Derived Picard groups and integration of Hochschild cohomology (arXiv:2405.14448)

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Proof relies on work by many people.

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Thank you for your attention!