Projective-injective modules of Temperley-Lieb algebras

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Temperly-Lieb algebras

K: any field

r: a natural number

For $\delta \in K$, the Temperley-Lieb algebra $TL_r(\delta)$ is a K-algebra defined by generators $\{u_1, \ldots, u_{r-1}\}$ and relations:

$$\begin{array}{lrll} (1) & u_i^2 & = \delta u_i, & \\ (2) & u_i u_j & = u_j u_i, & |i-j| > 1, \\ (3) & u_i u_{i\pm 1} u_i & = u_i. & \end{array}$$

Remark. There exists a ring epimorphism:

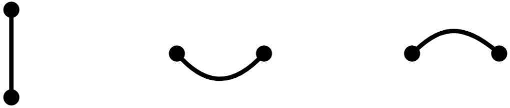
$$KS_r \twoheadrightarrow TL_r(-2)$$

which sends $(i, i+1)$ to u_i.

Temperley-Lieb algebras

In this talk, the following diagrammatic presentation is preferred.

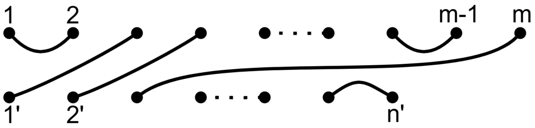
Temperley-Lieb category $TL(\delta)$ is a K-linear monoidal additive category generated by



Objects: $\{0, 1, 2, \cdots\}$

Morphisms: for $m, n \in \mathbb{N}$, $\mathrm{Hom}_{TL(\delta)}(m, n)$ is a K-vector space with the basis of diagrams of the form

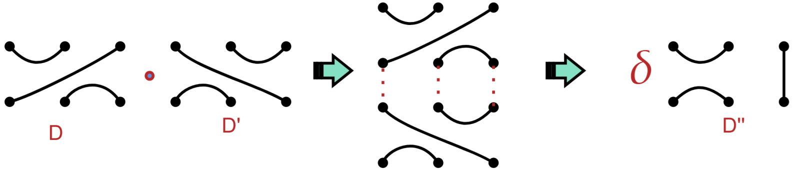
Temperley-Lieb algebras



(Recipe: pair points in two rows by curves without crossing!)

Remark. $\text{Hom}_{TL(\delta)}(m, n) \neq 0$ if and only if $m \equiv n \pmod 2$.

Composition Rule: concatenation of diagram, e.g.,



Temperley-Lieb algebras

In general, for diagrams $D : m \to n$ and $D' : n \to r$,

$$D \circ D' = \delta^{\#\text{circles}} D''$$

for a unique diagram $D'' : m \to r$.

Remark. For $r \geq 1$,

$$\text{End}_{TL(\delta)}(r) \cong TL_r(\delta) \qquad \text{(Temperley-Lieb algebra)}$$

In particular, for $m, n \in \mathbb{N}$,

$$\text{Hom}_{TL(\delta)}(m, n)$$

is a $(TL_m(\delta), TL_n(\delta))$-module.

Temperley-Lieb algebras

Temperley-Lieb algebras arise naturally in Schur-Weyl duality.

q: a nonzero element in K

G_q: quantum GL_n (i.e., general linear group $GL_n(K)$ if $q = 1$)

V: natural representation of G_q (i.e., vector space K^n if $q = 1$)

For $r \geq 1$, G_q acts on $V^{\otimes r}$, let

$$H_{n,r} = \mathrm{End}_{G_q}(V^{\otimes r})$$

Remark. If $n = 2$, then $H_{2,r} \cong TL_r(-q - q^{-1})$ as K-algebras

The (quantised) Schur algebra may be defined as:

$$S_q(n,r) = \mathrm{End}_{H_{n,r}}(V^{\otimes r}).$$

Projective-injective modules of $TL_r(\delta)$

A: finite dimensional K-algebra

An A-module is called projective-injective if it is both projective and injective.

