Positivity for quantum cluster algebras from orbifolds

Min Huang

School of Mathematics (Zhuhai)
Sun Yat-sen University

arXiv:2406.03362

ICRA 21, SJTU

08-06, 2024

Quantum cluster algebras from surfaces

Definition

A quantum cluster algebra $\mathscr{A}_v$ is called *coming from an orbifold* Σ if the commutative cluster algebra $\mathscr{A}_v \mid_{v=1}$ is coming from Σ.

Theorem (Fomin-Shapiro-Thurston [2], Berenstein-Zelevinsky [1])

Let $\mathscr{A}_v$ be a quantum cluster algebra from Σ.

(a) If Σ is not a closed orbifold with one puncture, then there are bijections,

$$\{\text{Tagged arcs in } \Sigma\} \to \{\text{Quantum cluster variables of } \mathscr{A}_v\}.$$

$$\{\text{Tagged triangulation of } \Sigma\} \to \{\text{Quantum seeds of } \mathscr{A}_v\}.$$

(b) If Σ is a closed orbifold with one puncture, then there are bijections

$$\{\text{Ordinary arcs in } \Sigma\} \to \{\text{Quantum cluster variables of } \mathscr{A}_v\}.$$

$$\{\text{Ideal triangulation of } \Sigma\} \to \{\text{Quantum seeds of } \mathscr{A}_v\}.$$

Objective

To give an expansion formula for X_β with respect to $X(\Delta)$, where β is a tagged/ordianary arc, Δ is a tagged/ideal triangulation.

Objective

To give an expansion formula for X_β with respect to $X(\Delta)$, where β is a tagged/ordianary arc, Δ is a tagged/ideal triangulation.

Objective

To give an expansion formula for X_β with respect to $X(\Delta)$, where β is a tagged/ordianary arc, Δ is a tagged/ideal triangulation.

- γ ordinary arc connection p and q, T^o ideal triangulation, T the tagged triangulation corresponding to T^o.

Objective

To give an expansion formula for X_β with respect to $X(\Delta)$, where β is a tagged/ordianary arc, Δ is a tagged/ideal triangulation.

- γ ordinary arc connection p and q, T^o ideal triangulation, T the tagged triangulation corresponding to T^o.

  1. $\beta = \gamma$, $\Delta = T$;

Objective

To give an expansion formula for X_β with respect to $X(\Delta)$, where β is a tagged/ordianary arc, Δ is a tagged/ideal triangulation.

- γ ordinary arc connection p and q, T^o ideal triangulation, T the tagged triangulation corresponding to T^o.
  1. $\beta = \gamma$, $\Delta = T$;
  2. $\beta = \gamma^{(q)}$, q puncture, $\Delta = T$ contains no arcs tagged notched at q;

Objective

To give an expansion formula for X_β with respect to $X(\Delta)$, where β is a tagged/ordianary arc, Δ is a tagged/ideal triangulation.

- γ ordinary arc connection p and q, T^o ideal triangulation, T the tagged triangulation corresponding to T^o.
  1. $\beta = \gamma$, $\Delta = T$;
  2. $\beta = \gamma^{(q)}$, q puncture, $\Delta = T$ contains no arcs tagged notched at q;
  3. $\beta = \gamma^{(p,q)}$, p, q punctures, $\Delta = T$ contains no arcs tagged notched at p or q.

Objective

To give an expansion formula for X_β with respect to $X(\Delta)$, where β is a tagged/ordianary arc, Δ is a tagged/ideal triangulation.

- γ ordinary arc connection p and q, T^o ideal triangulation, T the tagged triangulation corresponding to T^o.
  1. $\beta = \gamma$, $\Delta = T$;
  2. $\beta = \gamma^{(q)}$, q puncture, $\Delta = T$ contains no arcs tagged notched at q;
  3. $\beta = \gamma^{(p,q)}$, p, q punctures, $\Delta = T$ contains no arcs tagged notched at p or q.
- (γ, T) ordinary arc, $(\gamma^{(q)}, T)$ singly-notched arc, $(\gamma^{(p,q)}, T)$ doubly-notched arc.

