Simple Functors over the Green Biset Functor of Section Burnside Rings

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Category of Biset Functors

Definition

Let G and H be finite groups. The biset Burnside group $B(H, G)$ is the Burnside group $B(H \times G^{op})$, i.e. the Grothendieck group of the category of finite $(H, G)-$bisets.

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2. For finite groups G and H, $\mathrm{Hom}_{k\mathcal{C}}(G, H) = k \otimes_{\mathbb{Z}} B(H, G)$
3. If G, H, and K are finite groups, then the composition $v \circ u$ of the morphism $u \in \mathrm{Hom}_{\mathcal{C}}(G, H)$ and the morphism $v \in \mathrm{Hom}_{\mathcal{C}}(H, K)$ is equal to $v \times_H u$.
4. The composition of morphisms in $k\mathcal{C}$ is the k-linear extension of the composition in $\mathcal{C}$.

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4. The composition of morphisms in $k\mathcal{C}$ is the k-linear extension of the composition in $\mathcal{C}$.
5. For any finite group G, the identity morphism of G in $k\mathcal{C}$ is equal to $[k \otimes_{\mathbb{Z}} \mathrm{Id}_G]$.

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Example (Burnside functor)

For an object G of $\mathcal{C}$, the group $kB(G)$ defines a biset functor. If H is another object of $\mathcal{C}$, and if U is a finite (H, G)-biset, then the map $kB(U) : kB(G) \to kB(H)$ is induced by the correspondence sending a finite G-set X to the H-set $U \times_G X$.

Green Biset Functors

- Let A be a biset functor.
- We call A a *Green biset functor* if there are bilinear maps

$$A(G) \times A(H) \to A(G \times H), (a, b) \mapsto a \times b$$

  for each pair of finite groups G and H and an element $\epsilon_A \in A(1)$ with the properties, associativity, functoriality and identity element.
- **Functoriality** $A(\phi \times \psi)(a \times b) = A(\phi)(a) \times A(\psi)(b)$, where $a \in A(G)$, $b \in A(H)$ and $\phi : A(G) \to A(G')$, $\psi : A(H) \to A(H')$

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Example

Burnside functor B is a Green biset functor on $\mathcal{C}$, with values in $\mathbb{Z}$-Mod, for the bilinear product $B(G) \times B(H) \to B(G \times H)$, defined by sending (X, Y) to the $(G \times H)$-set $X \times Y$. The identity element is $1 \in B(1) = \mathbb{Z}$.

A-modules

An A-module F is an object of $\mathcal{F}_{\mathcal{C},k}$ such that for any object G and H of $\mathcal{C}$, there are product maps

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- We denote the (abelian) functor category of all A-modules by A-**Mod**.
- For instance for $A = kB$, the category kB-**Mod** is the same as the category $\mathcal{F}_{\mathcal{C},k}$ of all biset functors.

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- $F(G)$ is an $\hat{A}(G)$-module.
- If S is simple and G is minimal for S then $S(G)$ is a simple $\hat{A}(G)$-module.
- $(G, S(G))$ seed for S.
- There is a correspondence between isomorphism classes of simple A-modules and equivalence classes of seeds.

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In general two seeds (G, V) and (G', V') may induce isomorphic simple A-modules. In this case, we say that (G, V) and (G', V') are equivalent.

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1. *How to classify simple modules of essential algebra $\hat{A}_G$ for all G?*
2. *How one can define the equivalence relation on all these simple modules to determine isomorphism types of simple A-modules associated to them?*

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- For instance, the truncated algebra $\tilde{e}_G^G A(G \times G)\tilde{e}_G^G$ is a covering algebra. Here $\tilde{e}_G^G$ is the image of the primitive idempotent e_G^G of the Burnside ring $B(G)$ in the double Burnside ring $B(G \times G)$ as defined in [4, Section 6.5].

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- One aims to choose a covering algebra for which the intersection with the ideal I_G^A is easier to determine.

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- Therefore it is close to be an abstract example.
- It can help us to solve the problem of classification of simple modules over Green biset functors in more general cases.

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Theorem (10.6 of [5])

1. *For a finite (H,G)-biset U. The functor*

$$(X \xrightarrow{f} Y) \mapsto (U \times_G X \xrightarrow{U \times_G f} U \times_G Y)$$

from G-$\mathbf{Mor}^{Gal}$ to H-$\mathbf{Mor}^{Gal}$ induces a group homomorphism $\Gamma(U) : \Gamma(G) \to \Gamma(H)$.

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Since a basis for $\Gamma(G \times H)$ is given in terms of sections of $G \times H$, we first parametrize these sections in a way as Goursat Theorem parametrizes subgroups of direct products.