Projective-injective modules are of interest in study of

- Dominant dimension
- Schur-Weyl duality
- Tilting theory
- Algebraic Lie theory
- $\cdots\cdots$

Projective-injective modules of $TL_r(\delta)$

Q1: Classify indecomposable projective-injective modules for the Temperley-Lieb algebra $TL_r(\delta)$.

Q2: Determine the dominant dimension of $TL_r(\delta)$.

Q3: Determine the dominant dimension of $S_q(2, r)$.

Projective-injective modules of $TL_r(\delta)$

Motivation on the classification problem (Q1).

De Visscher and Donkin (2005):

$\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of n parts and $r = \lambda_1 + \cdots + \lambda_n$.

Conjecture: Projective $S_q(n, r)$-module $P_n(\lambda)$ is injective if and only if one of the following holds:

(1) $\lambda_1 < (n-1)(\ell - 1)$; (2) $\lambda = (\ell p^m - 1)\delta + w_0\mu + \ell p^m v$

where $m \geq 0$, μ and ν are partitions satisfying some technical conditions.

Here, $p = \mathrm{char}(K)$ and ℓ is the smallest number, s.t.,

$$1 + q + q^2 + \cdots + q^{\ell-1} = 0$$

Projective-injective modules of $TL_r(\delta)$

and $\delta = (n-1, n-2, \ldots, 1, 0)$, $w_0 \in S_n$ (symmetric group on n-letters) is the permutation s.t., $w_0(\mu_1, \ldots, \mu_n) = (\mu_n, \ldots, \mu_1)$.

This conjecture may be approached via :

- representations of quantum groups:
  (True for $n = 2, 3$, De Visscher-Donkin 2005)
- representations of (quantised) Schur algebras $S_q(n, r)$:
  (True if $r \leq n$ by Schur-Weyl duality;
  True if $r \leq n(p-1)$ and $q = 1$, F-2014)
- representations of Hecke algebras:
  (classification of indecomposable proj-injective modules of TL-algebras is the first step)

Projective-injective modules of $TL_r(\delta)$

Remark. De Visscher and Donkin's conjecture is about a set $\mathcal{X}_{\ell,p}(n,r)$ of partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of r, satisfying

(1) $\lambda_1 < (n-1)(\ell-1)$; or (2) $\lambda = (\ell p^m - 1)\delta + w_0\mu + \ell p^m v$

- It is generally indirect to tell whether a partition belongs to $\mathcal{X}_{\ell,p}(n,r)$;
- Combinatorial pattern of $\mathcal{X}_{\ell,p}(n,r)$ is not studied.

Projective-injective modules of TL-algebras

Motivation on the dominant dimension (Q2 and Q3)

The study of dominant dimension of (quantised) Schur algebras is closely related to the homological behavior of Schur functors (relating (Hochschild) cohomologies of quantum GL_n to that of Hecke algebras).

Assume $n \geq r$.

- $\operatorname{dom.dim} S_q(n,r) = 2(\ell - 1)$ if $\ell \leq r$ and ∞ otherwise. (F-Koenig 2011 for $q = 1$ and F-Miyachi 2019 for any q)
- For each non-semisimple block B of $S_q(n,r)$,

$$\operatorname{dom.dim} B = 2(\ell - 1).$$

(F-Hu-Koenig 2022)

If $n < r$, still little is known about $\operatorname{dom.dim} S_q(n,r)$.

Example. $p = 2, q = 1$, $\operatorname{dom.dim} S(2,4) = 0$.

If $q = 1$ and $r \leq n(p-1)$, then $\operatorname{dom.dim} S(n,r) \geq 2$ (F-2014).

For any n, r, p and q, $\operatorname{dom.dim} S_q(n,r) \geq 2$ iff $\operatorname{dom.dim} H_{n,r} \geq 2$ (F-Hu 2023)

In particular, $\operatorname{dom.dim} S_q(2,r) \geq 2$ iff $\operatorname{dom.dim} TL_r(\delta) \geq 2$.

