Fractional Brauer configuration algebras

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Introduction

- In 2017, Green and Schroll introduced a generalization of **Brauer graph algebras** (BGA) which they call **Brauer configuration algebras** (BCA).
- As each Brauer graph algebra is defined by a Brauer graph, each Brauer configuration algebra is defined by a Brauer configuration.
- (BGA=symmetric special biserial algebra) Every Brauer graph algebra is symmetric biserial, and the class of Brauer graph algebras coincides with the class of symmetric special biserial algebras over an algebraically closed field.
- (BCA=symmetric special multiserial algebra) Every Brauer configuration algebra is symmetric multiserial, and the class of Brauer configuration algebras coincides with the class of symmetric special multiserial algebras over an algebraically closed field (Green and Schroll, 2016).

Aim

We will

- give a further generalization of Brauer configurations which we call **fractional Brauer configurations** and
- define the corresponding **fractional Brauer configuration categories** and **fractional Brauer configuration algebras**.

Motivations

The reasons for us to introduce fractional Brauer configuration algebras are as follows:

- The class of Brauer graph algebras is closed under derived equivalence (Antipov and Zvonareva, 2022), but the class of Brauer configuration algebras is NOT closed under derived equivalence.
- We wish to generalize Brauer configuration algebras ($\subseteq$ **symmetric multiserial algebras**) to the scope of **self-injective algebras which are not necessarily multiserial**.

Brauer configuration

Definition (Green and Schroll, 2017)

Let $G = \langle g \rangle$ be an infinite cyclic group. A **Brauer configuration** (abbr. **BC**) is a finite G-set Γ (whose elements are called angles) together with

- a partition P of Γ such that each class of P contains at least two angles; (each class of P is called a polygon)
- a function $\mu : \Gamma \to \mathbb{Z}_+$ such that μ is constant on each G-orbit of Γ. (μ is called the multiplicity function)

A **Brauer graph** (abbr. **BG**) is a Brauer configuration such that each polygon of it contains exactly two angles (we usually use "half-edge" rather than "angle" in BG case).

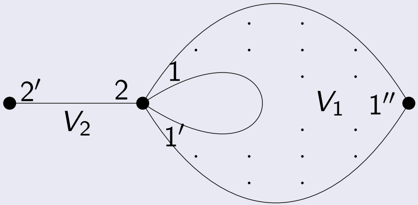
Example

Let $\Gamma = \{1, 1', 1'', 2, 2'\}$ be a Brauer configuration defined as follows:

- The G-set structure of Γ is given by
  $g \cdot 1 = 1', g \cdot 1' = 2, g \cdot 2 = 1, g \cdot 2' = 2', g \cdot 1'' = 1''$.
- The polygons of Γ are $V_1 = \{1, 1', 1''\}$, $V_2 = \{2, 2'\}$.
- The multiplicity function μ is given by $\mu(1'') = 2$ and
  $\mu(1) = \mu(1') = \mu(2) = \mu(2') = 1$.

Example

The above Brauer configuration can be realized by the following diagram:



Brauer configuration algebra

Green and Schroll associate each Brauer configuration Γ a quiver algebra kQ_Γ/I_Γ. The quiver Q_Γ is given as follows:

- The vertices of Q_Γ are in one to one correspondence with the polygons of Γ.
- The arrows of Q_Γ are in one to one correspondence with the elements of Γ. For every $a \in \Gamma$, the corresponding arrow α_a in Q_Γ has source $P(a)$ and terminal $P(g \cdot a)$, where $P(a)$ denotes the polygon of Γ that a belongs to.

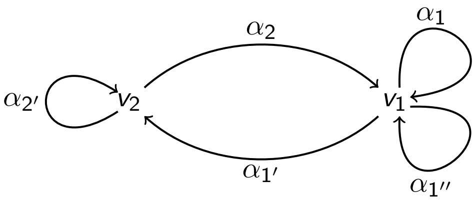
The ideal I_Γ is generated by the following two types of relations:

(R1) $(\alpha_{g^{|G\cdot a|-1}\cdot a}\cdots\alpha_{g\cdot a}\alpha_a)^{\mu(a)}-(\alpha_{g^{|G\cdot b|-1}\cdot b}\cdots\alpha_{g\cdot b}\alpha_b)^{\mu(b)}$, where a, b are elements of Γ which belong to the same polygon of Γ;

(R2) $\alpha_b\alpha_a$, where a, b elements of Γ such that $b\neq g\cdot a$.

