Simple Functors over the Green Biset Functor of Section Burnside Rings

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Category of Biset Functors

Definition

Let G and H be finite groups. The biset Burnside group $B(H, G)$ is the Burnside group $B(H \times G^{op})$, i.e. the Grothendieck group of the category of finite $(H, G)-$bisets.

Fix a commutative ring with unity k.
Biset category $k\mathcal{C}$ is an k-linear category such that

1. The objects of $k\mathcal{C}$ are finite groups.
2. For finite groups G and H, $\mathrm{Hom}_{k\mathcal{C}}(G, H) = k \otimes_{\mathbb{Z}} B(H, G)$
3. If G, H, and K are finite groups, then the composition $v \circ u$ of the morphism $u \in \mathrm{Hom}_{\mathcal{C}}(G, H)$ and the morphism $v \in \mathrm{Hom}_{\mathcal{C}}(H, K)$ is equal to $v \times_H u$.
4. The composition of morphisms in $k\mathcal{C}$ is the k-linear extension of the composition in $\mathcal{C}$.
5. For any finite group G, the identity morphism of G in $k\mathcal{C}$ is equal to $[k \otimes_{\mathbb{Z}} \mathrm{Id}_G]$.

Category of Biset Functor

- Biset functor defined over $k\mathcal{C}$, with values in k-Mod is an k-linear functor from $k\mathcal{C}$ to k-Mod.
- We denote by $\mathcal{F}_{\mathcal{C},k}$ the category whose objects are biset functors and morphisms are natural transformations.

Example (Burnside functor)

For an object G of $\mathcal{C}$, the group $kB(G)$ defines a biset functor. If H is another object of $\mathcal{C}$, and if U is a finite (H, G)-biset, then the map $kB(U) : kB(G) \to kB(H)$ is induced by the correspondence sending a finite G-set X to the H-set $U \times_G X$.

Green Biset Functors

- Let A be a biset functor.
- We call A a *Green biset functor* if there are bilinear maps

$$A(G) \times A(H) \to A(G \times H), (a, b) \mapsto a \times b$$

  for each pair of finite groups G and H and an element $\epsilon_A \in A(1)$ with the properties, associativity, functoriality and identity element.
- **Functoriality** $A(\phi \times \psi)(a \times b) = A(\phi)(a) \times A(\psi)(b)$, where $a \in A(G)$, $b \in A(H)$ and $\phi : A(G) \to A(G')$, $\psi : A(H) \to A(H')$
- A Green biset functor is a biset functor with a ring structure.
- Then Green biset functors on $\mathcal{C}$, with values on k-Mod, form a category $\mathbf{Green}_{\mathcal{C},k}$.

Example

Burnside functor B is a Green biset functor on $\mathcal{C}$, with values in $\mathbb{Z}$-Mod, for the bilinear product $B(G) \times B(H) \to B(G \times H)$, defined by sending (X, Y) to the $(G \times H)$-set $X \times Y$. The identity element is $1 \in B(1) = \mathbb{Z}$.

A-modules

An A-module F is an object of $\mathcal{F}_{\mathcal{C},k}$ such that for any object G and H of $\mathcal{C}$, there are product maps

$$A(G) \times M(H) \to M(G \times H) \quad (a, m) \mapsto a \times m$$

- An A-module is a k-linear functor $\mathcal{P}_A \to k$-mod.
- We denote the (abelian) functor category of all A-modules by A-**Mod**.
- For instance for $A = kB$, the category kB-**Mod** is the same as the category $\mathcal{F}_{\mathcal{C},k}$ of all biset functors.

General Theory

- Let F be an A-module,
- $F(G)$ is a module over the endomorphism ring $E_G^A = \mathrm{End}_{\mathcal{P}_A}(G) = A(G \times G)$ of G in $\mathcal{P}_A$
- If G is minimal order such that $F(G) \neq 0$,
- Then $F(G)$ is annihilated by all the endomorphisms that factor through a group of smaller order (I_G).
- $\hat{A}(G) = E_G^A / I_G^A$ the essential algebra of A at G.
- $F(G)$ is an $\hat{A}(G)$-module.
- If S is simple and G is minimal for S then $S(G)$ is a simple $\hat{A}(G)$-module.
- $(G, S(G))$ seed for S.
- There is a correspondence between isomorphism classes of simple A-modules and equivalence classes of seeds.

