Representation theory and positroid varieties
表征理论和正拟阵簇

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Slides: https://bit.ly/mdp-icra24

The totally positive Grassmannian

Definition

$M \in \mathbb{C}^{k\times n}$, $k < n$, is *totally positive* if its maximal minors $\Delta_I(M)$ are positive real numbers.

- Here $I \in \binom{[n]}{k}$ is a subset of k columns, $\Delta_I(M)$ its determinant.
- If $\operatorname{rk} M = k$, its row span $[M]$ is in $\mathrm{Gr}_{k,n}$, the *Grassmannian*.
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- A minimal positivity test needs only $\dim \widehat{\mathrm{Gr}}_{k,n} = k(n-k)+1$ minors ... chosen carefully!

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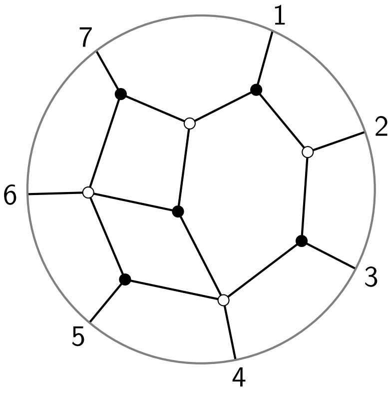
- $\overline{\mathrm{Gr}_{k,n}^{>0}} = \mathrm{Gr}_{k,n}^{\geqslant 0}$ decomposes into cells $\Pi_{\mathcal{P}}^{\circ} \cap \mathrm{Gr}_{k,n}^{\geqslant 0}$, indexed by *positroids* (正拟阵) $\mathcal{P} \subseteq \binom{[n]}{k}$.

Positroid varieties

Postnikov '06$^+$, Knutson–Lam–Speyer '13,...

To compute a positroid, use a Postnikov diagram / dimer model:



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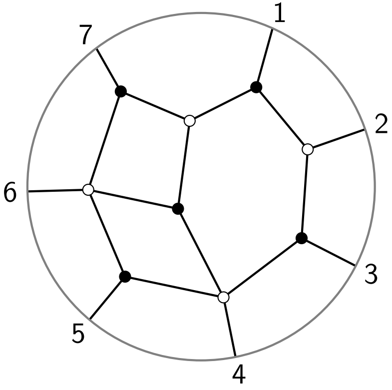

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$\mathcal{P} = \{\partial\mu : \mu \text{ perfect matching}\}$

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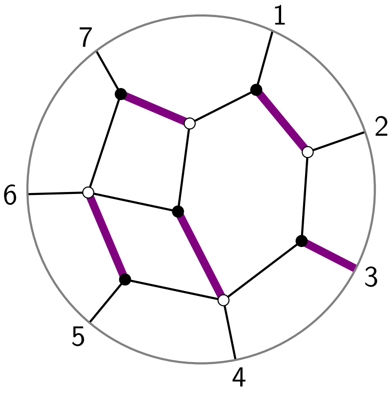

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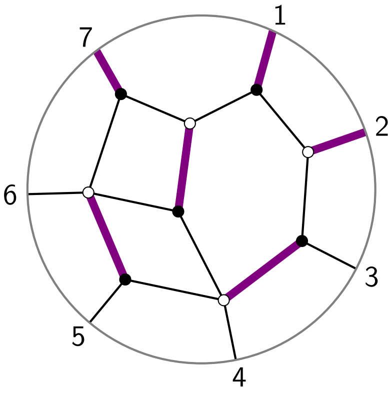

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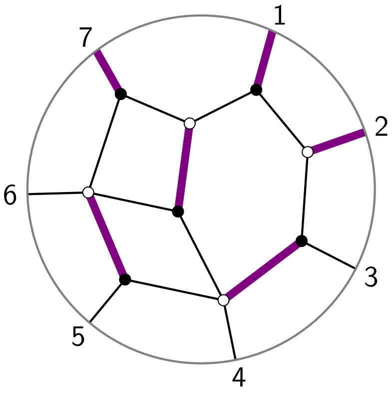

