2$	$2n-2, 2n-4, \ldots, 4, 2, n$
$\vec{E}_6$	12	$12, 9, 8, 6, 5, 2$
$\vec{E}_7$	18	$18, 14, 13, 10, 8, 6, 2$
$\vec{E}_8$	30	$30, 24, 20, 18, 14, 12, 8, 2$

More explicitly, we have

$$e(\mathcal{D}^b(\vec{\Delta})) = \begin{cases} (n+1)^{n-1}, & \vec{\Delta} = \vec{A}_n, \\ 2(n-1)^n, & \vec{\Delta} = \vec{D}_n, \\ 2^9 \cdot 3^4, & \vec{\Delta} = \vec{E}_6, \\ 2 \cdot 3^{12}, & \vec{\Delta} = \vec{E}_7, \\ 2 \cdot 3^5 \cdot 5^7, & \vec{\Delta} = \vec{E}_8. \end{cases}$$

In order to prove Theorem 2, we need the following recursive formula.

Theorem 3 (Obaid–Nauman–Shammakh–Fakieh–Ringel, Deligne).

$$e(\mathcal{D}^b(\vec{\Delta})) = \frac{h}{2} \sum_{v \in \vec{\Delta}_0} e(\mathcal{D}^b(\vec{\Delta}^{(v)})),$$

where $\vec{\Delta}^{(v)}$ *is the full subquiver of* $\vec{\Delta}$ *restricted to* $\vec{\Delta}_0 \setminus \{v\}$*.*

For any vertex $v \in \vec{\Delta}_0$, the full subquiver $\vec{\Delta}^{(v)}$ is a disjoint union of some Dynkin quivers. Therefore, the proof of Theorem 2 is done by the recursive formula and induction on the number of vertices.

Extended Dynkin case

Let $A = (a_1, a_2, a_3)$ be a tuple of positive integers satisfying

$$\chi_A := \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1 > 0.$$

Note that $\chi_A > 0$ if and only if

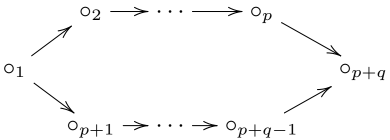
$$A = (1, p, q),\ (2, 2, r),\ (2, 3, 3),\ (2, 3, 4) \text{ or } (2, 3, 5),$$

where $p, q, r \in \mathbb{Z}_{\geq 1}$.

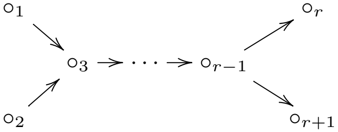
For $A = (a_1, a_2, a_3)$ with $\chi_A > 0$, we associate an extended Dynkin quiver $Q_A = ((Q_A)_0, (Q_A)_1)$ as follows:

A	$(1, p, q)$	$(2, 2, r)$	$(2, 3, 3)$	$(2, 3, 4)$	$(2, 3, 5)$
Q_A	$A^{(1)}_{p,q}$	$D^{(1)}_{r+2}$	$E^{(1)}_6$	$E^{(1)}_7$	$E^{(1)}_8$

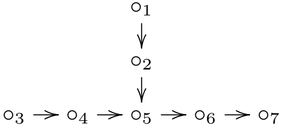
- $A_{p,q}^{(1)}$-quiver:



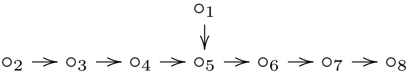
- $D_r^{(1)}$-quiver:



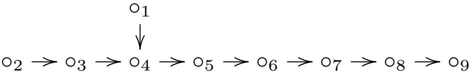
- $E_6^{(1)}$-quiver:



- $E_7^{(1)}$-quiver:



- $E_8^{(1)}$-quiver:



For $A = (a_1, a_2, a_3)$ with $\chi_A > 0$, one can define an orbifold projective line $\mathbb{P}^1_A$.
The orbifold projective line $\mathbb{P}^1_A$ can be regarded as the projective line $\mathbb{P}^1$ with isotropic points $0, 1, \infty \in \mathbb{P}^1$ and isotropic groups $\mathbb{Z}/a_1\mathbb{Z}$, $\mathbb{Z}/a_2\mathbb{Z}$, $\mathbb{Z}/a_3\mathbb{Z}$ respectively.
Moreover, the orbifold Euler characteristic of $\mathbb{P}^1_A$ is given by $\chi_A > 0$.

