Calabi-Yau completions and cluster categories for roots of τ

Comes from : tilting theory for projective varieties / singularity categories / reflection functors.

Example

(1) Beilinson's theorem : $D^b(\mathrm{coh}\,\mathbb{P}^n) \simeq D^b(\mathrm{mod}\,A)$ Beilinson algebra

$$T = \bigoplus_{i=0}^{n} \mathcal{O}(i) \longleftarrow \mathcal{O} \Rrightarrow \mathcal{O}(1) \Rrightarrow \cdots \Rrightarrow \mathcal{O}(n)$$

Compare the Serre functors

$$\circlearrowleft \qquad\qquad \circlearrowleft$$

$$(-n-1)[n] \qquad\qquad -\overset{L}{\otimes}_A DA = \nu$$

$$\circlearrowleft \qquad\qquad \circlearrowleft$$

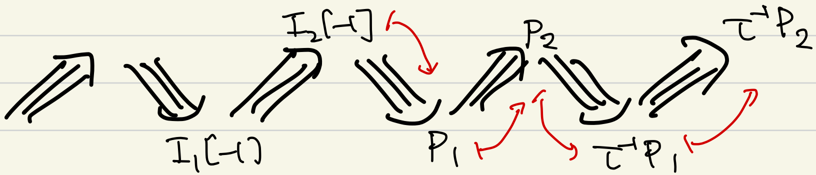
$$(-n-1) \qquad\qquad \nu_n := \nu \circ [-n] \ : n\text{-AR translation}$$

$$\circlearrowleft$$

$$(1) \longleftrightarrow \exists\ (n+1)\text{-st root of } \nu_n^{-1}$$

(2) [Keller - Murfet - Van den Bergh]

$A = \bullet \xrightarrow{m} \bullet$ m - Kronecker quiver



$\exists\ \tau^{-1/2}$ = one place to the right

§ Formulation

triangulated category $\longleftrightarrow$ dg algebra A

triangle functors $\longleftrightarrow$ dg bimodules

inverse Serre functors $\longleftrightarrow$ inverse dualizing bimodules

$\underbrace{\text{inverse Serre}}_{d\text{-shifted}}$ functors

$\mathbf{R}\mathrm{Hom}_{A^e}(A, A^e[d]) =: A^\vee[d]$

Def A : dg algebra, $d \in \mathbb{Z}_{>0}$

$- \overset{\mathbf{L}}{\otimes}_A U : D(A) \xrightarrow{\simeq} D(A)$

(1) An a-th root of $A^\vee[d]$ is $U \in D(A^e)$: invertible s.t.

$$U^{\otimes_A a} = U \overset{\mathbf{L}}{\otimes}_A \cdots \overset{\mathbf{L}}{\otimes}_A U \simeq A^\vee[d] \quad \text{in } D(A^e)$$

(2) An a-th root pair (U, P) consists of

- U : an a-th root of $A^\vee[d]$ as above
- $P \in \mathrm{add}\, A$ s.t.

semi-orth. decomposition

$$\mathrm{per}\, A = \langle \mathrm{thick}\, P,\ \mathrm{thick}(P \overset{\mathbf{L}}{\otimes}_A U),\ \cdots,\ \mathrm{thick}(P \overset{\mathbf{L}}{\otimes}_A U^{a-1}) \rangle$$

Examples

(1) $A = 0 \Rrightarrow 1 \Rrightarrow \cdots \Rrightarrow n$: Beilinson algebra

(U, P) with $\begin{cases} U = \mathrm{Hom}_{\mathbb{P}^n}(T, T(1)) \\ P = e_0 A \end{cases}$: $(n+1)$-st root pair.

(2) $A = 1 \xrightarrow{m} 2$: m-Kronecker quiver

$$\begin{tikzcd}[column sep=small] & & P_2 \arrow[dr, Rrightarrow] & & \cdots \\ \cdots \arrow[r, Rrightarrow] & P_1 \arrow[ur, Rrightarrow] & & \tau^{-1}P_1 \arrow[ur, Rrightarrow] & \end{tikzcd}$$

$P_2 \; \tau^{-1}P_1 =: U$

$\rightsquigarrow (U, e_1 A)$: (square) root pair.

