Calabi-Yau connected cochain DG algebras

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Homological properties of homologically smooth connected cochain DG algebras, https://arxiv.org/pdf/2407.14805

Definitions 1.1

Let $\mathscr{A}$ be a connected graded $\Bbbk$-algebra. If $\exists$ a $\Bbbk$-linear map $\partial_{\mathscr{A}}: \mathscr{A} \to \mathscr{A}$ of degree 1 such that $\partial_{\mathscr{A}} \circ \partial_{\mathscr{A}} = 0$, and

$$\partial_{\mathscr{A}}(ab) = \partial_{\mathscr{A}}(a)b + (-1)^{|a|}a\partial_{\mathscr{A}}(b),$$

for all graded elements $a, b \in \mathscr{A}$. Then $(\mathscr{A}, \partial_{\mathscr{A}})$ is called a connected cochain DG algebra.

$$(\mathscr{A}, \partial_{\mathscr{A}}): 0 \to \mathscr{A}^0 = \Bbbk \xrightarrow{\partial^0_{\mathscr{A}}=0} \mathscr{A}^1 \xrightarrow{\partial^1_{\mathscr{A}}} \mathscr{A}^2 \xrightarrow{\partial^2_{\mathscr{A}}} \cdots \xrightarrow{\partial^i_{\mathscr{A}}} \mathscr{A}^{i+1} \xrightarrow{\partial^{i+1}_{\mathscr{A}}} \cdots$$

1. Any connected graded algebra A can be considered as a connected cochain DG algebra with zero differential
$$\mathscr{A}: \quad 0 \to A^0 = \Bbbk \xrightarrow{0} A^1 \xrightarrow{0} A^2 \xrightarrow{0} \cdots \xrightarrow{0} A^{i+1} \xrightarrow{0} \cdots$$
2. $\forall$ complex of graded A-modules $\cdots \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} \cdots$ can be compressed as a DG $\mathscr{A}$-module
$$\left(\bigoplus_{i\in\mathbb{Z}} \Sigma^i X^i, ((-1)^i\Sigma^i(d^i))_{i\in\mathbb{Z}}\right). \qquad (\Sigma M)^i = M^{i+1}$$

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Comparison of two homology theories

DG homological algebra	graded hypercohomology algebra
compact DG module	perfect complex
tilting DG module	tilting complex
dualizing DG module	dualizing complex
semi-free resolution	free resolution
semi-projective resolution	projective resolution
semi-injective resolution	injective resolution
Gorenstein DG algebra	AS-Gorenstein algebra
Koszul DG algebra	Koszul algebra
homologically smooth	noetherian regular algebra
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- $\mathfrak{m}$: the maximal DG ideal
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- $\mathscr{A}^{op}$: opposite algebra of $\mathscr{A}$ with a product $\diamond$ is defined by
$$a_1 \diamond a_2 = (-1)^{|a_1|\cdot|a_2|} a_2 a_1;$$
- $\mathscr{A}^e$: enveloping DG algebra $\mathscr{A} \otimes \mathscr{A}^{op}$ of $\mathscr{A}$;
- $\mathscr{D}(\mathscr{A})$: derived category of DG left $\mathscr{A}$-modules;
- a DG $\mathscr{A}$-module M is called compact, if $\mathrm{Hom}_{\mathscr{D}(\mathscr{A})}(M, -)$ preserves all set-indexed coproducts in $\mathscr{D}(\mathscr{A})$;
- $\mathscr{D}^c(\mathscr{A})$: full subcat of $\mathscr{D}(\mathscr{A})$ consisting of compact objects;
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- Are there some relations between these four homological properties?
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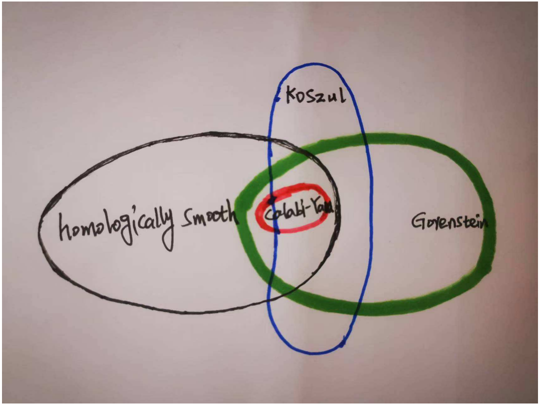
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$$\langle -, - \rangle : E \times E \to \Bbbk \quad \text{s.t.} \quad \langle xy, z \rangle = \langle x, yz \rangle, \forall x, y, z \in E.$$

Definition 1.10 If the Frobenius form $\langle -, - \rangle : E \times E \to \Bbbk$ of a Frobenius algebra E satisfies the condition: $\langle a, b \rangle = \langle b, a \rangle$, $\forall a, b \in E$, then E is called a symmetric Frobenius algebra .