The category $\mathrm{coh}(\mathbb{P}^1_A)$ of coherent sheaves on $\mathbb{P}^1_A$ is abelian and hereditary.
For simplicity, we put $\mathcal{D}^b(\mathbb{P}^1_A) := \mathcal{D}^b\mathrm{coh}(\mathbb{P}^1_A)$.

Proposition 4 (Geigle–Lenzing).

There exists an equivalence of triangulated categories

$$\mathcal{D}^b(Q_A) \cong \mathcal{D}^b(\mathbb{P}^1_A)$$

Octopus quiver

Define a quiver with relation $\widetilde{\mathbb{T}}_A = ((\widetilde{\mathbb{T}}_A)_0, (\widetilde{\mathbb{T}}_A)_1, I)$ as follows:

- The set of vertices is given by

$$(\widetilde{\mathbb{T}}_A)_0 := \{\mathbf{1}, \mathbf{1}^*\} \sqcup \{(i,j) \mid i = 1,2,3,\ j = 1, \dots, a_i - 1\}.$$

- Let $v, v' \in (\widetilde{\mathbb{T}}_A)_0$ be verticies.
    - If $(v, v') = (\mathbf{1}, (a_i, 1))$ or $((a_i, 1), \mathbf{1}^*)$, there is one arrow $\alpha_{v,v'} \in (\widetilde{\mathbb{T}}_A)_1$ from v to v'.
    - If $(v, v') = ((a_i, j), (a_i, j+1))$ for some i, j, there is one arrow $\alpha_{v,v'} \in (\widetilde{\mathbb{T}}_A)_1$ from v to v'.
    - Otherwise, there are no arrows.
- The relation I is given by

$$I := \langle \alpha_{(1,1),\mathbf{1}^*}\alpha_{\mathbf{1},(1,1)} + \alpha_{(2,1),\mathbf{1}^*}\alpha_{\mathbf{1},(2,1)}, \alpha_{(2,1),\mathbf{1}^*}\alpha_{\mathbf{1},(2,1)} + \alpha_{(3,1),\mathbf{1}^*}\alpha_{\mathbf{1},(3,1)} \rangle.$$

For simplicity, we put $\mathcal{D}^b(\widetilde{\mathbb{T}}_A) := \mathcal{D}^b\mathrm{mod}\big(\mathbb{C}\big((\widetilde{\mathbb{T}}_A)_0, (\widetilde{\mathbb{T}}_A)_1\big)/I\big)$.

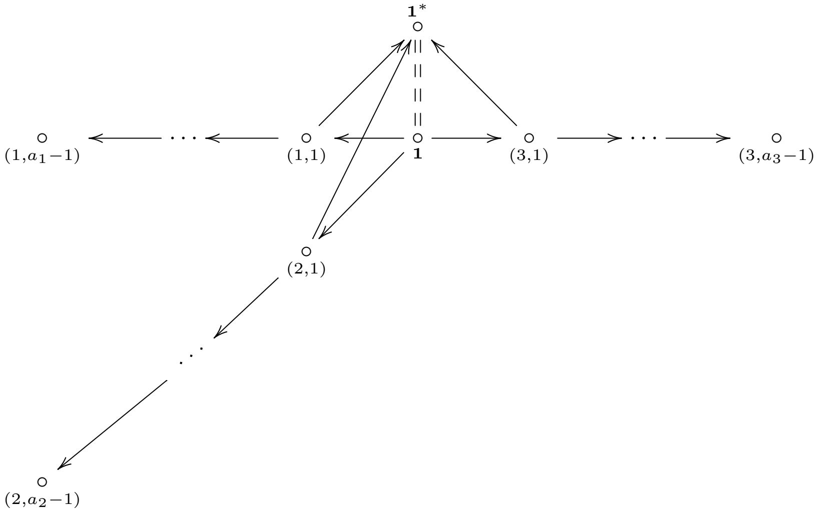


Figure: The octopus quiver with relation.