Example

The Brauer configuration algebra corresponding to the Brauer configuration in last example is given by the following quiver

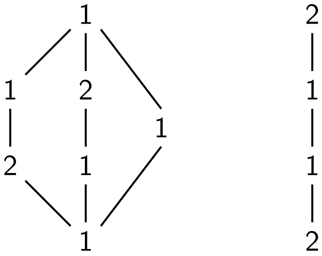


with relations

$$\alpha_2\alpha_{1'}\alpha_1 = \alpha_1\alpha_2\alpha_{1'} = \alpha_{1''}^2, \alpha_{1'}\alpha_1\alpha_2 = \alpha_{2'},$$
$$0 = \alpha_1^2 = \alpha_{1''}\alpha_1 = \alpha_{2'}\alpha_{1'} = \alpha_1\alpha_{1''} = \alpha_{1'}\alpha_{1''} = \alpha_{1'}\alpha_2 = \alpha_{1''}\alpha_2 = \alpha_2\alpha_{2'}.$$

Example

Therefore, the indecomposable projective modules have the following structures:



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fractional Brauer configuration

Definition

Let $G = \langle g \rangle$ be an infinite cyclic group. A **fractional Brauer configuration** (abbr. **f-BC**) is a quadruple $E = (E, P, L, d)$, where E is a G-set, P and L are two partitions of E, and $d : E \to \mathbb{Z}_+$ is a function, such that the following conditions hold.

$(f1)$ $L(e) \subseteq P(e)$ and $P(e)$ is a finite set for each $e \in E$.

$(f2)$ If $L(e_1) = L(e_2)$, then $P(g \cdot e_1) = P(g \cdot e_2)$.

$(f3)$ If e_1, e_2 belong to same $\langle g \rangle$-orbit, then $d(e_1) = d(e_2)$.

$(f4)$ $P(e_1) = P(e_2)$ if and only if $P(g^{d(e_1)} \cdot e_1) = P(g^{d(e_2)} \cdot e_2)$.

$(f5)$ $L(e_1) = L(e_2)$ if and only if $L(g^{d(e_1)} \cdot e_1) = L(g^{d(e_2)} \cdot e_2)$.

$(f6)$ The formal sequence $L(g^{d(e)-1} \cdot e) \cdots L(g \cdot e)L(e)$ is not a proper subsequence of the formal sequence $L(g^{d(h)-1} \cdot h) \cdots L(g \cdot h)L(h)$ for all $e, h \in E$.

fractional Brauer configuration

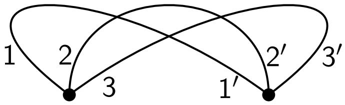
Remarks

- The elements in E are called **angles** of the f-BC. The $\langle g \rangle$-orbits of E are called **vertices** of the f-BC.
- The classes $P(e)$ of the partition P are called **polygons**.
- The partition L is said to be **trivial** if $L(e) = \{e\}$ for every $e \in E$.
- The function $d : E \to \mathbb{Z}_+$ is called **degree function**.
- Let E be a f-BC and v be a vertex such that v is a finite set, define the **fractional-degree** (abbr. **f-degree**) $d_f(v)$ of the vertex v to be the rational number $\frac{d(v)}{|v|}$. If the f-degree of each vertex of E is an integer, then E is said to have integral f-degree.
- Denote σ the map $E \to E$, $e \mapsto g^{d(e)} \cdot e$, which will be called the **Nakayama automorphism** of E.

Example

Example 1

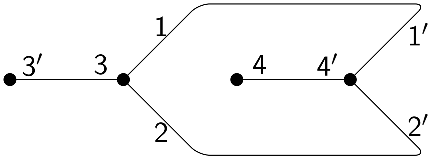
Let $E = \{1, 1', 2, 2', 3, 3'\}$. Define the group action on E by $g \cdot 1 = 2$, $g \cdot 2 = 3$, $g \cdot 3 = 1$, $g \cdot 1' = 2'$, $g \cdot 2' = 3'$, $g \cdot 3' = 1'$. Define $P(1) = \{1, 1'\}$, $P(2) = \{2, 2'\}$, $P(3) = \{3, 3'\}$ and $L(e) = \{e\}$ for every $e \in E$. The degree d of E is defined by $d(e) = 2$ for every $e \in E$. So the f-degree of E has constant value $\frac{2}{3}$.