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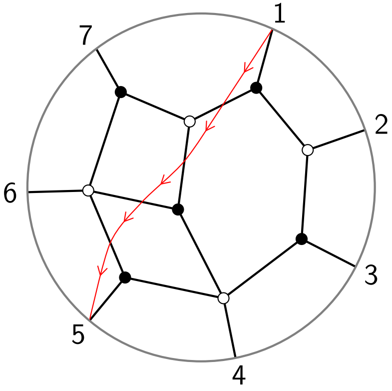

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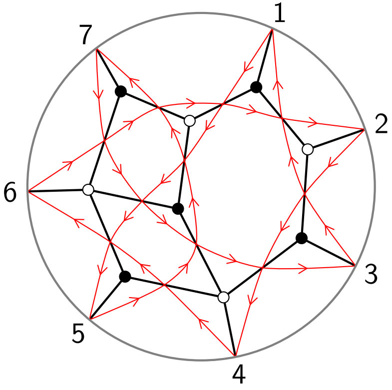

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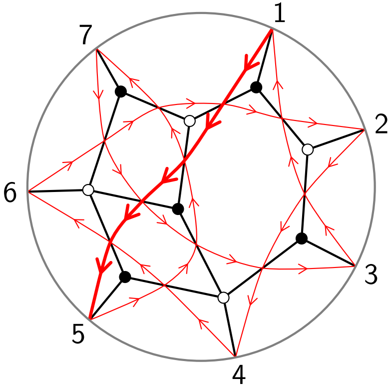

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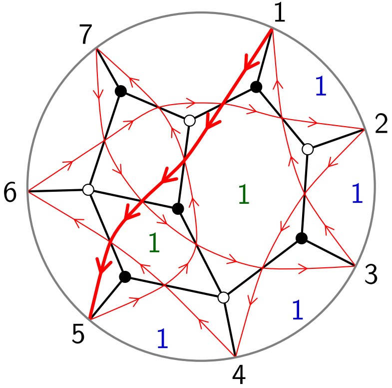

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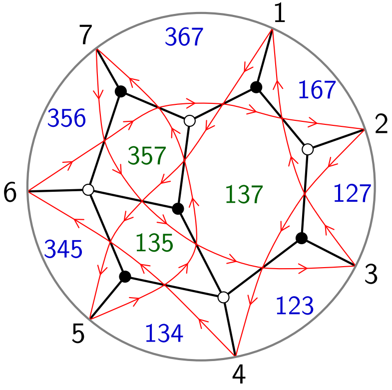

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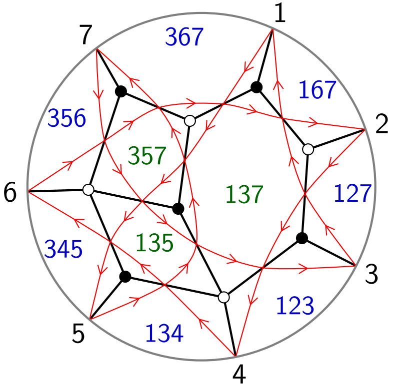

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$$\mathcal{F}^+ = \{\text{boundary labels}\}$$
$$= \{123, 134, 345, \dots\}$$

$$\Pi^\circ_{\mathcal{P}} = \{\Delta_I = 0 \text{ for all } I \notin \mathcal{P},\ \Delta_J \neq 0 \text{ for all } J \in \mathcal{F}^+\}$$

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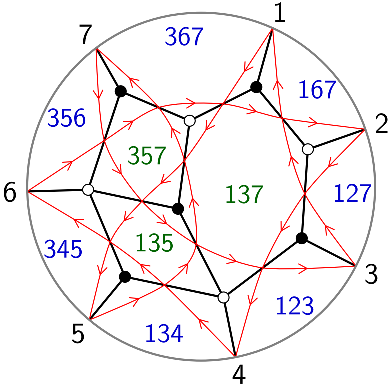