Q: Characterize when $\operatorname{dom.dim} TL_r(\delta) \geq 2$.

Remark. Erdmann and Tiago studied the relative dominant dimension of $TL_r(\delta)$.

Main results

$p = \mathrm{char}(K)$

$\delta \in K$

Spencer (2022): let $q : \mathbb{N}_{\geq 0} \to K$, s.t., $q(0) = 0, q(1) = 1$, and

$$q(n+1) = \delta q(n) - q(n-1)$$

n		n	
0	0	5	$\delta^4 - 3\delta^2 + 1$
1	1	6	$\delta^5 - 4\delta^3 + 3\delta$
2	δ	7	$\delta^6 - 5\delta^4 + 6\delta^2 - 1$
3	$\delta^2 - 1$	8	$\delta^7 - 6\delta^5 + 10\delta^3 - 4\delta$
4	$\delta^3 - 2\delta$	9	$\delta^8 - 7\delta^6 + 15\delta^4 - 10\delta^2 + 1$

Main results

Set $\ell = \min\{n \in \mathbb{N} \mid q(n) = 0\}$.

Theorem (Westbury(1995))

$TL_r(\delta)$ is semisimple if and only if $\ell = \infty$; Simple modules are of the form:

- $\delta \neq 0$, $\{L_r(m) \mid 0 \leq m \leq r, m \equiv r \pmod{2}\}$
- $\delta = 0$, $\{L_r(m) \mid 0 < m \leq r, m \equiv r \pmod{2}\}$

$P_r(m)$ is the projective cover of $L_r(m)$.

Definition

For $r \in \mathbb{N}$, let $r + 1 = a_0 + a_1\ell + a_2\ell p + \cdots + a_s\ell p^s$ be the (ℓ, p)-adic expansion of $r + 1$. Define

$$t(r) := 2a_s\ell p^s - r - 2.$$

Main results

Theorem (F-Yin(2024))

(1) *If* $P_r(m)$ *is projective-injective, then so are* $P_{r-1}(m+1)$ *and* $P_{r-2}(m)$.

(2) *For* $s \geq 1$, $r = \ell p^{s-1}$, $P_r(r-2)$ *is NOT projective-injective.*

(3) *For* $s \geq 0$, $P_r(\ell p^{s-1} - 1)$ *is projective-injective if*

$$r < \ell p^s + (p-1)\ell p^{s-1} - 1.$$

Example: r odd and $q = 1$ and $p = 3$,

✓ 3	-1	1																																											
5	-1	-1	1																																										
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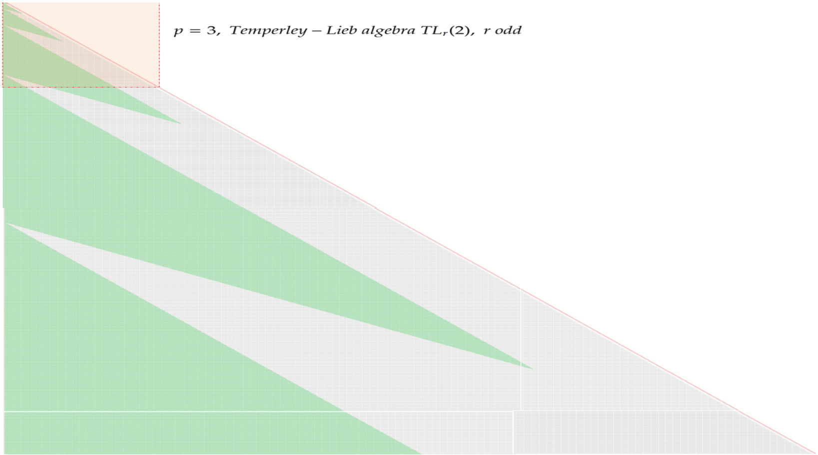
$P_3(1), P_9(7), P_{27}(25), \ldots$ are not projective-injective.
$P_1(1), P_{11}(3), P_{41}(9), \ldots$ are projective-injective.