- E is Frobenius iff ${}_E E \cong {}_E(E^*)$ or $E_E \cong (E^*)_E$
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Theorem 1.11 **He-Mao, Proc. AMS, (2017)**

Let $\mathscr{A}$ be a Koszul connected cochain DG algebra. Then $\mathscr{A}$ is Calabi-Yau iff its Ext-algebra $H(R\operatorname{Hom}_{\mathscr{A}}(\Bbbk,\Bbbk))$ is a symmetric Frobenius algebra.

Question 1.12: Can we drop the Koszul condition of the two theorems above?

Definition 1.13 Let E be a finite dimensional graded algebra. It is called a Frobenius graded algebra if any one of the following equivalent conditions holds.

1. $\exists j \in \mathbb{Z}$ and an isomorphism of left E-modules: $\Sigma^j E \to E^*$.
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3. $\exists d \in \mathbb{Z}$ and a graded non-degenerate bilinear form $\langle -,- \rangle : E \times E \to \Sigma^d \Bbbk$, s.t. $\langle ab, c\rangle = \langle a, bc\rangle, \forall a,b,c \in E$.

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Definition 1.14 If the Frobenius form $\langle -,-\rangle$ of a Frobenius graded algebra E satisfies the condition:

$$\langle a,b\rangle = (-1)^{ij}\langle b,a\rangle,\quad \forall a\in E^i, b\in E^j,$$

then E is called symmetric.

- A finite dimensional graded algebra E is Frobenius iff $\exists j\in\mathbb{Z}$ s.t. $\Sigma^j{}_E E \cong {}_E(E^*)$ or equivalently $\Sigma^j E_E \cong (E^*)_E$
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Aim

Calabi-Yauness of a connected cochain DG algebra $\mathscr{A}$

$$\Updownarrow$$

symmetric Frobenius properties of $H(R\,\mathrm{Hom}_{\mathscr{A}}(\mathbb{k},\mathbb{k}))$

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$\mathscr{A}$: a connected cochain DG algebra

- Let K be a minimal semi-free resolution of ${}_{\mathscr{A}}\mathbb{k}$.
- $\mathcal{E} = \mathrm{Hom}_{\mathscr{A}}(K, K)$: the Koszul dual DG algebra of $\mathscr{A}$
- Then $H(\mathcal{E}) = H(\mathrm{Hom}_{\mathscr{A}}(K, K))$ is just the Ext-algebra of $\mathscr{A}$.

Theorem 2.1 **arXiv:2407.14805**

Assume that $\mathscr{A}$ is a connected cochain DG algebra. Then $\mathscr{A}$ is Gorenstein and homologically smooth iff its Ext-algebra $H(\mathcal{E})$ is a graded Frobenius algebra.

Source of inspiration

[1] P. Jørgensen, Duality for cochain DG algebras, (2013)
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Theorem 2.2 **arXiv:2407.14805**

Let $\mathscr{A}$ be a homologically smooth connected cochain DG algebra. Then following statements are equivalent.

1. The Ext-algebra $H(\mathcal{E})$ is a Frobenius graded algebra;
2. $\mathscr{A}$ is left Gorenstein;
3. $\mathscr{A}$ is right Gorenstein;
4. $(\mathcal{E}^*)_{\mathcal{E}} \in \mathscr{D}^c(\mathcal{E}^{op})$ and ${}_{\mathcal{E}}(\mathcal{E}^*) \in \mathscr{D}^c(\mathcal{E})$;
5. $\dim_{\mathbb{k}} H(R\,\mathrm{Hom}_{\mathcal{E}}(\mathbb{k},\mathcal{E})) < \infty,\ \dim_{\mathbb{k}} H(R\,\mathrm{Hom}_{\mathcal{E}^{op}}(\mathbb{k},\mathcal{E})) < \infty$;
6. $\dim_{\mathbb{k}} H(R\,\mathrm{Hom}_{\mathcal{E}}(\mathbb{k},\mathcal{E})) = 1,\ \dim_{\mathbb{k}} H(R\,\mathrm{Hom}_{\mathcal{E}^{op}}(\mathbb{k},\mathcal{E})) = 1$;
7. $\mathscr{D}^c(\mathcal{E})$ and $\mathscr{D}^c(\mathcal{E}^{op})$ admit Auslander-Reiten triangles;
8. $\mathscr{D}^b_{lf}(\mathscr{A})$ and $\mathscr{D}^b_{lf}(\mathscr{A}^{op})$ admit Auslander-Reiten triangles;

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5. $\dim_{\Bbbk} H(R\operatorname{Hom}_{\mathcal{E}}(\Bbbk, \mathcal{E})) < \infty$, $\dim_{\Bbbk} H(R\operatorname{Hom}_{\mathcal{E}^{op}}(\Bbbk, \mathcal{E})) < \infty$;
6. $\dim_{\Bbbk} H(R\operatorname{Hom}_{\mathcal{E}}(\Bbbk, \mathcal{E})) = 1$, $\dim_{\Bbbk} H(R\operatorname{Hom}_{\mathcal{E}^{op}}(\Bbbk, \mathcal{E})) = 1$;
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Theorem 2.3 arXiv:2407.14805

Let $\mathscr{A}$ be a homologically smooth and Gorenstein DG algebra. Then the following statements are equivalent.