(3) (U, P) : a-th root pair of $A^\vee[d]$

V : a-th root of $B^\vee[e]$

$\Rightarrow (U \otimes_k V, P \otimes_k B)$: a-th root pair of $(A \otimes B)^\vee[d+e]$

§ CY completions

Def-Thm (Keller) A : smooth dg algebra

(1) $\Pi = \Pi_{d+1}(A) = T^{L}_{A}(A^{\vee}[d])$: $(d+1)$-Calabi-Yau completion of A

derived tensor algebra

(2) Π is $(d+1)$-Calabi-Yau i.e. $\Pi^{\vee}[d+1] \xrightarrow{\sim} \Pi$ in $D(\Pi^{e})$

Def-Thm A : smooth dg algebra, (U, eA) : a-th root pair of $A^{\vee}[d]$

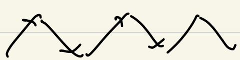
(1) $\Pi = \Pi^{(1/a)}_{d+1}(A) = e(T^{L}_{A} U)e$: $(d+1)$-CY completion of (U, P)

(2) Π is $(d+1)$-Calabi-Yau, up to twist.

Example

(1) A : Beilinson $\rightsquigarrow$ $\Pi = k[x_0, \cdots, x_n]$: polynomial ring

(2) $A = \bullet \longrightarrow \bullet$, $U = \tau^{-1/2}$ $(= DA)$ $\rightsquigarrow$ Π :



$|a| = 0$

$|t| = -1$

$dt = a^2$

§ Cluster categories (= cosingularity categories)

Def - Thm (Amiot)

Π : $(d+1)$-Calabi-Yau dg algebra s.t. $H^{>0}\Pi = 0$, $H^0\Pi$: f.d

connective

(1) $\mathcal{C}(\Pi) = \mathrm{per}\,\Pi / \mathrm{pvd}\,\Pi$: cluster category of Π

(2) $\mathcal{C}(\Pi)$: d-CY triang. cat., $\Pi \in \mathcal{C}(\Pi)$: d-CT.

(3) When $\Pi = \Pi_{d+1}(A)$, then $\mathrm{per}\,A / - \overset{L}{\otimes}_A A^\vee[d] \hookrightarrow \mathcal{C}(\Pi) =: \mathcal{C}_d(A)$

triangulated hull

$$A \longmapsto \Pi$$

Thm A : smooth dg alg, (U, P) : a-th root pair

s.t. $\Pi = \Pi^{(1/a)}_{d+1}(A)$ satisfies $H^{>0}\Pi = 0$, $H^0\Pi$: f.d.

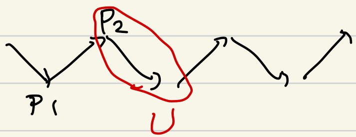
Then $\mathrm{per}\,A / - \overset{L}{\otimes}_A U \hookrightarrow \mathcal{C}(\Pi)$: triang. hull

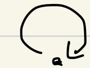
$P \longmapsto \Pi$: d-CT

$\mathcal{C}(\Pi) =: \mathcal{C}^{(1/a)}_d(A)$

Example

(1) $A = 1 \longrightarrow 2$, $U = \tau^{-1/2} = DA$, $P = e_1 A$.

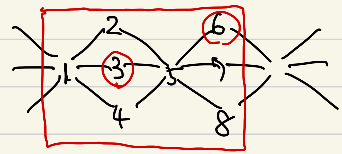


$\rightsquigarrow$ $\mathcal{C}(\pi) = D^b(A_2)/\tau^{-1/2} =$  $= \mathcal{C}_1^{(1/2)}(A_2)$

1-CT (tw) 1-CY

(2) A : as above, (U, P) : square root pair of $A^\vee[1]$

tensor product $\rightsquigarrow$ $(U \otimes U, P \otimes A)$: square root pair of $(A \otimes A)^\vee[2]$

$\rightsquigarrow$ $\mathcal{C}(\pi) = D^b(A \otimes A)/\sqrt{\nu_2} =$  $= \mathcal{C}_2^{(1/2)}(D_4)$

2-CY

2-CT