Proposition 5 (Geigle–Lenzing, Shiraishi–Takahashi–Wada).
There exists an equivalence of triangulated categories

$$\mathcal{D}^b(\mathbb{P}^1_A) \cong \mathcal{D}^b(\widetilde{\mathbb{T}}_A).$$

Hence, we have two equivalences

$$\mathcal{D}^b(Q_A) \cong \mathcal{D}^b(\mathbb{P}^1_A) \cong \mathcal{D}^b(\widetilde{\mathbb{T}}_A),$$

which play an important role in a recursive formula to count the number of full exceptional collections.

Definition of $e(\mathcal{D}^b(Q_A))$

Put $n_A := a_1 + a_2 + a_3 - 1$, which is the maximal length of a full exceptional collection in $\mathcal{D}^b(Q_A)$. To define a finite number $e(\mathcal{D}^b(Q_A))$, we consider spherical twists.

Definition 6 (Seidel–Thomas).

Let $\mathbb{S} \in \mathrm{Aut}(\mathcal{D}^b(Q_A))$ be the Serre functor.
An object $S \in \mathcal{D}^b(Q_A)$ is (1-)spherical if $\mathbb{S}(S) \cong S[1]$ and

$$\mathrm{Hom}(S, S[p]) \cong \begin{cases} \mathbb{C}, & p = 0, 1, \\ 0, & p \neq 0, 1. \end{cases}$$

Proposition 7 (Seidel–Thomas).

Let $S \in \mathcal{D}^b(Q_A)$ be spherical. There exists an autoequivalence $\mathrm{Tw}_S \in \mathrm{Aut}(\mathcal{D}^b(Q_A))$ defined by the exact triangle

$$\mathbb{R}\mathrm{Hom}(S, X) \otimes S \longrightarrow X \longrightarrow \mathrm{Tw}_S(X)$$

for any object $X \in \mathcal{D}$. The inverse functor $\mathrm{Tw}_S^{-1} \in \mathrm{Aut}(\mathcal{D}^b(Q_A))$ is given by

$$\mathrm{Tw}_S^{-1}(X) \longrightarrow X \longrightarrow S \otimes \mathbb{R}\mathrm{Hom}(X, S)^*.$$

Define a subgroup $\mathrm{ST}(\mathcal{D}^b(Q_A)) \subset \mathrm{Aut}(\mathcal{D}^b(Q_A))$ by

$$\mathrm{ST}(\mathcal{D}^b(Q_A)) := \left\langle \mathrm{Tw}_S \mid S \text{ is spherical in } \mathcal{D}^b(Q_A) \right\rangle$$

By direct calculations, we can prove the following

Proposition 8.

$$\mathrm{ST}(\mathcal{D}^b(Q_A)) \cong \mathbb{Z}$$

Now, we are ready to define the number $e(\mathcal{D}^b(Q_A))$.

Definition 9.

$$e(\mathcal{D}^b(Q_A)) := \left| \mathrm{FEC}(\mathcal{D}^b(Q_A)) \Big/ \langle \mathrm{ST}(\mathcal{D}^b(Q_A)), \mathbb{Z}^{n_A} \rangle \right|$$

Theorem 10 (O–Shiraishi–Takahashi).

$$e(\mathcal{D}^b(Q_A)) = \frac{n_A!}{a_1!a_2!a_3!\chi_A} a_1^{a_1} a_2^{a_2} a_3^{a_3}$$

More explicitly, we have

$$e(\mathcal{D}^b(Q_A)) = \begin{cases} \dfrac{(p+q-1)!}{(p-1)!(q-1)!} p^p q^q, & A = (1,p,q), \\ 4(r+1)(r+2)(r+3)r^{r+1}, & A = (2,2,r), \\ 1224720, & A = (2,3,3), \\ 46448640, & A = (2,3,4), \\ 2551500000, & A = (2,3,5). \end{cases}$$

Remark 11.
Takahashi–Zhang proved a similar result in the cases of $\chi_A = 0$*.*

Theorem 10 is proved by the following recursive formula for the orbifold projective line.

