Generic bases of cluster algebras in affine types

Lang Mou
(joint work with Xiuping Su)

University of Cologne

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Affine space of quiver representations

Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{|Q_0|} = \mathbb{N}^n$. There is an affine space

$$\mathrm{rep}(Q, \mathbf{d}) = \prod_{a \in Q_1} \mathrm{Hom}_K(K^{d_{t(a)}}, K^{d_{h(a)}})$$

of K-representations of Q of dimension $\mathbf{d}$. The group

$$\mathrm{GL}_{\mathbf{d}} = \prod_{i \in Q_0} \mathrm{GL}_{d_i}(K)$$

acts on $\mathrm{rep}(Q, \mathbf{d})$ by

$$(g_i)_{i \in Q_0} \cdot (M_a)_{a \in Q_1} = \left(g_{h(a)}^{-1} \cdot M_a \cdot g_{t(a)} \right)_{a \in Q_1}.$$

Generic F-polynomial

Quiver grassmannian:

$$\mathrm{Gr}_{\mathbf{e}}\, M := \{N \subset M \mid \dim N = \mathbf{e}\} \subset \prod_{i=1}^{n} \mathrm{Gr}(e_i, d_i).$$

F-polynomial (take $K = \mathbb{C}$):

$$F_M = \sum_{\mathbf{e}} \chi_{\mathrm{an}}(\mathrm{Gr}_{\mathbf{e}}\, M) \prod_{i=1}^{n} y_i^{e_i} \in \mathbb{Z}[y_1, \ldots, y_n].$$

The function $M \mapsto F_M$ is constructible on $\mathrm{rep}(Q, \mathbf{d})$. Denote the generic value by $F_{\mathbf{d}}$.

Generic Caldero–Chapoton function

Let Q be acyclic and $B \in \mathbb{Z}^{n\times n}$ the skew-symmetric matrix of Q. Define another vector

$$\mathbf{g} = (-d_i + \sum_j [-b_{ij}]_+ d_j)_i \in \mathbb{Z}^n.$$

Definition

The generic Caldero–Chapoton function is

$$X_{\mathbf{g}} = F_{\mathbf{d}(\mathbf{g})}(\hat{y}_1, \ldots, \hat{y}_n) \prod_{i \in Q_0} x_i^{g_i} \quad \text{where} \quad \hat{y}_i = \prod_j x_j^{b_{ji}}.$$

Many people have considered generic CC functions in various contexts including Dupont, Ding–Xiao–Xu, Geiss–Leclerc–Schröer, Plamondon etc.

Generic Caldero–Chapoton function

Theorem (Geiss–Leclerc–Schröer)

The generic CC functions $\{X_{\mathbf{g}} \mid \mathbf{g} \in \mathbb{Z}^n\}$ form a basis of the cluster algebra $\mathcal{A}(B)$ and contain all cluster monomials.

The function $X_{\mathbf{g}}$ is a cluster monomial when there is rigid $M \in \mathrm{rep}(Q, \mathbf{d})$, in which case the orbit $\mathcal{O}_M$ is open (Caldero–Keller).

Skew-symmetrizable case

Let B be skew-symmetrizable with symmetrizer $D = \operatorname{diag}(c_i)$ and acyclic.
Geiss, Leclerc and Schröer defined a fin. dim. K-algebra $H(B, D)$.

$$H_i = \mathbb{C}[\varepsilon_i]/\varepsilon_i^{c_i}, \quad H_{ij} = \left(H_i \otimes H_j / \varepsilon_i^{\frac{|b_{ij}|}{g_{ij}}} \otimes 1 - 1 \otimes \varepsilon_j^{\frac{|b_{ji}|}{g_{ij}}} \right)^{g_{ij}}.$$

The algebra H can be defined as the tensor algebra (over $S = \prod H_i$)

$$\bigoplus_{k \geq 0} \left(\bigoplus_{b_{ij} > 0} H_{ij} \right)^{\otimes_S^k}.$$

Skew-symmetrizable case

There is an affine space of **locally free** H-modules of rank $\mathbf{r} \in \mathbb{N}^n$

$$\operatorname{rep}(H, \mathbf{r}) := \prod_{b_{ij}>0} \operatorname{Hom}_{H_i}(H_{ij} \otimes_{H_j} H_j^{r_j}, H_i^{r_i}),$$

with an action of $\prod \operatorname{GL}_{r_i}(H_i)$.
Replace quiver grassmaniann with the quasi-projective variety

$$\operatorname{Gr}_{\mathbf{e}}^{\mathrm{lf}}(M) = \{N \subset M \mid N \text{ loc. free, } \operatorname{rank} N = \mathbf{e}\}.$$

There is also generic (l.f.) F-polynomial $F_{\mathbf{r}}$ on $\operatorname{rep}(H, \mathbf{r})$.

Main theorem

Theorem (Su–M.)

If $\widetilde{B} \in \mathbb{Z}^{m \times n}$ is of affine type and of full rank, then

$$\left\{ X_{\mathbf{g}} = F_{\mathbf{r}(\mathbf{g})}(\hat{y}_1, \ldots, \hat{y}_n) \prod_{i=1}^{m} x_i^{g_i} \mid \mathbf{g} \in \mathbb{Z}^m \right\}$$

is a $\mathbb{Z}$-basis of $\mathcal{A}(\widetilde{B})$ and contains all cluster monomials.

In finite types, the theorem is due to Geiss–Leclerc–Schröer.