Ideal

Theorem (Musiker-Schiffler-Williams [3])

Let $\mathscr{A}$ be a commutative cluster algebra from Σ. For any ordinary arc γ and ideal triangulation T^o, we have

$$x_\gamma = \sum_{P \in \mathscr{P}(G_{T^o,\gamma})} x(P).$$

ideal

For each perfect matching P, associative with an integer $w(P)$ and quantum Laurent monomial $X(P)$ such that

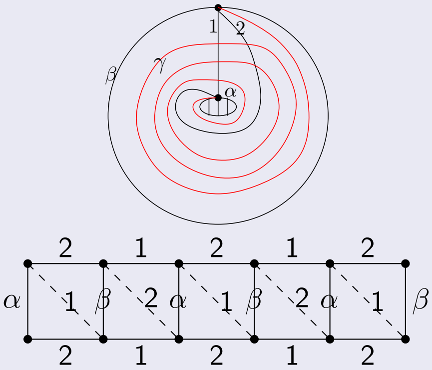
$$X_\gamma = \sum_{P \in \mathscr{P}(G_{T^o,\gamma})} v^{w(p)} X(P).$$

Expansion formula for ordinary arcs

Lattice $\mathscr{L}(T, \gamma)$

- Fix γ ordinary arc, T^o ideal triangulation, T the tagged triangulation corresponding to T^o
- Snake graph $G_{T,\gamma}$ is constructed via the following:
  - γ crosses T^o at $p_1, \cdots, p_c$ in order;
  - Associate a tile G_i with each point p_i;
  - Glue $G_i, i = 1, \cdots, c$ to obtain $G_{T,\gamma}$.
- A *perfect matching* of $G_{T,\gamma}$ is a set P of edges such that each vertex incident to exactly one edge in P;
- $\mathscr{L}(T, \gamma)$:={the set of all perfect matchings of $G_{T,\gamma}$}.

Snake graph – An example



Lattice $\mathscr{L}(T, \gamma)$

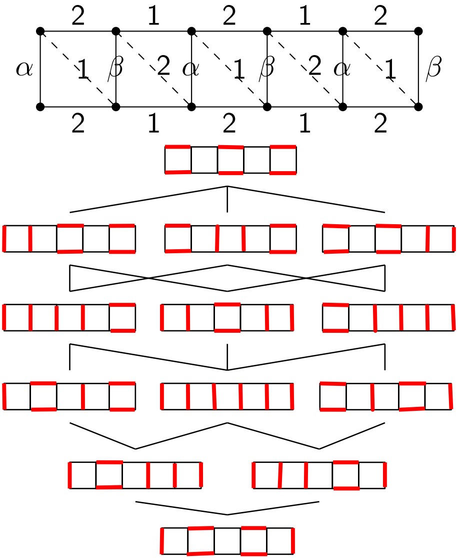
Definition

- We say that $P \in \mathscr{L}(T, \gamma)$ can *twist* on a tile G_i if there are two edges of G_i in P. The perfect matching obtained by replacing the two edges by the remaining two edges of G_i is called the *twist* of P at G_i, denoted by $\mu_{G_i}(P)$.
- let $P < \mu_{G_i}(P)$ if $W(G_i), E(G_i) \in P, rel(G_i, T^o) = 1$ or $N(G_i), S(G_i) \in P, rel(G_i, T^o) = -1$.

Proposition

$\mathscr{L}(T, \gamma)$ is a lattice with maximum/minimum element P_+/P_-.

An example



Valuation map on $\mathscr{L}(T,\gamma)$

P can twist at a tile G_l with diagonal labeled τ_{i_l}.

- $G_l^- := G_1 \cup G_2 \cup \cdots \cup G_{l-1}$, $G_l^+ := G_{l+1} \cup G_{l+2} \cup \cdots \cup G_c$
- $m^{\pm}(P, G_l; \alpha) :=$
  number of edges labeled α in $P \cap edge(G_l^{\pm}) \setminus edge(G_l)$.
- $n(G,\alpha) :=$ number of diagonals labeled α of G.
- $$\begin{aligned} \Omega(P, G_l) : &= d(\tau_{i_l})[(m^+(P, G_l; \tau_{i_l}) - m^-(P, G_l; \tau_{i_l})) \\ &- (n(G_l^+; \tau_{i_l}) - n(G_l^-; \tau_{i_l}))]. \end{aligned}$$

Valuation map on $\mathscr{L}(T, \gamma)$

Proposition

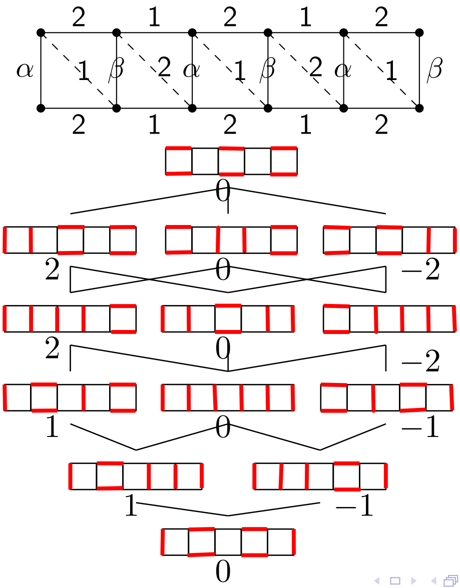
There is a unique map $w : \mathscr{L}(T, \gamma) \to \mathbb{Z}$ such that

1. (Initial condition) $w(P_-) = 0$, where P_- is the minimum element in $\mathscr{L}(T, \gamma)$,
2. (Recurrence condition)

$$w(\mu_{G_l}(P)) = w(P) + \Omega(P, G_l)$$

for any $P \in \mathscr{L}(T, \gamma)$ such that P can twist on G_l with $P < \mu_{G_l}(P)$.