Example

Example 2

Let $E = \{1, 1', 2, 2', 3, 3', 4, 4'\}$. Define the group action on E by $g \cdot 1 = 2$, $g \cdot 2 = 3$, $g \cdot 3 = 1$, $g \cdot 1' = 2'$, $g \cdot 2' = 4'$, $g \cdot 4' = 1'$, $g \cdot 3' = 3'$, $g \cdot 4 = 4$. Define $P(1) = \{1, 1'\}$, $P(2) = \{2, 2'\}$, $P(3) = \{3, 3'\}$, $P(4) = \{4, 4'\}$, $L(1) = \{1, 1'\}$ and $L(e) = \{e\}$ for $e \neq 1, 1'$. The f-degree of E is defined to have constant value 1.



Sequence

Definition

- Let n be a positive integer. We call $p = (e_n, \cdots, e_2, e_1)$ a sequence of angles of length n in E, if e_i's are angles in E and $P(g \cdot e_i) = P(e_{i+1})$ for all $1 \leq i \leq n-1$.
- Moreover, for every $e \in E$, we call $()_e$ a sequence of angles of length 0 in E at e, or a trivial sequence of E at e.

Standard sequence

- A sequence of the form $p = (g^{n-1} \cdot e, \cdots, g \cdot e, e)$ with $e \in E$ and $0 \leq n \leq d(e)$ is called a **standard sequence** of E (we define $p = ()_e$ when $n = 0$). When $n = d(e)$, we call it a **full sequence**.
- For $n > 0$, the source (resp. terminal) of standard sequence $p = (g^{n-1} \cdot e, \cdots, g \cdot e, e)$ is defined to be e (resp. $g^n \cdot e$); both the source and the terminal of the trivial sequence $()_e$ are defined to be e.
- The composition of two standard sequences is defined in a natural way.

Standard sequence

For a standard sequence $p = (g^{n-1} \cdot e, \cdots, g \cdot e, e)$, we can define two associated standard sequences

$$
{}^{\wedge}p = \begin{cases} (g^{d(e)-1} \cdot e, \cdots, g^{n+1} \cdot e, g^{n} \cdot e), \text{ if } 0 < n < d(e); \\ ()_{g^{d(e)}(e)}, \text{ if } n = d(e); \\ (g^{d(e)-1} \cdot e, \cdots, g \cdot e, e), \text{ if } n = 0 \text{ and } p = ()_e, \end{cases}
$$

and

$$
p^{\wedge} = \begin{cases} (g^{-1} \cdot e, g^{-2} \cdot e, \cdots, g^{n-d(e)} \cdot e), \text{ if } 0 < n < d(e); \\ ()_e, \text{ if } n = d(e); \\ (g^{-1} \cdot e, \cdots, g^{-d(e)} \cdot e), \text{ if } n = 0 \text{ and } p = ()_e, \end{cases}
$$

which is called the left complement and the right complement of p respectively. For a set $\mathscr{X}$ of standard sequences, denote ${}^{\wedge}\mathscr{X} = \{{}^{\wedge}p \mid p \in \mathscr{X}\}$ (resp. $\mathscr{X}^{\wedge} = \{p^{\wedge} \mid p \in \mathscr{X}\}$).

Associated formal sequence

- For a sequence $p = (e_n, \cdots, e_2, e_1)$ of length > 0 in E, we associate a formal sequence $L(p) := L(e_n) \cdots L(e_2) L(e_1)$.
- Moreover, for a trivial sequence $p = ()_e$, we associate a formal sequence $L(p) := 1_{P(e)}$.
- For a set $\mathscr{X}$ of sequences, define $L(\mathscr{X}) = \{L(p) \mid p \in \mathscr{X}\}$.

Identical sequence

Definition

Let E be a f-BC, p, q be two sequences of angles in E. Denote $p \equiv q$ if the associated formal sequences $L(p)$ and $L(q)$ are equal. In this case we say that p, q are **identical**.

For a set $\mathscr{X}$ of standard sequences, denote $[\mathscr{X}] = \{$ standard sequence $q \mid q$ is identical to some $p \in \mathscr{X} \}$.

Fractional Brauer configuration of type S

Definition

A f-BC E is said to be of **type S** (or E is a f_s-**BC** in short) if it satisfies additionally the following condition:
$(f7)$ For standard sequences $p \equiv q$, $[[^{\wedge}p]^{\wedge}] = [[^{\wedge}q]^{\wedge}]$ or $[^{\wedge}[p^{\wedge}]] = [^{\wedge}[q^{\wedge}]]$.

In particular, if E is a f_s-BC such that each polygon of E has exactly two elements, then we call E a fractional Brauer graph of type S (abbr. f_s-BG).