Example: r odd and $q = 1$ and $p = 3$,



Remark. The set $X_{\ell,p}(2, r)$ exhibits a nice pattern (shadow of Frobenius twist in Lie theory).

Main results

Corollary

Set $\ell p^{-1} = 1$ *and* $\ell p^{-2} = 0$.

(1) *If* $r \in [\ell p^{s-1} - 1, \ell p^{s-1} + (p-1)\ell p^{s-2} - 3]$ *for some* $s \geq 0$, *then* $P_r(m)$ *is projective-injective, iff*

$$m \in [\ell p^{s-2} - 1, t(r) - 2] \cup [\ell p^{s-1} - 1, r].$$

(2) *If* $r \in (\ell p^{s-1} + (p-1)\ell p^{s-2} - 3, \ell p^{s} - 1)$ *for some* $s \geq 0$, *then* $P_r(m)$ *is projective-injective iff*

$$m \in [\ell p^{s-1} - 1, r].$$

Idea: relate projective-injective modules of $TL_r(\delta)$ for different r.

(a) Use the fact $TL_r(\delta) \hookrightarrow TL_{r+1}(\delta)$ and $TL_r(\delta) \cong eTL_{r+1}(\delta)e$ for some idempotent e.

Main results

(b) Analysis on the bimodule $Q_n(m) = \mathrm{Hom}(n, m)$:

$$Q_{n+s}(m-s)\downarrow_{TL_n(\delta)} \cong Q_n(m), \qquad s \leq m \leq n.$$

If $\delta \neq 0$, then $Q_n(m)$ is projective for $0 \leq m \leq n, m \equiv n (\mathrm{mod} 2)$.

If $\delta = 0$, then $Q_n(m)$ is projective for $0 < m \leq n, m \equiv n (\mathrm{mod} 2)$.

Remark. If $\delta = -q - q^{-1}$ for some $q \in K^*$, the classification of indecomposable $TL_r(\delta)$-modules can be obtained from the fact that De Visscher and Donkin's conjecture holds true for $n = 2$ and the Schur-Weyl duality.

Main results

The dominant dimension of an algebra A is at least t, written $\operatorname{dom.dim} A \geq t$, if in a minimal injective coresolution:

$$0 \to {}_AA \to I^0 \to I^1 \to \cdots \to I^{t-1} \to I^t \cdots$$

$I^0, I^1 \ldots, I^{t-1}$ are all projective-injective.

$S_q(2,r)$ is a quasi-hereditary algebra with the standard modules $\Delta(\lambda)$ (Weyl modules), where $\lambda = (\lambda_1, \lambda_2)$ is a partition of r.

Proposition (Deriziotist(1981), Doty(1985), Erdmann(2023))

$\operatorname{soc}\Delta(\lambda) = L(\mu)$, *where* $\mu = (\mu_1, \mu_2)$ *is a partition of* r *with*

$$\mu_1 - \mu_2 = t(\lambda_1 - \lambda_2).$$

Theorem (F-Yin(2024))

$\operatorname{dom.dim} TL_r(\delta) \geq 2$ *iff* $\operatorname{dom.dim} S_q(2,r) \geq 2$ *iff*

$$r \notin [\ell p^{s-1}, 2\ell p^{s-1} - 4], \quad \text{for any } s \geq 2.$$

Idea: Combine the last proposition with the first main result and the following facts:

- $\operatorname{dom.dim} S_q(2, r) \geq 1$ if each Weyl module is embedded into a projective-injective module, known as torsionless;
- $\operatorname{dom.dim} S_q(2, r) \geq 2$ implies that each Weyl module is torsionless (F-Koenig 2011);
- $\operatorname{dom.dim} S_q(2, r) = 0, \infty$ or $2m$ for some $m \in \mathbb{N}$ (F-2014);
- $\operatorname{dom.dim} S_q(2, r) \geq 2$ iff $\operatorname{dom.dim} TL_r(\delta) \geq 2$ (F-Hu 2023);

Thank you for your attention!