An example



Expansion formula

Theorem

For any $P \in \mathscr{L}(T, \gamma)$, we can associate with a quantum Laurent monomial $X(P)$ in $\mathscr{T}(T)$. Then

$$X_\gamma = \sum_{P \in \mathscr{L}(T,\gamma)} v^{w(P)} X(P).$$

Ideal of the proof

An observation

- $f = x + y + x^{-2}y^2 + x^{-2}yz + x^{-2}zy + x^{-2}z^2 = x'^{-1}y + x'^{-1}z + y + x'^2$ with $xx' = y + z$.
- $\mathscr{P} = \{x, y, x^{-2}y^2, x^{-2}yz, x^{-2}zy, x^{-2}z^2\}$ and $\mathscr{P}' = \{x'^{-1}y, x'^{-1}z, y, x'^2\}$.
- $\{x\} \leftrightarrow \{x'^{-1}y, x'^{-1}z\}$, $\{y\} \leftrightarrow \{y\}$, $\{x^{-2}y^2, x^{-2}yz, x^{-2}zy, x^{-2}z^2\} \leftrightarrow \{x'^2\}$.

Definition

Let $\mathscr{P}, \mathscr{P}'$ be finite sets.

Definition

Let $\mathscr{P}, \mathscr{P}'$ be finite sets.

(1) A *partition* of $\mathscr{P}$ is a finite collection of subsets $\mathscr{P}_i, i \in I$ such that $\cup_{i\in I}\mathscr{P}_i = \mathscr{P}$ and $\mathscr{P}_i \cap \mathscr{P}_j = \emptyset$ for any $i \neq j \in I$.

Definition

Let $\mathscr{P}, \mathscr{P}'$ be finite sets.

(1) A *partition* of $\mathscr{P}$ is a finite collection of subsets $\mathscr{P}_i, i \in I$ such that $\cup_{i\in I}\mathscr{P}_i = \mathscr{P}$ and $\mathscr{P}_i \cap \mathscr{P}_j = \emptyset$ for any $i \neq j \in I$.

(2) A *partition bijection* from $\mathscr{P}$ to $\mathscr{P}'$ is a bijection from some partition of $\mathscr{P}$ to some partition of $\mathscr{P}'$, denoted by $\varphi : \mathscr{P} \overset{par}{\to} \mathscr{P}'$.

Definition

Let $\mathscr{P}, \mathscr{P}'$ be finite sets.

(1) A *partition* of $\mathscr{P}$ is a finite collection of subsets $\mathscr{P}_i, i \in I$ such that $\cup_{i\in I}\mathscr{P}_i = \mathscr{P}$ and $\mathscr{P}_i \cap \mathscr{P}_j = \emptyset$ for any $i \neq j \in I$.

(2) A *partition bijection* from $\mathscr{P}$ to $\mathscr{P}'$ is a bijection from some partition of $\mathscr{P}$ to some partition of $\mathscr{P}'$, denoted by $\varphi : \mathscr{P} \overset{par}{\to} \mathscr{P}'$.

Remark

To give a partition bijection from $\mathscr{P}$ to $\mathscr{P}'$ is equivalent to associate each $P \in \mathscr{P}$ with a non-empty subset $\varphi(P) \subset \mathscr{P}'$ such that the following conditions hold.

Definition

Let $\mathscr{P}, \mathscr{P}'$ be finite sets.

(1) A *partition* of $\mathscr{P}$ is a finite collection of subsets $\mathscr{P}_i, i \in I$ such that $\cup_{i\in I}\mathscr{P}_i = \mathscr{P}$ and $\mathscr{P}_i \cap \mathscr{P}_j = \emptyset$ for any $i \neq j \in I$.

(2) A *partition bijection* from $\mathscr{P}$ to $\mathscr{P}'$ is a bijection from some partition of $\mathscr{P}$ to some partition of $\mathscr{P}'$, denoted by $\varphi : \mathscr{P} \overset{par}{\to} \mathscr{P}'$.

Remark

To give a partition bijection from $\mathscr{P}$ to $\mathscr{P}'$ is equivalent to associate each $P \in \mathscr{P}$ with a non-empty subset $\varphi(P) \subset \mathscr{P}'$ such that the following conditions hold.