Theorem 12 (O–Shiraishi–Takahashi).

$$
\begin{aligned}
e(\mathcal{D}^b(\mathbb{P}^1_A)) &= \frac{1}{\chi_A} \sum_{v \in (Q_A)_0} e(\mathcal{D}^b(Q_A^{(v)})) \\
&\quad + \sum_{i=1}^{3} a_i \sum_{j=1}^{a_i - 1} \binom{n_A - 1}{a_i - j - 1} \cdot e(\mathcal{D}^b(\mathbb{P}^1_{A_{(i,j)}})) \cdot e(\mathcal{D}^b(\vec{A}_{a_i - j - 1})),
\end{aligned}
$$

where $Q_A^{(v)}$ is the full subquiver of Q_A restricted to $(Q_A)_0 \setminus \{v\}$, and $A_{(i,j)} = (a'_1, a'_2, a'_3)$ is defined by $a'_i = j$ and $a'_k = a_k$ for $k \neq i$.

Note that the subquiver $Q_A^{(v)}$ is a disjoint union of some Dynkin quivers and extended Dynkin quivers.

The red term comes from the derived equivalence $\mathcal{D}^b(\mathbb{P}^1_A) \cong \mathcal{D}^b(Q_A)$, and the blue term comes from the derived equivalence $\mathcal{D}^b(\mathbb{P}^1_A) \cong \mathcal{D}^b(\widetilde{\mathbb{T}}_A)$.

Lyashko–Looijenga map

Let $f\colon \mathbb{C}^3 \longrightarrow \mathbb{C}$ be the normal form of a simple singularity (i.e., ADE singularity) and n its Milnor number, i.e.,

$$\begin{array}{llcl} A_n: & f(x_1,x_2,x_3) & = & x_1^{n+1}+x_2^2+x_3^2 \\ D_n: & f(x_1,x_2,x_3) & = & x_1^{n-1}+x_1x_2^2+x_3^2 \\ E_6: & f(x_1,x_2,x_3) & = & x_1^4+x_2^3+x_3^2 \\ E_7: & f(x_1,x_2,x_3) & = & x_1^3+x_1x_2^3+x_3^2 \\ E_8: & f(x_1,x_2,x_3) & = & x_1^5+x_2^3+x_3^2 \end{array}$$

Then, there exists the universal unfolding $F\colon M \times \mathbb{C}^3 \longrightarrow \mathbb{C}$ of f equipped with the parameter space $M := \mathbb{C}^n$.

The Lyashko–Looijenga map LL is a branched covering map defined by

$$LL\colon M \longrightarrow \mathbb{C}^n, \quad s \mapsto (b_1(s), \ldots, b_n(s)),$$

where $b_1, \ldots, b_n$ are coefficients of the polynomial

$$\prod_{i=1}^{n}(u-u_i(s)) = u^n + b_1(s)u^{n-1} + \cdots + b_n(s).$$

and $u_1(s), \ldots, u_n(s)$ are critical values of $F(s,-)\colon \mathbb{C}^3 \longrightarrow \mathbb{C}$ for $s \in M$.

Theorem 13 (Looijenga, Lyashko).

$$\deg LL = \frac{n!}{d_1 \cdots d_n} h^n.$$

Recall that Theorem 2 says that

$$e(\mathcal{D}^b(\vec{\Delta})) = \frac{n!}{d_1 \cdots d_n} h^n.$$

Hence we have

Corollary 14.

$$e(\mathcal{D}^b(\vec{\Delta})) = \deg LL.$$

We generalized the equation to the cases of extended Dynkin quivers.

For $A = (a_1, a_2, a_3)$, define $f_A \in \mathbb{C}[x_1, x_2, x_3]$ by

$$f_A(\mathbf{x}) := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1} \cdot x_1 x_2 x_3$$

for a nonzero complex number $q \in \mathbb{C}^*$. There exists the universal unfolding of f_A equipped with the parameter space $M := \mathbb{C}^{n_A - 1} \times \mathbb{C}^*$.

We can define the Lyashko–Looijenga map $LL \colon M \longrightarrow \mathbb{C}^{n_A}$ in the same way of a simple singularity.

By mirror symmetry, one can define this Lyashko–Looijenga map for the quantum cohomology of $\mathbb{P}^1_A$ and the Weyl group invariant theory of Q_A.

Theorem 15 (Dubrovin–Zhang).

$$\deg LL = \frac{n_A!}{a_1! a_2! a_3! \chi_A} a_1^{a_1} a_2^{a_2} a_3^{a_3}.$$

Corollary 16.

$$e(\mathcal{D}^b(Q_A)) = \deg LL$$

Thank you very much!