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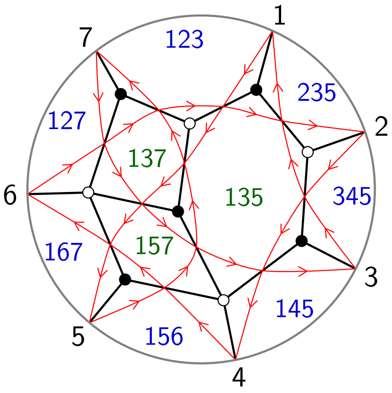

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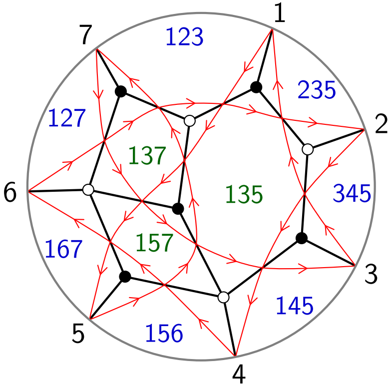

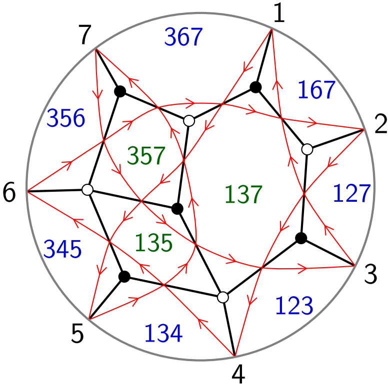

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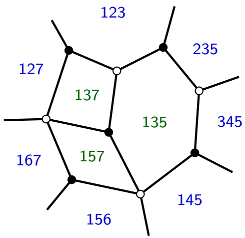
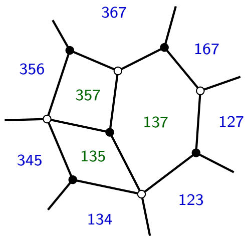
Cluster structures





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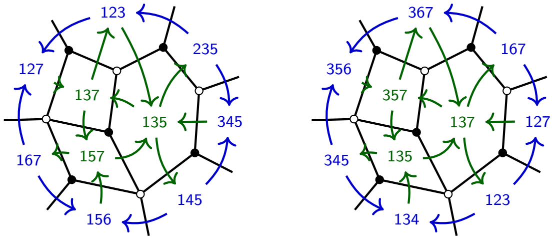

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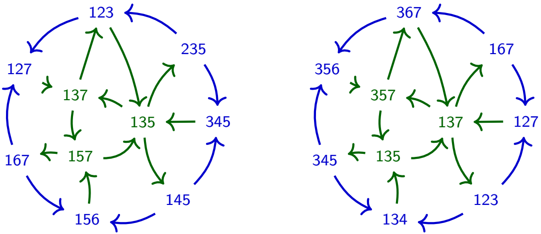


Theorem (Galashin–Lam '23)

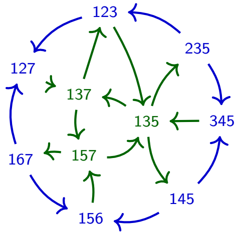
$\mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$ *has two natural cluster algebra structures:*
one cluster algebra $\mathscr{A}_{\mathcal{P}}$, *two isomorphisms* $\eta^{\pm}\colon \mathscr{A}_{\mathcal{P}} \xrightarrow{\sim} \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$.