(i) For any $P, Q \in \mathscr{P}$, we have either $\varphi(P) \cap \varphi(Q) = \emptyset$ or $\varphi(P) = \varphi(Q)$;

Definition

Let $\mathscr{P}, \mathscr{P}'$ be finite sets.

(1) A *partition* of $\mathscr{P}$ is a finite collection of subsets $\mathscr{P}_i, i \in I$ such that $\cup_{i\in I}\mathscr{P}_i = \mathscr{P}$ and $\mathscr{P}_i \cap \mathscr{P}_j = \emptyset$ for any $i \neq j \in I$.

(2) A *partition bijection* from $\mathscr{P}$ to $\mathscr{P}'$ is a bijection from some partition of $\mathscr{P}$ to some partition of $\mathscr{P}'$, denoted by $\varphi : \mathscr{P} \overset{par}{\to} \mathscr{P}'$.

Remark

To give a partition bijection from $\mathscr{P}$ to $\mathscr{P}'$ is equivalent to associate each $P \in \mathscr{P}$ with a non-empty subset $\varphi(P) \subset \mathscr{P}'$ such that the following conditions hold.

(i) For any $P, Q \in \mathscr{P}$, we have either $\varphi(P) \cap \varphi(Q) = \emptyset$ or $\varphi(P) = \varphi(Q)$;

(ii) $\bigcup_{P\in\mathscr{P}} \varphi(P) = \mathscr{P}'$.

Theorem

Let T^o be an ideal triangulation and $\alpha \in T^o$ be a flippable arc. Denote $T'^o = \mu_\alpha(T^o)$ and T (resp. T') the corresponding tagged triangulation of T^o (resp. T'^o). There exists a partition bijection

$$\pi : \mathscr{L}(T, \beta) \to \mathscr{L}(T', \beta)$$

such that for any $P \in \mathscr{L}(T, \beta)$ we have

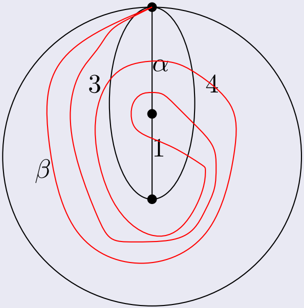
$$\sum_{P' \in \pi(P)} v^{w(P')} X(P') = \sum_{Q \in \pi^{-1}\pi(P)} v^{w(Q)} X(Q).$$

Consequently, we have

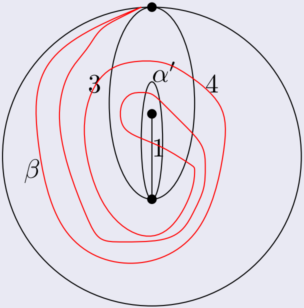
$$\sum_{P' \in \mathscr{L}(T', \beta)} v^{w(P')} X(P') = \sum_{P \in \mathscr{L}(T, \beta)} v^{w(P)} X(P).$$

An example

Let Σ, T^o, T'^o, α, α' and β be as shown in the following figure.

