Categorification in Representation Theory

Vanessa Miemietz
University of East Anglia

Motivation

Observations:

- Translation functors on Category $\mathcal{O}$ of a Lie algebra satisfy relations of a Hecke algebra. *[Soergel]*
- Certain induction and restriction functors on (affine) Hecke algebras satisfy relations of a Lie algebra. *[Lascoux–Leclerc–Thibon, Ariki, Grojnowski]*

$\rightsquigarrow$ Categorification in representation theory.

Why?

More information in the higher structure: now have additional information about natural transformations between these functors $\rightsquigarrow$ new information about the decategorified object.

Motivation

Examples in representation theory

- categorification of Kac–Moody algebras *[Khovanov–Lauda, Rouquier]* ($\rightsquigarrow$ 4-dimensional topological quantum field theories (TQFT)?)
- categorification of Heisenberg algebras *[Khovanov]*
- categorification of Lie superalgebras *[Brundan–Stroppel]*
- categorification of Hall algebras (for cyclic quivers) *[Stroppel–Webster]*
- categorification of Hecke algebras via **Soergel bimodules** *[Soergel, Elias–Williamson]*

$\rightsquigarrow$ proof of Broué's abelian defect group conjecture for symmetric groups, proof of Kazhdan–Lusztig conjectures for all Coxeter systems, counterexample to James' conjecture for Hecke algebras, counterexamples to (and refinements of) Lusztig's conjectures

Classical representation theory

Let $\Bbbk$ be a(n algebraically closed) field.

Algebra over $\Bbbk$: A $\Bbbk$-linear category $\mathcal{A}$ with one (or finitely many) object(s), say $\bullet$.
Representation of $\mathcal{A}$: A $\Bbbk$-linear functor from $\mathcal{A}$ to $\mathcal{V}ect_{\Bbbk}$.

Observe:

- $A := \mathrm{End}_{\mathcal{A}}(\bullet)$ is an associative $\Bbbk$-algebra.
- If the functor describing a representation is given by $\bullet \mapsto V$, $\mathrm{End}_{\mathcal{A}}(\bullet) \ni a \mapsto \rho(a) \in \mathrm{End}_{\mathcal{V}ect_{\Bbbk}}(V)$, V is an A-module and ρ is a representation of A.
- If $\mathcal{A}$ has several objects $\mathtt{1}, \dots, \mathtt{n}$, their identities are idempotents in the algebra $A = \mathrm{End}_{\mathcal{A}}(\bigoplus_{\mathtt{i}=\mathtt{1}}^{\mathtt{n}} \mathtt{i})$.

2-categories

A 2-**category** $\mathscr{C}$ is a category enriched over the monoidal category $\mathbf{Cat}$ of small categories, i.e. it consists of

- a class (or set) $\mathscr{C}$ of objects;
- for every $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ a small category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ of morphisms from $\mathtt{i}$ to $\mathtt{j}$
  - objects in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are called 1-**morphisms**
  - morphisms in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are called 2-**morphisms**;
- functorial composition $\mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$;
- identity 1-morphisms $\mathbb{1}_{\mathtt{i}}$ for every $\mathtt{i} \in \mathscr{C}$;
- natural (strict) axioms.

Remark. Everything I will say has a bicategorical analogue.

2-categories

Examples.

- The 2-category $\mathbf{Cat}$:
  - objects are small categories;
  - 1-morphisms functors;
  - 2-morphisms are natural transformations.
- The 2-category $\mathfrak{A}^f_{\Bbbk}$:
  - objects are small idempotent complete $\Bbbk$-linear additive categories with finitely many indecomposable objects up to isomorphism and finite-dimensional morphism spaces (that is, equivalent to the category of finitely generated projective modules over a finite-dimensional $\Bbbk$-algebra);
  - 1-morphisms are $\Bbbk$-linear (additive) functors;
  - 2-morphisms are natural transformations.

2-categories

A 2-category $\mathscr{C}$ is **finitary** over $\Bbbk$ if

- $\mathscr{C}$ has finitely many objects;
- each $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is in $\mathfrak{A}_{\Bbbk}^{f}$;
- composition is biadditive and $\Bbbk$-bilinear;
- identity 1-morphisms are indecomposable.