Question

In general two seeds (G, V) and (G', V') may induce isomorphic simple A-modules. In this case, we say that (G, V) and (G', V') are equivalent.

Questions

1. *How to classify simple modules of essential algebra $\hat{A}_G$ for all G?*
2. *How one can define the equivalence relation on all these simple modules to determine isomorphism types of simple A-modules associated to them?*

General Theory

- Let E_G^c be a subalgebra of E_G^A such that

$$E_G^A = E_G^c + I_G^A.$$

- So $\hat{A}(G) = E_G^c/(E_G^c \cap I_G^A)$
- For instance, the truncated algebra $\tilde{e}_G^G A(G \times G)\tilde{e}_G^G$ is a covering algebra. Here $\tilde{e}_G^G$ is the image of the primitive idempotent e_G^G of the Burnside ring $B(G)$ in the double Burnside ring $B(G \times G)$ as defined in [4, Section 6.5].
- One aims to choose a covering algebra for which the intersection with the ideal I_G^A is easier to determine.

The Aim and Reason

- Our aim is to solve the classification problem for the functor of section Burnside rings.
- Section Burnside rings are introduced by Bouc in [5] as the Grothendieck rings of the subcategory of “Galois” morphisms in the category of morphisms of G-sets.
- It is different from the previous known examples
- The previous examples are known to have strong connections with the theory of representations of finite groups
- On the other hand section Burnside rings are relatively new and these kind of deep connections are not known yet
- Therefore it is close to be an abstract example.
- It can help us to solve the problem of classification of simple modules over Green biset functors in more general cases.

Section Burnside Functor

- The ring generated by the isomorphism classes of Galois morphisms of G-sets is called the section Burnside ring and is denoted by $\Gamma(G)$.
- The section Burnside ring $\Gamma(G)$ has a basis consisting of G-conjugacy classes of sections (T, S) of G. Here $S \trianglelefteq T \leqslant G$.

Theorem (10.6 of [5])

1. *For a finite (H, G)-biset U. The functor*

$$(X \xrightarrow{f} Y) \mapsto (U \times_G X \xrightarrow{U \times_G f} U \times_G Y)$$

from G-$\mathbf{Mor}^{Gal}$ to H-$\mathbf{Mor}^{Gal}$ induces a group homomorphism $\Gamma(U) : \Gamma(G) \to \Gamma(H)$.

2. *The correspondence $G \mapsto \Gamma(G)$ is a Green biset functor.*

Since a basis for $\Gamma(G \times H)$ is given in terms of sections of $G \times H$, we first parametrize these sections in a way as Goursat Theorem parametrizes subgroups of direct products.

Goursat Theorem for Sections

Theorem

There is a bijective correspondence between

1. the set of all sections $S \trianglelefteq T$ of $G \times H$ and
2. the set of all pairs $((P_1, K_1, \eta_1, L_1, Q_1), (P_2, K_2, \eta_2, L_2, Q_2))$ of Goursat quintuples satisfying the following conditions.

(i) $P_2 \subseteq P_1, K_2 \subseteq K_1, L_2 \subseteq L_1, Q_2 \subseteq Q_1$.

(ii) $(P_2/K_2, P_1/K_1, \partial)$ and $(Q_2/L_2, Q_1/L_1, \partial')$ are crossed modules where ∂ and ∂' is given by $\partial(xK_2) = xK_1$ and $\partial(xL_2) = xL_1$ and the actions are given by

$$aK_1 \cdot cK_2 = aca^{-1}K_2, \quad a'L_1 \cdot c'L_2 = a'c'a'^{-1}L_2$$

(iii) $(\eta_2, \eta_1) : (Q_2/L_2, Q_1/L_1, \partial') \to (P_2/K_2, P_1/K_1, \partial)$ is an isomorphism of crossed modules.