Theorem (P '23$^+$, conj. Muller–Speyer '17)

The cluster structures $\eta^{\pm}\colon \mathscr{A}_{\mathcal{P}} \xrightarrow{\sim} \mathbb{C}[\widehat{\Pi}^{\circ}_{\mathcal{P}}]$ *quasi-coincide.*

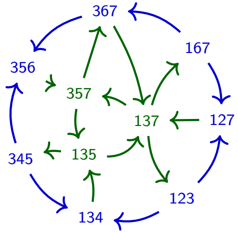
Casals–Le–Sherman-Bennett–Weng '23$^+$: alt. proof

Example



Target-labelled structure

Frozen	$\Delta_{123}, \Delta_{235}, \Delta_{345}, \Delta_{145}, \Delta_{156}, \Delta_{167}, \Delta_{127}$
Mutable, degree 1	$\Delta_{137}, \Delta_{136}, \Delta_{135}, \Delta_{126}, \Delta_{125}, \Delta_{245}, \Delta_{157}$
Mutable, degree 2	$\Delta_{147}\Delta_{235}, \Delta_{145}\Delta_{236}$



Source-labelled structure

Frozen	$\Delta_{167}, \Delta_{127}, \Delta_{123}, \Delta_{134}, \Delta_{345}, \Delta_{356}, \Delta_{367}$
Mutable, degree 1	$\Delta_{357}, \Delta_{347}, \Delta_{137}, \Delta_{346}, \Delta_{136}, \Delta_{126}, \Delta_{135}$
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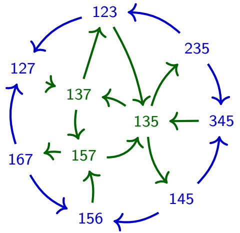

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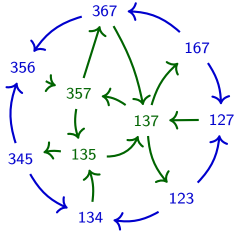

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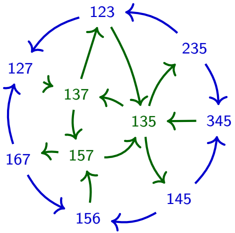

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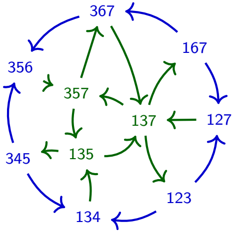

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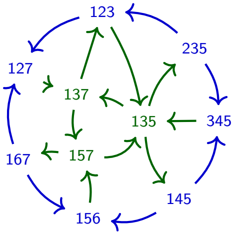

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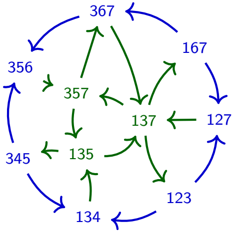

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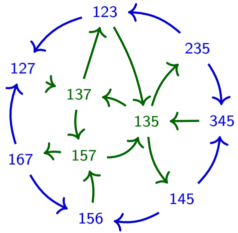

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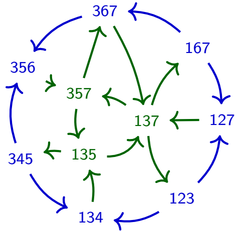

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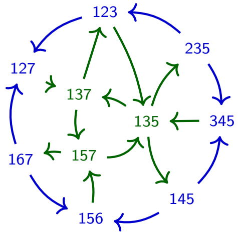

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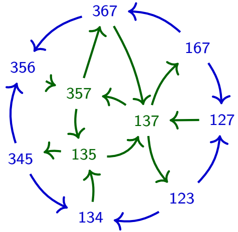

Source-labelled structure

Frozen	${\color{blue}\Delta_{167}}, \Delta_{127}, \Delta_{123}, \Delta_{134}, \Delta_{345}, \Delta_{356}, {\color{blue}\Delta_{367}}$
Mutable, degree 1	${\color{green}\Delta_{357}}, \Delta_{347}, \Delta_{137}, \Delta_{346}, \Delta_{136}, \Delta_{126}, \Delta_{135}$
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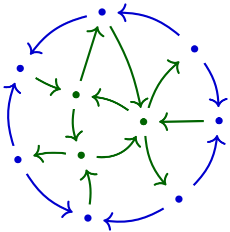
Categorification, part 1: combinatorics