Moral: Finitary 2-categories are 2-analogues of finite dimensional algebras.

A 2-category $\mathscr{C}$ is **fiat** (**f**initary - **i**nvolution - **a**djunction - **t**wo-category) if

- it is finitary;
- there is a weak involutive equivalence $(-)^* \colon \mathscr{C} \to \mathscr{C}^{\mathrm{op},\mathrm{op}}$ such that there exist adjunction morphisms $F \circ F^* \to \mathbb{1}_{\mathtt{i}}$ and $\mathbb{1}_{\mathtt{j}} \to F^* \circ F$.

Example. Let A be a connected finite-dimensional $\Bbbk$-algebra. The 2-category $\mathscr{C}_A$ has

- one object $\bullet$ (identified with A-proj);
- 1-morphisms are endofunctors of $\varnothing$ isomorphic to tensoring with bimodules in the additive closure of $A \oplus A \otimes_{\Bbbk} A$;
- 2-morphisms are natural transformations (bimodule homomorphisms).

Observe:

- $\mathscr{C}_A$ is finitary.
- If A is basic with complete set of idempotents $e_1, \ldots, e_n$, the indecomposable 1 morphisms correspond to the bimodules A and $Ae_i \otimes_{\Bbbk} e_j A$, for $i, j = 1, \ldots n$.
- If A is weakly symmetric, $\mathscr{C}_A$ is fiat with involution given by $(Ae_i \otimes_{\Bbbk} e_j A)^* \cong Ae_j \otimes_{\Bbbk} e_i A$.

Soergel bimodules or the Hecke 2-category

(W, S) Coxeter group, $W = \langle s_i | \, s_i \in S, s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$, $m_{ij} \geq 2$

V faithful reflection representation

$R = \mathbb{C}[V]/(\mathbb{C}[V]^W)_+$ coinvariant algebra

$R_i := R \otimes_{R^{s_i}} R$ for $s_i \in S$

The 2-category $\mathscr{S} = \mathscr{S}_{W,S,V}$ of **Soergel bimodules** or **Hecke 2-category** has

- one object $\varnothing$ (identified with $R\text{-proj}$);
- 1-morphisms are endofunctors of $\varnothing$ isomorphic to tensoring with bimodules in the additive closure of finite tensor products (over R) of the R_i;
- 2-morphisms are all natural transformations (bimodule morphisms).

Fact: $\mathscr{S}$ is fiat (for W finite) and categorifies the Hecke algebra.

2-representations

A **finitary 2-representation** $\mathbf{M}$ of a finitary 2-category $\mathscr{C}$ is a (strict) 2-functor $\mathscr{C} \to \mathfrak{A}_{\Bbbk}^{f}$, i.e.

- $\mathbf{M}(\mathtt{i}) \approx B_{\mathtt{i}}\text{-proj}$ for some algebra $B_{\mathtt{i}}$;
- for $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, $\mathbf{M}(\mathrm{F})\colon \mathbf{M}(\mathtt{i}) \to \mathbf{M}(\mathtt{j})$ is an additive functor;
- for $\alpha\colon \mathrm{F} \to \mathrm{G}$, $\mathbf{M}(\alpha)\colon \mathbf{M}(\mathrm{F}) \to \mathbf{M}(\mathrm{G})$ is a natural transformation.

Examples.

- For $\mathtt{i} \in \mathscr{C}$, we have $\mathbf{P}_{\mathtt{i}} = \mathscr{C}(\mathtt{i}, -)$.
- $\mathscr{C}_A$ and $\mathscr{S}$ were defined via their **natural** 2-representations on $A\text{-proj}$, resp. $R\text{-proj}$.

A 2-representation $\mathbf{M}$ is **simple** if $\coprod_{\mathtt{i} \in \mathscr{C}} \mathbf{M}(\mathtt{i})$ has no proper $\mathscr{C}$-stable ideals.

Goal. Classify simple 2-representations for interesting 2-categories.

Cell combinatorics for 2-categories

From now on, let $\mathscr{C}$ be a fiat 2-category.