Remarks

- For standard sequences $p \equiv q$, it can be shown that the two conditions $[[^{\wedge}p]^{\wedge}] = [[^{\wedge}q]^{\wedge}]$ and $[^{\wedge}[p^{\wedge}]] = [^{\wedge}[q^{\wedge}]]$ are equivalent.
- The f-BCs in Example 1 and Example 2 are both f_s-BGs.

Fractional Brauer configuration of type MS

Definition

A f-BC $E = (E, P, L, d)$ is said to be of **type MS** (or E is a f_{ms}-**BC** in short) if partition L of E is trivial.

In particular, if E is a f_{ms}-BC such that each polygon of E has exactly two elements, then we call E a fractional Brauer graph of type MS (abbr. f_{ms}-BG).

Remarks

- A f-BC of type MS must be of type S.
- A BC can be regarded as a finite f_{ms}-BC with integral f-degree by identifying the multiplicity function of the BC with the f-degree of the corresponding f_{ms}-BC.

Relations

We have the following diagram:

$$
\begin{array}{ccccccc}
BC & \Longrightarrow & f_{ms}\text{-}BC & \Longrightarrow & f_s\text{-}BC & \Longrightarrow & f\text{-}BC \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
BG & \Longrightarrow & f_{ms}\text{-}BG & \Longrightarrow & f_s\text{-}BG & &
\end{array}
$$

Fractional Brauer configuration category

We associate every f-BC E a k-category $\Lambda_E = kQ_E/I_E$ which we call a **fractional Brauer configuration category**. The quiver Q_E is defined as follows:

- The set $(Q_E)_0$ of vertices is given by $\{P(e) \mid e \in E\}$;
- The set $(Q_E)_1$ of arrows is given by $\{L(e) \mid e \in E\}$, where the arrow $L(e)$ has the source $P(e)$ and the terminal $P(g \cdot e)$.

Fractional Brauer configuration category

The ideal I_E of the path category kQ_E is generated by the following three types of relations:

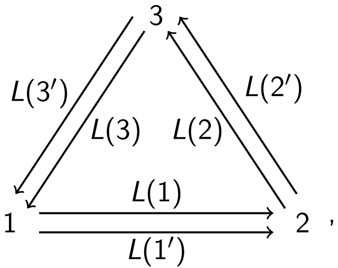
(*fR*1) $L(g^{d(e)-1-k} \cdot e) \cdots L(g \cdot e)L(e) - L(g^{d(h)-1-k} \cdot h) \cdots L(g \cdot h)L(h)$, where $k \geq 0$, $P(e) = P(h)$ and $L(g^{d(e)-i} \cdot e) = L(g^{d(h)-i} \cdot h)$ for $1 \leq i \leq k$. ("identical below" $\Longrightarrow$ "commutative above")

(*fR*2) Paths of the form $L(e_n) \cdots L(e_2)L(e_1)$ with $\bigcap_{i=1}^{n} g^{n-i} \cdot L(e_i) = \emptyset$ for $n > 1$.

(*fR*3) Paths of the form $L(g^{n-1} \cdot e) \cdots L(g \cdot e)L(e)$ for $n > d(e)$.

Example

In Example 1, Q_E is the following quiver



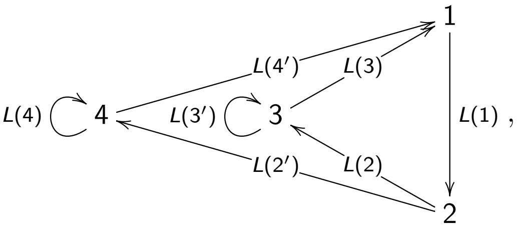
and I_E is generated by the following relations:

$$L(2)L(1) - L(2')L(1'), L(3)L(2) - L(3')L(2'), L(1)L(3) - L(1')L(3'),$$

$$L(2')L(1), L(2)L(1'), L(3')L(2), L(3)L(2'), L(1')L(3), L(1)L(3').$$

Example

In Example 2, Q_E is the following quiver



and I_E is generated by the following relations:

$$L(3') - L(2)L(1)L(3), L(4) - L(2')L(1)L(4'), L(3)L(2) - L(4')L(2'), L(3')L(2),$$

$$L(3)L(3'), L(4)L(2'), L(4')L(4),$$

$$L(2)L(1)L(4'), L(2')L(1)L(3), L(1)L(3)L(2)L(1).$$

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Locally bounded category

Definition (Bongartz and Gabriel, 1982)

A **locally bounded category** is a k-category Λ satisfying the following three conditions:

- For each $x \in \Lambda$, the endomorphism algebra $\Lambda(x, x)$ is local.
- Distinct objects of Λ are not isomorphic.
- For each $x \in \Lambda$, $\sum_{y \in \Lambda} \mathrm{dim}_k \Lambda(x, y) < \infty$ and $\sum_{y \in \Lambda} \mathrm{dim}_k \Lambda(y, x) < \infty$.

