Factorization theory of cluster algebras

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ICRA21, August 8, 2024

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The history of factorization

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$$D_n = \{a_0 + a_1\omega + \cdots + a_{n-1}\omega^{n-1}\} \quad \text{has unique factorization}$$

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1994	Andrew Wiles's proof

Factorizations

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Irreducibles

Factorizations

$$42 = 2 \cdot 3 \cdot 7 = 3 \cdot (-2) \cdot (-7)$$

$$x^3 - x = x(x-1)(x+1)$$

factorization: $a = c_1 \cdot c_2 \cdots c_r = d_1 \cdot d_2 \cdots d_s$

- r ... length of factorization
- the same if $r = s$ and (up to reordering) $c_i \sim d_i$

Krull domains

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One possibile solution: Krull domains and their class groups

- Integrally closed noetherian domains are Krull domains.
- To each Krull domain we can attach an invariant, the class group.
- Factorization theory of a Krull domain is completely determined by its class group (and one of its subgroup).

Class groups

- $\mathcal{C}(A) = \langle\{\text{ height-1 primes }\}\rangle/\mathbf{q}(A)^{\times}$.
- If A is an integrally closed noetherian domain, then the class group $\mathcal{C}(A)$ is the divisor class group of the corresponding affine variety[1]
- If the class group is infinite, then for each finite set of integers $2 \leq l_1 < \cdots < l_k$ there exists an element of A whose lenghts of factorizations are exactly $l_1, \ldots, l_k$.

Theorem

For a domain A the following statements are equivalent.

1. A is a UFD.
2. A is a Krull domain with trivial class group.

[1] group of Weil divisors modulo principal divisors.

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Our setting: cluster algebras of geometric type

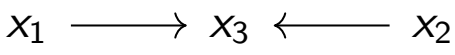
Definition

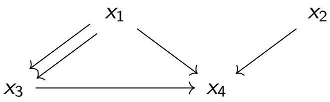
A **seed** $(\mathbf{x}, \mathcal{Q})$ is

- a **cluster**: a set $\mathbf{x} = \{x_1, \ldots, x_n\}$ of algebraically independent indeterminates over $\mathbb{Z}$;
- the cluster is identified with the vertices of a **quiver**[a] $\mathcal{Q}$.

[a] a finite directed graph without loops and 2-cycles

Two seeds:





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Parallel mutation of **cluster**:

$$\{x_1, \dots, x_{k-1}, x_k, , x_{k+1}, \dots, x_n\} \rightsquigarrow \{x_1, \dots, x_{k-1}, x_k', x_{k+1}, \dots, x_n\}$$

with

$$x_k x_k' = \underbrace{\prod_{j \to k} x_j + \prod_{k \to j} x_j}_{\mathbf{f_k \text{ exchange polynomial}}} .$$

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Mutations produce a collection of seeds (possibly infinitely many).

Cluster algebras

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Definition

The **cluster algebra** $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathcal{Q})$ is the subalgebra of the rational functions $\mathbb{Z}(x_1, \ldots, x_n)$ generated by all the cluster variables.

Cluster algebras: an example

Let $A_2 = \; x_1 \longrightarrow x_2$.

$$\begin{array}{ccccccccccc} x_1 & \longrightarrow & x_2 & \overset{1}{\Rightarrow} & \frac{x_2+1}{x_1} & \longleftarrow & x_2 & \overset{2}{\Rightarrow} & \frac{x_2+1}{x_1} & \longrightarrow & \frac{1+x_1+x_2}{x_1x_2} \\ & & & & & & & & & & \Downarrow 1 \\ x_2 & \longleftarrow & x_1 & \overset{1}{\Leftarrow} & \frac{x_1+1}{x_2} & \longrightarrow & x_1 & \overset{2}{\Leftarrow} & \frac{x_1+1}{x_2} & \longleftarrow & \frac{1+x_1+x_2}{x_1x_2} \end{array}$$

Then

$$\mathcal{A}(\mathbf{x}, A_2) = \mathbb{Z}\left[x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}\right]$$

Theorem (Fomin-Zelevinsky 2002)

Given a cluster $\mathbf{x} = \{x_1, x_2, ..., x_n\}$ in a cluster algebra $\mathcal{A}$, every element of $\mathcal{A}$ can be written as a Laurent polynomial in $\mathbf{x}$.

$$\mathbb{Z}[x_1, \ldots, x_{n+m}] \subseteq \mathcal{A} \subseteq \mathbb{Z}[\mathbf{x_1^{\pm 1}}, \ldots, \mathbf{x_n^{\pm 1}}] \subseteq \mathbb{Z}(x_1, \ldots, x_n)$$

where $\mathbb{Z}[\mathbf{x_1^{\pm 1}}, \ldots, \mathbf{x_n^{\pm 1}}] = \{\frac{f}{x_1^{a_1} \ldots x_n^{a_n}} \mid f \in \mathbb{Z}[x_1, \ldots, x_n],\ a_i \in \mathbb{N}_0\}$ is the Laurent polynomial ring (associated to $x_1, \ldots, x_n$).

Upper cluster algebras

Definition

The **upper cluster algebra** associated to a seed $(\mathbf{x}, \mathcal{Q})$ is

$$\mathcal{U} = \bigcap_{\mathbf{y}\text{ cluster}} \mathbb{Z}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}].$$

Notice that the Laurent phenomenon is equivalent to $\mathcal{A} \subseteq \mathcal{U}$.