What we have done

- So to study simple modules over section Burnside functor we identify covering algebra for it.
- Our main theorem shows that covering algebra for section Burnside functor is Morita equivalent to the product of some matrix algebras.
- Essential algebra is isomorphic to the subalgebra of covering algebra.

Theorem

There exists a k-algebra isomorphism

$$E_G^c \xrightarrow{\sim} \bigoplus_{\{K,P\}_G \in \mathcal{G}_G/\sim} Mat_{\,|\{K,P\}_G|} k\Gamma_{(G,K,P)}$$

with the following property:

For every $\{K, P\}_G = \{(K_1, P_1), \cdots, (K_n, P_n)\} \in \mathcal{G}_G/\sim$, the isomorphism maps the element $f_{(K_i,P_i)} \in E_G^c$ to the diagonal idempotent matrices $e_i = diag(0, \cdots, 0, 1, 0, \cdots, 0) \in \text{Mat}_{\,|\{K,P\}_G|} k\Gamma_{(G,K,P)}$, $i = 1, \cdots, |\{K, P\}_G|$, in the $\{K, P\}_G$-component.

Essential Algebra

- We gave the intersection of the ideal I_G with covering algebra

Proposition

$$E_G^c \cap I_G = \bigoplus_{\substack{\{K,P\}_G \in \mathcal{G}_G/\sim \\ (K,P)\notin \mathcal{R}_G}} f_{\{K,P\}_G} E_G^c.$$

- Hence, we determined essential algebra for section Burnside ring,
- and showed that it is isomorphic to the subalgebra of covering algebra

$$\bigoplus_{\{K,P\}_G \in \mathcal{R}_G/\sim} f_{\{K,P\}_G} E_G^c.$$

Simple $\bar{E}_G$-modules

Finally we parameterize simple modules over $\bar{E}_G$.

Definition

$$\mathcal{S}_G = \mathcal{S}_k(G) := \{((K,P),[V]) \mid (K,P) \in \mathcal{R}_G, [V] \in \mathrm{Irr}(k\Gamma_{(G,K,P)})\}.$$

$((K,P),[V]) \sim ((K',P'),[V'])$ *if* (K,P) *and* (K',P') *are* G*-linked and the canonical bijection* $\mathrm{Irr}(k\Gamma_{(G,K',P')}) \xrightarrow{\sim} \mathrm{Irr}(k\Gamma_{(G,K,P)})$ *maps* $[V']$ *to* $[V]$.
$\tilde{\mathcal{S}}_G = \mathcal{S}_G / \sim$, *that is,*

$$\tilde{\mathcal{S}}_G := \{((K,P),[V]) \mid (K,P) \in \tilde{\mathcal{R}}_G, [V] \in \mathrm{Irr}(k\Gamma_{(G,K,P)})\}.$$

Theorem

Then there is a bijection between the set $\tilde{\mathcal{S}}_G$ *and* $\mathrm{Irr}(\bar{E}_G)$.

Main Theorem

Definition

We write the set of all seeds for $k\Gamma$ as

$$\mathrm{Seeds}(k\Gamma) = \{(G, K, P, [V]) | G \in Ob(\mathcal{P}_\Gamma), (K, P, [V]) \in \tilde{\mathcal{S}}_G\}.$$

Definition (Linkage)

$$(G, K, P, [V]) \sim (H, L, Q, [W])$$

if and only if $(G, K, P, 1) \sim (H, L, Q, 1)$ *and* $V \cong k[_{(G,K,P)}\Gamma_{(H,L,Q)}] \otimes_{k\Gamma_{(H,L,Q)}} W$.

Theorem

There is a bijective correspondence between

a. *the set $Irr(k\Gamma)$ of isomorphism classes of simple section biset functors*
b. *the set $Seeds(k\Gamma)/\sim$ of linkage classes of quadruples $(G, K, P, [V])$.*

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