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$$B = eAe$$
$$R = \mathbb{C}[\![t]\!],\ R \subseteq B \subseteq A$$
$$\operatorname{CM} B = \{X \in \operatorname{mod} B : {}_R X \text{ free} + \text{f.g.}\}$$
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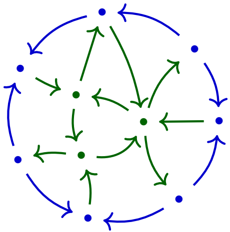

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Theorem (P '22)

For each (connected) positroid $\mathcal{P}$, the Frobenius exact category $\operatorname{gproj} \operatorname{CM} B$ categorifies the cluster algebra $\mathscr{A}_{\mathcal{P}}$.

- Key fact: A is internally 3-Calabi–Yau.
- $\operatorname{ginj} \operatorname{CM} B = \{X \in \operatorname{CM} B : \operatorname{Ext}_B^{>0}(B^\vee, X) = 0\}$ also categorifies.
- $\operatorname{gproj} \operatorname{CM} B \simeq \operatorname{ginj} \operatorname{CM} B$, but different subcategories of $\operatorname{CM} B$!

Categorification, part 2: geometry

- Jensen–King–Su '16: categorification CM C of $\mathbb{C}[\widehat{\mathrm{Gr}}_{k,n}]$, with $M_I \in \mathrm{CM}\, C$ for each Δ_I.

Theorem (Çanakçı–King–P '24, P '22)

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Categorification, part 2: geometry

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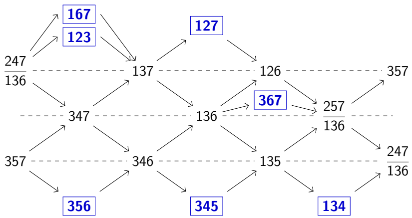

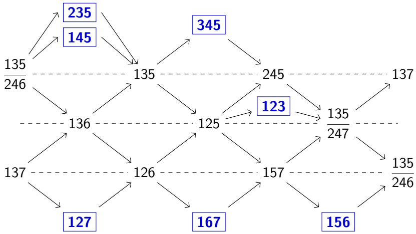

Proving the Muller–Speyer conjecture

- Reduce (geometrically) to connected positroids, for access to categorifications.
- Key fact: inclusions induce *derived* equivalences

$$\mathcal{D}^{\mathrm{b}}(\operatorname{gproj}\mathsf{CM}\,B) \xhookrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\mathsf{CM}\,B) \xhookleftarrow{\sim} \mathcal{D}^{\mathrm{b}}(\operatorname{ginj}\mathsf{CM}\,B)$$

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Theorem (P '23$^+$, conj. Muller–Speyer '16)

The cluster structures η^+ and η^- quasi-coincide.

- Independent proof: (Casals–Le–Sherman-Bennett–Weng '23$^+$) Inspired by symplectic topology!

Computation

$$0 \longrightarrow \Omega X \longrightarrow P \longrightarrow X \longrightarrow 0$$

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 & & \Big\| & & \Big\uparrow & & \Big\uparrow & & \\
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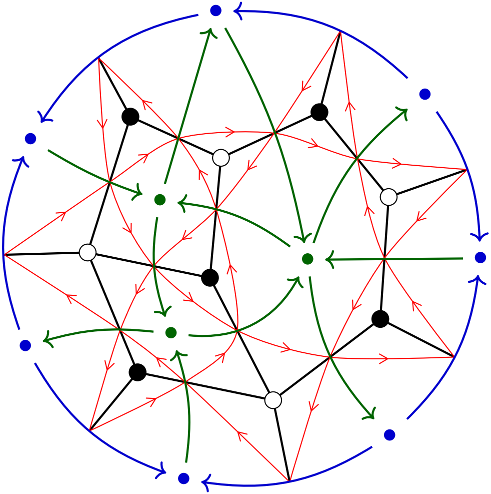

谢谢!