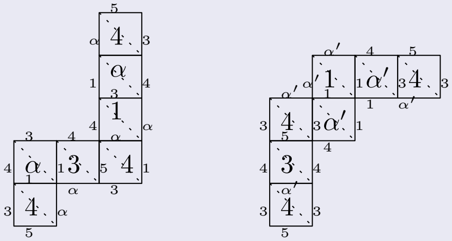


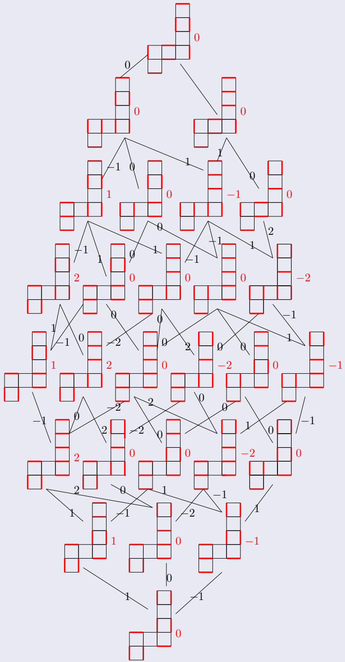
T^o

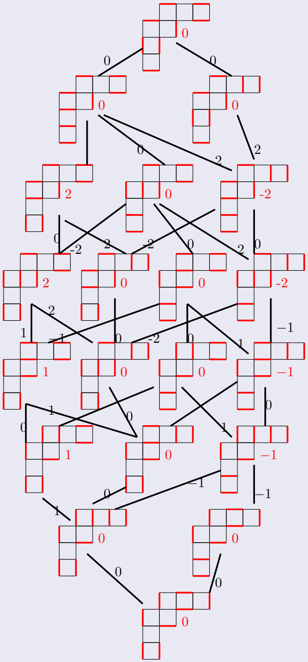


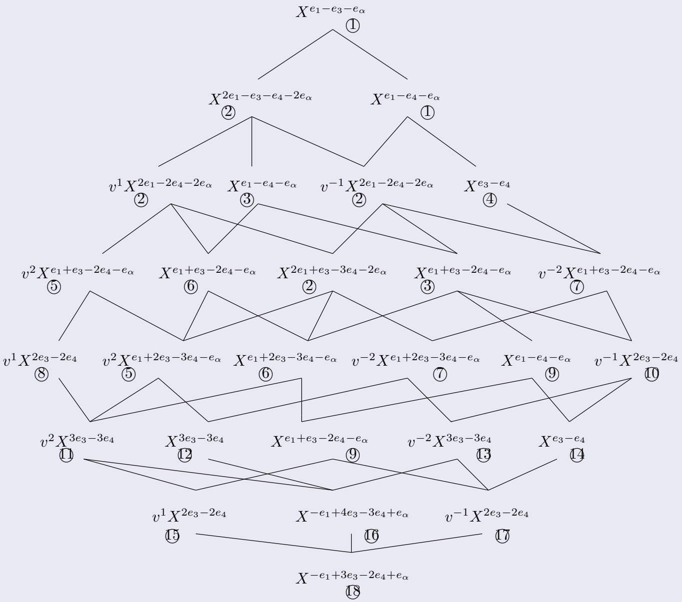
T'^o

Then the snake graphs $G_{T,\beta}$ and $G_{T',\beta}$ are the graphs in the following Figure.

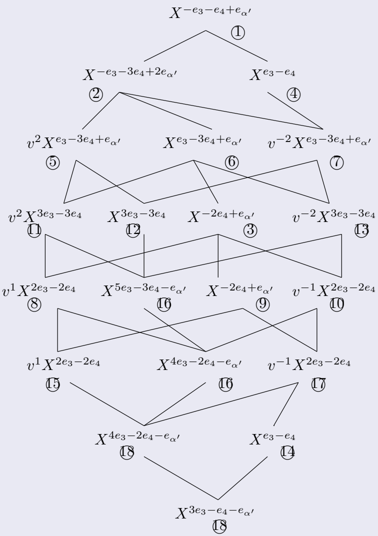








$X^{e_1-e_3-e_\alpha}$ ①
$X^{2e_1-e_3-e_4-2e_\alpha}$ ②
$X^{e_1-e_4-e_\alpha}$ ①
$v^1X^{2e_1-2e_4-2e_\alpha}$ ②
$X^{e_1-e_4-e_\alpha}$ ③
$v^{-1}X^{2e_1-2e_4-2e_\alpha}$ ②
$X^{e_3-e_4}$ ④
$v^2X^{e_1+e_3-2e_4-e_\alpha}$ ⑤
$X^{e_1+e_3-2e_4-e_\alpha}$ ⑥
$X^{2e_1+e_3-3e_4-2e_\alpha}$ ②
$X^{e_1+e_3-2e_4-e_\alpha}$ ③
$v^{-2}X^{e_1+e_3-2e_4-e_\alpha}$ ⑦
$v^1X^{2e_3-2e_4}$ ⑧
$v^2X^{e_1+2e_3-3e_4-e_\alpha}$ ⑤
$X^{e_1+2e_3-3e_4-e_\alpha}$ ⑥
$v^{-2}X^{e_1+2e_3-3e_4-e_\alpha}$ ⑦
$X^{e_1-e_4-e_\alpha}$ ⑨
$v^{-1}X^{2e_3-2e_4}$ ⑩
$v^2X^{3e_3-3e_4}$ ⑪
$X^{3e_3-3e_4}$ ⑫
$X^{e_1+e_3-2e_4-e_\alpha}$ ⑨
$v^{-2}X^{3e_3-3e_4}$ ⑬
$X^{e_3-e_4}$ ⑭
$v^1X^{2e_3-2e_4}$ ⑮
$X^{-e_1+4e_3-3e_4+e_\alpha}$ ⑯
$v^{-1}X^{2e_3-2e_4}$ ⑰
$X^{-e_1+3e_3-2e_4+e_\alpha}$ ⑱