On (iso-classes of) indecomposable 1-morphisms in $\mathscr{C}$, define

left preorder: $\mathrm{F} \leq_L \mathrm{G}$ if $\exists\, \mathrm{H}$ such that G is a direct summand of HF

left cells: equivalence classes w.r.t. $\geq_L$

Similarly:

right preorder: $\mathrm{F} \leq_R \mathrm{G}$ if $\exists\, \mathrm{H}$ such that G is a direct summand of FH

right cells: equivalence classes w.r.t. $\geq_R$

two-sided preorder: $\mathrm{F} \leq_J \mathrm{G}$ if $\exists\, \mathrm{H}_1, \mathrm{H}_2$ such that G is a direct summand of $\mathrm{H}_1\mathrm{F}\mathrm{H}_2$

two-sided cells: equivalence classes w.r.t. $\geq_J$

H-cells: intersections of left and right cells

Example $\mathscr{C}_A$

Example. For $\mathscr{C}_A$, the cells are given by

$$\boxed{\mathbb{1} = A}$$

$Ae_1 \otimes e_1 A$	$Ae_1 \otimes e_2 A$	$\cdots$	$Ae_1 \otimes e_n A$
$Ae_2 \otimes e_1 A$	$Ae_2 \otimes e_2 A$	$\cdots$	$Ae_2 \otimes e_n A$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$Ae_n \otimes e_1 A$	$Ae_n \otimes e_2 A$	$\cdots$	$Ae_n \otimes e_n A$

Example $\mathscr{S}$

Fact: Indecomposable 1-morphisms in $\mathscr{S}$ are labelled by W, denoted by $\theta_w, w \in W$. They descend to a cellular basis (the KL-basis) .
$\rightsquigarrow$ cell structure: left, right, two-sided, H-cells (**Kazhdan–Lusztig cells**)

Example $\mathscr{S}_{B_2}$. $W = \langle s,t | s^2 = 1 = t^2, stst = tsts\rangle$ of type $B_2 = C_2$.
Cells are

$$\boxed{1}$$

$$\begin{array}{|c|c|} \hline s, sts & st \\ \hline ts & t, tst \\ \hline \end{array}$$

$$\boxed{stst}$$

$\mathcal{H}$-cell reduction

Let $\mathcal{H}$ be a *diagonal* H-cell in $\mathscr{C}$, contained in a two-sided cell $\mathcal{J}$.

Construct $\mathscr{C}_{\mathcal{H}}$ in several steps:

- take quotients by all two-sided cells $\mathcal{J}' \not\leq \mathcal{J}$;
- inside quotient, take additive closure of $\mathbb{1}$ and the F in $\mathcal{H}$;
- factor out the maximal 2-ideal not containing id_{F} for $\mathrm{F} \in \mathcal{H}$.

Examples.

- For $\mathscr{C} = \mathscr{C}_A$, take $\mathcal{H} = \{Ae_1 \otimes e_1 A\}$, then $\mathscr{C}_{\mathcal{H}}$ has cell structure

$$\boxed{\mathbb{1} = A}$$

$$\boxed{Ae_1 \otimes e_1 A}$$

- For $\mathscr{S} = \mathscr{S}_{B_2}$, take $\mathcal{H} = \{\theta_s, \theta_{sts}\}$, then $\mathscr{S}_{\mathcal{H}}$ has cell structure

$$\boxed{\mathbb{1} = \theta_1}$$

$$\boxed{\theta_s, \theta_{sts}}$$

$\mathcal{H}$-cell reduction

Theorem. [Mackaay–Mazorchuk–M–Zhang] There is a bijection

$$\{\text{nontrivial simple } 2\text{-representations of } \mathscr{C}\}$$
$$\updownarrow$$
$$\{\text{nontrivial simple } 2\text{-representations of the } \mathscr{C}_{\mathcal{H}}\}$$

where $\mathcal{H}$ runs over a choice of diagonal $\mathcal{H}$-cell in every two-sided cell.

Upshot: concentrate on $\mathscr{C}_{\mathcal{H}} \rightsquigarrow$ smaller! We call this **$\mathcal{H}$-cell reduction**.