Goursat Theorem for Sections

Theorem

There is a bijective correspondence between

1. the set of all sections $S \trianglelefteq T$ of $G \times H$ and
2. the set of all pairs $((P_1, K_1, \eta_1, L_1, Q_1), (P_2, K_2, \eta_2, L_2, Q_2))$ of Goursat quintuples satisfying the following conditions.

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(i) $P_2 \subseteq P_1, K_2 \subseteq K_1, L_2 \subseteq L_1, Q_2 \subseteq Q_1$.

(ii) $(P_2/K_2, P_1/K_1, \partial)$ and $(Q_2/L_2, Q_1/L_1, \partial')$ are crossed modules where ∂ and ∂' is given by $\partial(xK_2) = xK_1$ and $\partial(xL_2) = xL_1$ and the actions are given by

$$aK_1 \cdot cK_2 = aca^{-1}K_2, \quad a'L_1 \cdot c'L_2 = a'c'a'^{-1}L_2$$

(iii) $(\eta_2, \eta_1) : (Q_2/L_2, Q_1/L_1, \partial') \to (P_2/K_2, P_1/K_1, \partial)$ is an isomorphism of crossed modules.

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Theorem

There exists a k-algebra isomorphism

$$E_G^c \xrightarrow{\sim} \bigoplus_{\{K,P\}_G \in \mathcal{G}_G/\sim} Mat_{\,|\{K,P\}_G|} k\Gamma_{(G,K,P)}$$

with the following property:

For every $\{K,P\}_G = \{(K_1,P_1),\cdots,(K_n,P_n)\} \in \mathcal{G}_G/\sim$, the isomorphism maps the element $f_{(K_i,P_i)} \in E_G^c$ to the diagonal idempotent matrices $e_i = diag(0,\cdots,0,1,0,\cdots,0) \in \text{Mat}_{\,|\{K,P\}_G|} k\Gamma_{(G,K,P)}$,
$i = 1,\cdots,|\{K,P\}_G|$, in the $\{K,P\}_G$-component.

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Proposition

$$E_G^c \cap I_G = \bigoplus_{\substack{\{K,P\}_G \in \mathcal{G}_G/\sim \\ (K,P) \notin \mathcal{R}_G}} f_{\{K,P\}_G} E_G^c.$$

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Simple $\bar{E}_G$-modules

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Definition

$$\mathcal{S}_G = \mathcal{S}_k(G) := \{((K,P),[V]) \mid (K,P) \in \mathcal{R}_G, [V] \in \mathrm{Irr}(k\Gamma_{(G,K,P)})\}.$$

$((K,P),[V]) \sim ((K',P'),[V'])$ *if* (K,P) *and* (K',P') *are* G*-linked and the canonical bijection* $\mathrm{Irr}(k\Gamma_{(G,K',P')}) \xrightarrow{\sim} \mathrm{Irr}(k\Gamma_{(G,K,P)})$ *maps* $[V']$ *to* $[V]$.
$\tilde{\mathcal{S}}_G = \mathcal{S}_G / \sim$, *that is,*

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Theorem

Then there is a bijection between the set $\tilde{\mathcal{S}}_G$ *and* $\mathrm{Irr}(\bar{E}_G)$.

Main Theorem

Definition

We write the set of all seeds for $k\Gamma$ as

$$\mathrm{Seeds}(k\Gamma) = \{(G, K, P, [V]) | G \in Ob(\mathcal{P}_\Gamma), (K, P, [V]) \in \tilde{\mathcal{S}}_G\}.$$

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Definition (Linkage)

$$(G, K, P, [V]) \sim (H, L, Q, [W])$$

if and only if $(G, K, P, 1) \sim (H, L, Q, 1)$ *and*
$V \cong k[{}_{(G,K,P)}\Gamma_{(H,L,Q)}] \otimes_{k\Gamma_{(H,L,Q)}} W$.

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Definition

We write the set of all seeds for $k\Gamma$ as

$$\mathrm{Seeds}(k\Gamma) = \{(G, K, P, [V]) | G \in Ob(\mathcal{P}_\Gamma), (K, P, [V]) \in \tilde{\mathcal{S}}_G\}.$$

Definition (Linkage)

$$(G, K, P, [V]) \sim (H, L, Q, [W])$$

if and only if $(G, K, P, 1) \sim (H, L, Q, 1)$ *and*
$V \cong k[{}_{(G,K,P)}\Gamma_{(H,L,Q)}] \otimes_{k\Gamma_{(H,L,Q)}} W$.

Theorem

There is a bijective correspondence between

- a. *the set $Irr(k\Gamma)$ of isomorphism classes of simple section biset functors*
- b. *the set $Seeds(k\Gamma)/\sim$ of linkage classes of quadruples $(G, K, P, [V])$.*

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