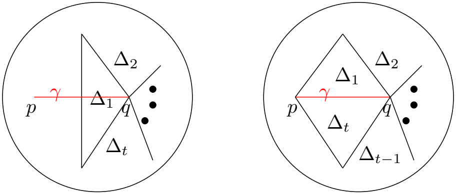

$X^{-e_3-e_4+e_{\alpha'}}$ ①
$X^{-e_3-3e_4+2e_{\alpha'}}$ ②
$X^{e_3-e_4}$ ④
$v^2X^{e_3-3e_4+e_{\alpha'}}$ ⑤
$X^{e_3-3e_4+e_{\alpha'}}$ ⑥
$v^{-2}X^{e_3-3e_4+e_{\alpha'}}$ ⑦
$v^2X^{3e_3-3e_4}$ ⑪
$X^{3e_3-3e_4}$ ⑫
$X^{-2e_4+e_{\alpha'}}$ ③
$v^{-2}X^{3e_3-3e_4}$ ⑬
$v^1X^{2e_3-2e_4}$ ⑧
$X^{5e_3-3e_4-e_{\alpha'}}$ ⑯
$X^{-2e_4+e_{\alpha'}}$ ⑨
$v^{-1}X^{2e_3-2e_4}$ ⑩
$v^1X^{2e_3-2e_4}$ ⑮
$X^{4e_3-2e_4-e_{\alpha'}}$ ⑯
$v^{-1}X^{2e_3-2e_4}$ ⑰
$X^{4e_3-2e_4-e_{\alpha'}}$ ⑱
$X^{e_3-e_4}$ ⑭
$X^{3e_3-e_4-e_{\alpha'}}$ ⑱

Expansion formula for singly-notched and doubly-notched arcs

Lattice $\mathscr{L}(T, \gamma^{(q)})$

The ending point q of γ is a puncture. T contains no arcs tagged notched at q.

- $\Delta(q) = \{\Delta_1(q), \cdots, \Delta_t(q)\}$: triangles incident to q in T^o.
- $\mathscr{L}(T, \gamma^{(q)}) = \mathscr{L}(T, \gamma) \times \Delta(q)$.



Lattice $\mathscr{L}(T, \gamma^{(q)})$

$$E_1(q) := \begin{cases} N(G_c), & \text{if } rel(G_c, T^o) = 1, \\ E(G_c), & \text{if } rel(G_c, T^o) = -1, \end{cases}$$

$$E_2(q) := \begin{cases} E(G_c), & \text{if } rel(G_c, T^o) = 1, \\ N(G_c), & \text{if } rel(G_c, T^o) = -1. \end{cases}$$

Remark

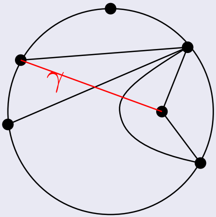
$E_1(q) \in P_-$, $E_2(q) \in P_+$.

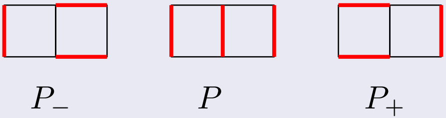
Lattice $\mathscr{L}(T, \gamma^{(q)})$

- For any $P \in \mathscr{P}(G_{T,\gamma})$,
  - if $E_1(q) \in P$ then
  $$(P, \Delta_1(q)) < (P, \Delta_2(q)) < (P, \Delta_3(q)) < \cdots < (P, \Delta_t(q));$$
  - if $E_2(q) \in P$ then
  $$(P, \Delta_2(q)) < \cdots < (P, \Delta_{t-1}(q)) < (P, \Delta_t(q)) < (P, \Delta_1(q)).$$
- For any $j \in \{1, \cdots, t\}$, $(P, \Delta_j(q)) < (Q, \Delta_j(q))$ if $P < Q$.

Proposition

$\mathscr{L}(T, \gamma^{(q)})$ is a lattice with minimum element $(P_-, \Delta_1(q))$ and maximum element $(P_+, \Delta_1(q))$.





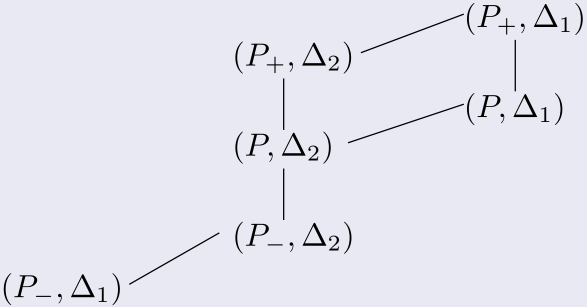


Figure: Hasse graph of $\mathscr{L}(T, \gamma^{(q)})$

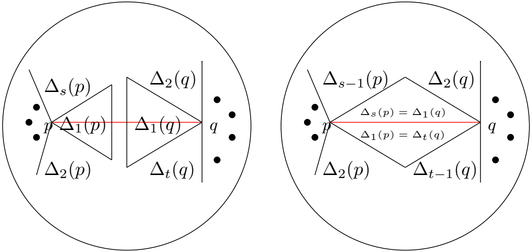
Cover relations

- $(P, \Delta_{j+1}(q))$ covers $(P, \Delta_j(q))$; or
- $(\mu_{G_l} P, \Delta_j(q))$ covers $(P, \Delta_j(q))$.