Double Centraliser Theorem

Let $\mathbf{M}$ be a non-trivial simple 2-representation of $\mathscr{C}_{\mathcal{H}}$. There is a canonical 2-functor

$$\mathrm{can}\colon \mathscr{C}_{\mathcal{H}} \to \mathscr{E}nd_{\mathscr{E}nd_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})}(\mathbf{M}).$$

Double Centraliser Theorem.
[Mackaay–Mazorchuk–M.–Tubbenhauer–Zhang] There is an equivalence

$$\mathscr{E}nd^{\mathrm{inj}}_{\mathscr{E}nd_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})}(\mathbf{M}) \simeq \mathrm{add}(\mathcal{H}).$$

Under some technical conditions, we obtain a bijection

$$\{\text{nontrivial simple } 2\text{-representations of the } \mathscr{C}_{\mathcal{H}}\}$$
$$\updownarrow$$
$$\{\text{nontrivial simple } 2\text{-representations of } \mathscr{E}nd_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})\}$$

Representations of Hecke algebras

[Lusztig]: (W, S) Coxeter group

$\mathcal{H}$ a two-sided cell or diagonal H-cell $\rightsquigarrow$ **asymptotic algebra** $\mathsf{A}_{\mathcal{H}}$ (via $q \to 0$)

Theorem. *[Lusztig]* There is a bijection

$$\{\text{simple representations of the Hecke algebra}\}$$
$$\updownarrow$$
$$\{\text{simple representations of the asymptotic algebras}\}$$

where the asymptotic algebras run over all two-sided cells or a choice of diagonal $\mathcal{H}$-cell in each two-sided cell.

Idea: Asymptotic algebras are easier to understand.

Representations of Hecke 2-categories

[Lusztig] $\mathcal{H}$ a two-sided cell or diagonal H-cell $\rightsquigarrow$ **asymptotic bicategory** $\mathscr{A}_{\mathcal{H}}$

- $\mathscr{A}_{\mathcal{H}}$ categorifies $\mathrm{A}_{\mathcal{H}}$.
- $\mathscr{A}_{\mathcal{H}}$ is a fusion category. *[Lusztig, Elias–Williamson]*
- W any finite Weyl group: $\mathscr{A}_{\mathcal{H}}$ is well-understood; simple 2-representations have been classified. *[Etingof, Ostrik et al.]*

To classify simple 2-representations of $\mathscr{S}$, want to relate 2-representations of $\mathscr{S}_{\mathcal{H}}$ to those of $\mathscr{A}_{\mathcal{H}}$.

From now on, assume (W, S) is a finite Coxeter group and $\Bbbk = \mathbb{C}$.

Representations of Hecke 2-categories

Let $\mathbf{C}$ be the so-called **cell 2-representation** of $\mathscr{S}_{\mathcal{H}}$ corresponding to $\mathcal{H}$. This is simple.

Proposition. *[Mackaay–Mazorchuk–M.–Tubbenhauer–Zhang]*
$\mathscr{E}nd_{\mathscr{S}_{\mathcal{H}}}(\mathbf{C}) \cong \mathscr{A}_{\mathcal{H}}$.

Combining the proposition with H-cell reduction and the double centraliser theorem, we obtain

Main Theorem. *[Mackaay–Mazorchuk–M.–Tubbenhauer–Zhang]*
There is a bijection

$$\{\text{simple 2-representations of } \mathscr{S}\}$$
$$\updownarrow$$
$$\{\text{simple 2-representations of the } \mathscr{A}_{\mathcal{H}}\}$$

where $\mathcal{H}$ runs over a choice of diagonal $\mathcal{H}$-cell in every two-sided cell.

Representations of Hecke 2-categories

Main Theorem. *[Mackaay–Mazorchuk–M.–Tubbenhauer–Zhang]*
There is a bijection

$$\{\text{simple } 2\text{-representations of } \mathscr{S}\}$$
$$\updownarrow$$
$$\{\text{simple } 2\text{-representations of the } \mathscr{A}_{\mathcal{H}}\}$$

where $\mathcal{H}$ runs over a choice of diagonal $\mathcal{H}$-cell in every two-sided cell.

Remarks

- completes classification in all finite Coxeter types apart form H_3, H_4
- for few H-cells in types H_3, H_4, $\mathscr{A}_{\mathcal{H}}$ is not well-understood

Thank you!

Thank you for your attention!