They conjecture that for any acyclic B,

$$X_M = F_M(\hat{y}_1, \dots, \hat{y}_n) \cdot x^{\mathbf{g}(M)}$$

gives the corresponding cluster monomial when M is locally free and rigid. Our main theorem verifies the conjecture for all affine types.

Reflections of generic F-polynomials

Let k be a sink and $H' = H(\mu_k(B), D)$.
Let $\mathbf{v}, \mathbf{v}' \in \mathbb{Z}^n$ be (decorated) rank vectors related by 'mutation' at k.

Proposition (Su–M.)

The generic F-polynomials satisfy the recurrence

$$(1 + y_k)^{-[v_k]_+} F_{\mathbf{v}}^{H}(y_1, \ldots, y_n) = (1 + y_k')^{-[-v_k]_+} F_{\mathbf{v}'}^{H'}(y_1', \ldots, y_n')$$

where $y_i' = y_i y_k^{[b_{ki}]_+}(y_k + 1)^{-b_{ki}}$ for $i \neq k$ and $y_k' = y_k^{-1}$.

Corollary

For $((x_1, \ldots, x_m), \widetilde{B}) \xrightarrow{\mu_k} ((x_1', \ldots, x_m'), \widetilde{B}')$, we have

$$X_{\mathbf{g}}^{\widetilde{B}}(x_1, \ldots, x_m) = X_{\mathbf{g}'}^{\widetilde{B}'}(x_1', \ldots, x_m').$$

Proof: $X_{\mathbf{g}}$ is universal Laurent

1. Perform sink/source mutations to reach a bipartite seed t_0.
2. Mutate in every direction $t_0 \overset{k}{\text{———}} t_k$ for $k = 1, \dots, n$.
3. Berenstein–Fomin–Zelevinsky (Corollary 1.9)

$$\bigcap_{k=0}^{n} \mathbb{Z}[x_{1;t_k}^{\pm}, \dots, x_{m;t_k}^{\pm}] = \bigcap_{\text{all seeds}} \mathbb{Z}[x_{1;t}^{\pm}, \dots, x_{m;t}^{\pm}].$$

Proof: $\{X_{\mathbf{g}} \mid \mathbf{g} \in \mathbb{Z}^m\}$ is a $\mathbb{Z}$-basis

Apply a maximal green (source mutation) sequence

$$v_0 \xrightarrow{\ n\ } v_1 \xrightarrow{\ n-1\ } \cdots \xrightarrow{\ 2\ } v_{n-1} \xrightarrow{\ 1\ } v_n.$$

Under the full rank assumption, every $X_{\mathbf{g}}$ is *compatibly pointed* in each seed $v_0, \ldots, v_n$. A powerful theorem of Fan Qin then applies:

$$\bigoplus_{\mathbf{g} \in \mathbb{Z}^m} \mathbb{Z} \cdot X_{\mathbf{g}} = \bigcap_{\text{all seeds}} \mathbb{Z}[x_{1;t}^{\pm}, \ldots, x_{m;t}^{\pm}],$$

which is known to equal $\mathcal{A}(\widetilde{B})$.

Proof: CC formula for cluster monomials

Which are the cluster variables in $\{X_{\mathbf{g}}\}$?

$$\{\text{non-ini. cluster variables}\} \xrightarrow{\sim} \{\text{ind. l.f. rigid } H\text{-modules}\}$$

$$x \mapsto \mathbf{d}(x) \qquad \Delta_{\mathrm{rS}} \qquad M \mapsto \operatorname{rank} M$$

In affine types Real Schur roots have linear algebraic classification (Reading–Stella) by considering the Coxeter action on the root system.

$$\Delta_{\mathrm{rS}} = \{\text{preprojectives}\} \sqcup \{\text{preinjectives}\} \sqcup \{\text{regular Schur roots}\}$$

Proof: CC formula for cluster monomials

1. **Preprojectives and preinjectives**.

- Directly verify X_{E_i} where rank $E_i = \alpha_i$ simple root. Its F-polynomial is simply $1 + y_i$.
- Apply sink/source mutations to reach indecomposable injective or projective modules.
- Apply Coxeter transformations to reach other preprojectives and preinjectives.

Proof: CC formula for cluster monomials

2. **Regular Schur roots**. They are organized as follows.

1. $\{\beta_{n,m} \mid (n,m) \in \mathbb{Z} \times \mathbb{Z}/d\mathbb{Z},\ n \leq d-1\}$
2. $\beta_{n,m} + \beta_{n,m+1} = \beta_{n+1,m} + \beta_{n-1,m+1}$ (mesh relation)
3. $c(\beta_{n,m}) = \beta_{n+1,m}$ (Coxeter action)
4. $\beta_{1,1}, \ldots, \beta_{1,d-1}$ are positive roots of B_{fin}, generate A_{d-1}-root system Φ as simple roots.

If $\beta \in \Phi$, then F_β can be computed using H_{fin}-modules.
There is at least one β in Φ on each level! Then apply Coxeter transformation to reach all $\beta_{n,m}$.

For example, let $B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$ of type $\widetilde{\mathbb{B}}_3$.

Regular Schur roots are

$$\text{level 2:} \quad (0,1,1,0) \xrightarrow{c} (2,1,2,2) \xrightarrow{c} (2,2,1,2) \xrightarrow{c} (0,1,1,0)$$

$$\text{level 1:} \quad (0,1,0,0) \xrightarrow{c} (0,0,1,0) \xrightarrow{c} (2,1,1,2) \xrightarrow{c} (0,1,0,0).$$

Thank you!