Lattice $\mathscr{L}(T, \gamma^{(p,q)})$

The starting point and ending point of γ are punctures p and q, respectively. T contains no arcs tagged notched at p or q.

- $\Delta(p) = \{\Delta_1(p), \cdots, \Delta_s(p)\}$: triangles incident to p in T^o.
- $\Delta(q) = \{\Delta_1(q), \cdots, \Delta_t(q)\}$: triangles incident to q in T^o.
- $\mathscr{L}(T, \gamma^{(p,q)}) = \Delta(p) \times \mathscr{L}(T, \gamma) \times \Delta(q)$.



Lattice $\mathscr{L}(T, \gamma^{(p,q)})$

- For any $(P, \Delta_j(q)) \in \mathscr{P}(G_{T,\gamma}) \times \Delta(q)$,
  - if $E_1(p) \in P$ then

$$(\Delta_1(p), P, \Delta_j(q)) < (\Delta_2(p), P, \Delta_j(q)) < \cdots < (\Delta_s(p), P, \Delta_j(q));$$

  - if $E_2(p) \in P$ then

$$(\Delta_2(p), P, \Delta_j(q)) < (\Delta_3(p), P, \Delta_j(q)) < \cdots (\Delta_s(p), P, \Delta_j(q)) < (\Delta_1(p), P, \Delta_j(q));$$

- For any $i \in \{1, \cdots, t\}$, $(\Delta_i(p), P, \Delta_a(q)) < (\Delta_i(p), Q, \Delta_b(q))$ if $(P, \Delta_a(q)) < (Q, \Delta_b(q))$ in $\mathscr{L}(T, \gamma^{(q)})$.

Proposition

$\mathscr{L}(T, \gamma^{(p,q)})$ is a lattice with minimum element $(\Delta_1(p), P_-, \Delta_1(q))$ and maximum element $(\Delta_1(p), P_+, \Delta_1(q))$.

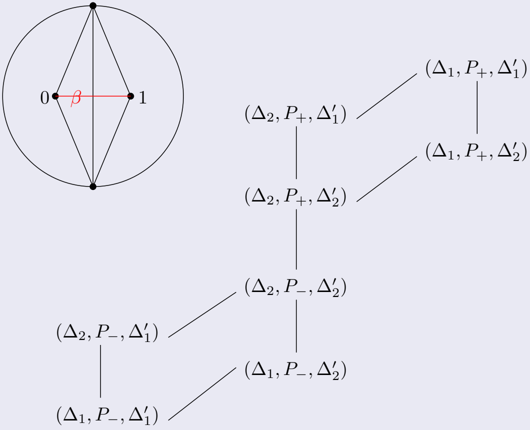


Figure: Hasse graph of $\mathscr{L}(T, \beta^{(0,1)})$

Cover relations

- $(\Delta_i(p), P, \Delta_{j+1}(q))$ covers $(\Delta_i(p), P, \Delta_j(q))$; or
- $(\Delta_{i+1}(p), P, \Delta_{j+1}(q))$ covers $(\Delta_i(p), P, \Delta_j(q))$; or
- $(\Delta_i(p), \mu_{G_l} P, \Delta_j(q))$ covers $(\Delta_i(p), P, \Delta_j(q))$.

Valuation map on $\mathscr{L}(T, \gamma^{(q)})$

Assume the edges of $\Delta_j(q)$ are $\tau_{j-1}(q), \tau_j(q), \tau_{[j]}(q)$.

- $$m(\Delta_j(q);\alpha) = \text{number of edges labeled } \alpha \text{ in } \{\tau_{[j]}(q)\}$$
  $$- \text{number of edges labeled } \alpha \text{ in } \{\tau_{j-1}(q), \tau_j(q)\}.$$
- $\forall P \in \mathscr{L}(T, \gamma)$ and arc $\alpha \in T^o$,
  $$m(P, \alpha) = \text{number of edges labeled } \alpha \text{ in } P.$$

Valuation map on $\mathscr{L}(T, \gamma^{(q)})$

$\forall \ (P, \Delta_j(q)) \in \mathscr{L}(T, \gamma^{(q)})$,

- $$\Omega^{(q)}(P, \underline{\Delta_j(q)}) = -d(\tau_j(q))m(P; \tau_j(q)).$$
- If P can twist at G_l,
$$\Omega^{(q)}(P, \Delta_j(q); G_l) = \Omega(P, G_l) + d(\tau_j(q))m(\Delta_j(q); \tau_{i_l}).$$

Valuation map on $\mathscr{L}(T, \gamma^{(q)})$

Proposition

There is a unique map $w : \mathscr{L}(T, \gamma^{(q)}) \to \mathbb{Z}$ such that