Full rank upper cluster algebras

The **signed adjacency matrix** associated to a quiver $\mathcal{Q}$ is the matrix $B = B(\mathcal{Q})$ given by

$$b_{ij} = \#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\}.$$

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Theorem (Berenstein-Fomin-Zelevinsky 2005)

If $\mathcal{U}$ is a full rank upper cluster algebra, then

$$\mathcal{U} = \bigcap_{k=0}^{n} \mathbb{Z}[x_1^{\pm 1}, \ldots, x_k'^{\pm 1}, \ldots, x_n^{\pm 1}].$$

In particular, it is a Krull domain.

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- 2019 — A. Garcia Elsener, P. Lampe, D. Smertnig: class group of an acyclic cluster algebra
- 2023 — P. Cao, B. Keller, F. Qin: factoriality of full rank upper cluster algebras

Atomicity

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Every $a \in A$ can be written in only finitely many different ways as a product of atoms:

$$a = u_1 \cdots u_k \quad \text{with } u_i \text{ atoms.}$$

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- $x_1 \Rrightarrow x_2 \qquad x_1 x_1' = x_2^3 + 1 = (x_2 + 1)(x_2^2 + x_2 + 1)$

Earlier results on factoriality

Proposition (Geiss-Leclerc-Schröer, 2012)

If A is a UFD, all exchange polynomials $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$ are irreducible and pairwise distinct.

Earlier results on factoriality

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Theorem (Cao-Keller-Qin, 2023)

Let $\mathcal{U}$ be a full rank upper cluster algebra. Then $\mathcal{U}$ is a UFD if and only if all the exchange polynomials associated to its initial seed are irreducible.

Class groups of (upper) cluster algebras

Theorem (Garcia Elsener-Lampe-Smertnig, 2019 & P. 2023)

Let A be an (upper) cluster algebra, with initial cluster $\{x_1, \ldots, x_n\}$. Assume that A is a Krull domain.

1. $\mathcal{C}(A) \cong \mathbb{Z}^r$, for some $r \geq 0$,

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1. $\mathcal{C}(A) \cong \mathbb{Z}^r$, for some $r \geq 0$,
2. $r = t - n$, with t the number of height-1 prime ideals that contain (at least) one of the x_i.

And the rank?

Can we say something more about the rank of this class group?

Class group of a full rank upper cluster algebra

Remember: Full rank upper cluster algebras are Krull domains!

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Theorem (P. 2023)

Let $\mathcal{U}$ be a full rank upper cluster algebra. Let $f_1, \ldots, f_n$ be the exchange polynomials associated to the initial seed. Then

$$\mathcal{C}(\mathcal{U}) \cong \mathbb{Z}^{t-n} \quad \text{with } t = \sum_{i=1}^{n} l_i$$

where l_i is the number of irreducible factors of f_i.

Factorial upper cluster algebras

Corollary

Let $\mathcal{U}$ be a full rank upper cluster algebra. Let $f_1, \ldots, f_n$ be the exchange polynomials associated to the initial seed. Then $\mathcal{U}$ is a UFD if and only if $f_1, \ldots, f_n$ are irreducible.

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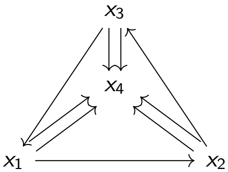
Proof:

Remember: $\mathcal{C}(\mathcal{U}) \cong \mathbb{Z}^{t-n}$, $t = \#$irreducible factors

$$\mathcal{C}(\mathcal{U}) = 0 \iff t = n \iff f_1, \ldots, f_n \text{ are irreducible.}$$

Some examples I

Let $\mathcal{Q} =$



$$B(\mathcal{Q}) = \begin{pmatrix} 0 & 1 & -1 & 2 \\ -1 & 0 & 1 & 2 \\ 1 & -1 & 0 & 2 \\ -2 & -2 & -2 & 0 \end{pmatrix}$$

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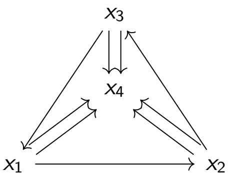

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$$\begin{aligned} f_1 &= x_3 + x_2 x_4^2 \\ f_2 &= x_1 + x_3 x_4^2 \\ f_3 &= x_1 x_4^2 + x_2 \\ f_4 &= x_2^2 x_1^2 x_3^2 + 1 \end{aligned}$$

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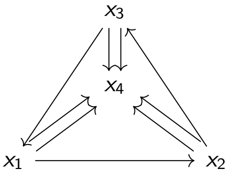

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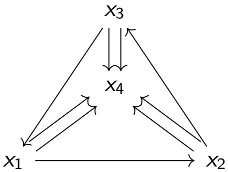

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$\mathcal{C}(\mathcal{Q}) = 0$

Some examples II

$$\mathcal{Q} = \quad x_1 \Rrightarrow x_2 \rightrightarrows x_3 \rightrightarrows x_4 \qquad\qquad B(\mathcal{Q}) = \begin{pmatrix} 0 & 3 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

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$t = 5, n = 4$

$\mathcal{C}(\mathcal{Q}) = \mathbb{Z}$

Summary

- The class groups of (upper) cluster algebras that are Krull domains are always of the type $\mathbb{Z}^r$.

- For full rank upper cluster algebras, this r can be computed counting the number of irreducible factors of the exchange polynomials.

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Similar results hold over fields of characteristic 0 as ground ring,
and allowing frozen variables.

Open questions:

- What about non full rank upper cluster algebras?

- Are all upper cluster algebras Krull domains? If no, how to characterize them?

Thank you for your attention!