Fractional Brauer configuration categories are locally bounded

Theorem

Let $E = (E, P, L, d)$ be a **f-BC**, and let $\Lambda_E = kQ_E/I_E$ be the associated fractional Brauer configuration category. Then we have the following.

(1) Λ_E is **locally bounded** k**-category**.

(2) Let J be the ideal of Λ_E generated by the arrows of Q_E. Then J is the radical of Λ_E.

(3) The Nakayama automorphism σ of E induces an automorphism of the category Λ_E which is also denoted by σ.

Locally bounded Frobenius category

Definition

We call a locally bounded category Λ **Frobenius** if for every object x of Λ, there exists some objects y, z of Λ such that $\Lambda(-, x) \cong D\Lambda(y, -)$ and $\Lambda(x, -) \cong D\Lambda(-, z)$, where D denotes the usual k-duality on vector spaces.

Remarks

- Such objects y, z in the definition above are also unique by Yoneda's lemma.
- If Λ is a locally bounded Frobenius category, then the module category modΛ is a Frobenius category in the sense of Happel.

The category Λ_E in type S is a locally bounded Frobenius category

Theorem

Let E be a f_s-**BC**. Then $\Lambda_E(-, \sigma(x)) \cong D\Lambda_E(x, -)$ for all object x of Λ_E. Therefore the associated fractional Brauer configuration category Λ_E is a **locally bounded Frobenius category**.

Fractional Brauer configuration algebra

Definition

Let E be a f-BC, and let $A_E = \bigoplus_{x,y \in (Q_E)_0} \Lambda_E(x, y)$ be the algebra corresponding to the category Λ_E. We call A_E the **fractional Brauer configuration algebra** of E.

Fractional Brauer configuration algebra

Proposition

If E is a **finite** f_s**-BC**, then A_E is a **finite-dimensional Frobenius algebra** with some linear form $\epsilon : A_E \to k$.

Proposition

If E is a **finite** f_s**-BC of integral f-degree**, then A_E is a **finite-dimensional symmetric algebra**.

Locally bounded special multiserial category

Definition (Green and Schroll, 2016)

A locally bounded category Λ is said to be **special multiserial** if $\Lambda \cong kQ/I$ for some locally finite quiver Q and some admissible ideal I, such that for each arrow α of Q, there exists at most one arrow β (resp. γ) of Q such that $\beta\alpha \notin I$ (resp. $\alpha\gamma \notin I$).

Proposition

If E is a f_{ms}-**BC**, then Λ_E is a **locally bounded special multiserial Frobenius category**. In particular, if E is a f_{ms}-BG, then Λ_E is a locally bounded special biserial Frobenius category.

Locally representation-finite category

From now, let k be an algebraically closed field.

Definition (Bongartz and Gabriel, 1982)

A locally bounded category Λ is called **locally representation-finite** if for every object x of Λ, the number of isomorphism classes of finitely generated indecomposable Λ-module I such that $I(x) \neq 0$ is finite.

Standard locally representation-finite category

- For a translation quiver Γ, let $k\Gamma$ be its path category, and let $k(\Gamma)$ be the **mesh category** of Γ, which is a factor category of $k\Gamma$ by the mesh ideal.
- For a locally bounded category Λ, we denote by $\text{ind}\Lambda$ the category formed by chosen representatives of the finitely generated indecomposable modules.

Standard locally representation-finite category

Definition (Bongartz and Gabriel, 1982)

A locally representation-finite category Λ is said to be **standard** if $k(\Gamma_\Lambda) \cong \mathrm{ind}\Lambda$, where Γ_Λ is the AR-quiver of Λ.

Proposition

If E is a f_s-BC such that Λ_E is locally representation-finite, then Λ_E is standard.

Connection with RFS algebras

We abbreviate indecomposable, basic, representation-finite self-injective algebra over k (not isomorphic to the underlying field k) by RFS algebra.

Theorem

The class of standard RFS algebras is equal to the class of finite-dimensional indecomposable representation-finite fractional Brauer configuration algebras in type S.

Corollary

The class of finite-dimensional representation-finite fractional Brauer configuration algebras in type S is closed under derived equivalence.

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Thank you!