1. (Initial condition) $w(P_-, \Delta_1(q)) = 0$,
2. (Recurrence conditions)
   1. For any $(P, \Delta_j(q))$ such that $(P, \Delta_{j+1}(q))$ covers $(P, \Delta_j(q))$,
   $$w(P, \Delta_{j+1}(q)) - w(P, \Delta_j(q)) = \Omega^{(q)}(P, \underline{\Delta_j(q)}). \tag{1}$$
   2. For any $(P, \Delta_j(q)), (Q, \Delta_j(q)) \in \mathscr{L}(T, \gamma^{(q)})$ such that $(Q, \Delta_j(q))$ covers $(P, \Delta_j(q))$, in particular $P < Q$ are related by a twist on some tile G_l,
   $$w(Q, \Delta_j(q)) - w(P, \Delta_j(q)) = \Omega^{(q)}(P, \Delta_j(q); G_l). \tag{2}$$

Valuation map on $\mathscr{L}(T, \gamma^{(p,q)})$

1. $$\Omega^{(p,q)}(\underline{\Delta_i(p)}, P, \Delta_j(q)) = d(\tau_i(p))[m(P; \tau_i(p)) + m(\Delta_j(q); \tau_i(p))].$$

2. $$\Omega^{(p,q)}(\Delta_i(p), P, \underline{\Delta_j(q)}) = -d(\tau_i(p))[m(P; \tau_j(q)) + m(\Delta_i(p); \tau_j(q))].$$

3. If P can twist at G_l,

$$\Omega^{(p,q)}(\Delta_i(p), P, \Delta_j(q); G_l) = \Omega^{(q)}(P, \Delta_j(q); G_l) - d(\tau_i(p))m(\Delta_i(p); \tau_{i_l}).$$

Valuation map on $\mathscr{L}(T, \gamma^{(p,q)})$

Proposition

There is a unique map $w : \mathscr{L}(T, \gamma^{(p,q)}) \to \mathbb{Z}$ such that

1. (Initial condition) $w(\Delta_1(p), P_-, \Delta_1(q)) = 0$,
2. (Recurrence conditions)
   - For any $(\Delta_i(p), P, \Delta_j(q))$ such that $(\Delta_{i+1}(p), P, \Delta_j(q))$ covers $(\Delta_i(p), P, \Delta_j(q))$, we have

$$w(\Delta_{i+1}(p), P, \Delta_j(q)) - w(\Delta_i(p), P, \Delta_j(q)) = \Omega^{(p,q)}(\underline{\Delta_i(p)}, P, \Delta_j(q)).$$

Valuation map on $\mathscr{L}(T^o, \gamma^{(p,q)})$

Continue

- For any $(\Delta_i(p), P, \Delta_j(q))$ such that $(\Delta_i(p), P, \Delta_{j+1}(q))$ covers $(\Delta_i(p), P, \Delta_j(q))$, we have

$$w(\Delta_i(p), P, \Delta_{j+1}(q)) - w(\Delta_i(p), P, \Delta_j(q)) = \Omega^{(p,q)}(\Delta_i(p), P, \underline{\Delta_j(q)}).$$

- For any $(\Delta_i(p), Q, \Delta_j(q)), (\Delta_i(p), P, \Delta_j(q)) \in \mathscr{L}(T^o, \beta^{(p,q)})$ such that $(\Delta_i(p), Q, \Delta_j(q))$ covers $(\Delta_i(p), P, \Delta_j(q))$, in particular, $Q > P$ are related by a twist on some tile G_l,

$$w(\Delta_i(p), Q, \Delta_j(q)) - w(\Delta_i(p), P, \Delta_j(q)) = \Omega^{(p,q)}(\Delta_i(p), P, \Delta_j(q); G_l).$$

Expansion formula

Theorem

Let $\beta = \gamma^{(q)}$ or $\gamma^{(p,q)}$. For any $\mathbf{P} \in \mathscr{L}(T, \beta)$, we can associate with a quantum Laurent monomial $X(\mathbf{P})$ in $\mathscr{T}(T)$. Then

$$X_\beta = \sum_{\mathbf{P} \in \mathscr{L}(T,\beta)} v^{w(\mathbf{P})} X(\mathbf{P}).$$

Theorem

The positivity conjecture holds for quantum cluster algebras from orbifolds.

Reference

A. Berenstein and A. Zelevinsky, Quantum cluster algebras. Adv Math 195 (2) (2005) 405-455.

S. Fomin, M. Shapiro, and D. Thurston, Cluster algebras and triangulated surfaces. Part I: Cluster complexes, Acta Math. 201 (2008) 83-146.

G. Musiker, R. Schiffler, and L. Williams, Positivity for cluster algebras from surfaces, Adv. Math. 227 (2011) 2241-2308.

Thanks!