1. $\mathscr{A}$ is Calabi-Yau;
2. The Ext-algebra $H(\mathcal{E})$ is a symmetric Frobenius graded algebra;
3. The triangulated categories $\mathscr{D}^c(\mathcal{E})$ and $\mathscr{D}^c(\mathcal{E}^{op})$ are Calabi-Yau;
4. The triangulated categories $\mathscr{D}^b_{lf}(\mathscr{A})$ and $\mathscr{D}^b_{lf}(\mathscr{A}^{op})$ are Calabi-Yau.

Theorem 2.4 arXiv:2407.14805

A connected cochain DG algebra $\mathscr{A}$ is Calabi-Yau if and only if its Ext-algebra $H(\mathcal{E})$ is a symmetric Frobenius graded algebra.

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Theorem 2.4 **arXiv:2407.14805**

A connected cochain DG algebra $\mathscr{A}$ is Calabi-Yau if and only if its Ext-algebra $H(\mathcal{E})$ is a symmetric Frobenius graded algebra.

Example 3.1 Let $\mathscr{A}$ be a connected cochain DG algebra s.t.

$$\mathscr{A}^{\#} = \Bbbk\langle x, y\rangle/(x^2y - yx^2, xy^2 - y^2x), |x| = |y| = 1$$

with a differential defined by $\partial_{\mathscr{A}}(x) = y^2, \partial_{\mathscr{A}}(y) = 0$.
Then $\mathscr{A}$ is a non-Koszul Calabi-Yau DG algebra.

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Then $\mathscr{A}$ is a non-Koszul Calabi-Yau DG algebra.

Step 1: ${}_{\mathscr{A}}\Bbbk$ admits a minimal semi-free resolution F s.t.

$$F^{\#} = \mathscr{A}^{\#} \oplus \mathscr{A}^{\#}\Sigma e_y \oplus \mathscr{A}^{\#}\Sigma e_z \oplus \mathscr{A}^{\#}\Sigma e_{x^2} \oplus \mathscr{A}^{\#}\Sigma e_t \oplus \mathscr{A}^{\#}\Sigma e_r,$$

$$\partial_F(\Sigma e_y) = y, \quad \partial_F(\Sigma e_z) = x + y\Sigma e_y, \quad \partial_F(\Sigma e_{x^2}) = x^2,$$

$$\partial_F(\Sigma e_t) = x^2\Sigma e_y + y\Sigma e_{x^2}, \partial_F(\Sigma e_r) = y\Sigma e_t + x\Sigma e_{x^2} + x^2\Sigma e_Z.$$

Step 2: $H(R\operatorname{Hom}_{\mathscr{A}}(\Bbbk, \Bbbk)) \cong H(\operatorname{Hom}_{\mathscr{A}}(F, \Bbbk)) = \operatorname{Hom}_{\mathscr{A}}(F, \Bbbk)$

$$\cong \Bbbk 1^* \oplus \Bbbk(\Sigma e_y)^* \oplus \Bbbk(\Sigma e_z)^* \oplus \Bbbk(\Sigma e_{x^2})^* \oplus \Bbbk(\Sigma e_t)^* \oplus \Bbbk(\Sigma e_r)^*$$

Note that $\begin{cases} |1^*| = |(\Sigma e_y)^*| = |(\Sigma e_z)^*| = 0 \\ |(\Sigma e_{x^2})^*| = |(\Sigma e_t)^*| = |(\Sigma e_r)^*| = -1. \end{cases}$

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Step 3: Compute $H(R\operatorname{Hom}_{\mathscr{A}}(\mathbb{k}, \mathbb{k})) = H(\operatorname{Hom}_{\mathscr{A}}(K, K))$.
It is isomorphic to the algebra

$$\left\{ \begin{pmatrix} d & 0 & 0 & 0 & 0 & 0 \\ e & d & 0 & 0 & 0 & 0 \\ q & e & d & 0 & 0 & 0 \\ a & 0 & 0 & d & 0 & 0 \\ b & a & 0 & e & d & 0 \\ c & b & a & q & e & d \end{pmatrix} \quad \middle| \quad a, b, c, d, e, q \in \mathbb{k} \right\} = \bigoplus_{i=0}^{5} \mathbb{k} e_i,$$

where $e_0 = \sum\limits_{i=1}^{6} E_{ii}$, $e_1 = E_{21} + E_{32} + E_{54} + E_{65}$, $e_2 = E_{31} + E_{64}$, $e_3 = E_{41} + E_{52} + E_{63}$, $e_4 = E_{51} + E_{62}$ and $e_5 = E_{61}$.

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Step 4: $H(R\operatorname{Hom}_{\mathscr{A}}(\Bbbk, \Bbbk)) \cong \bigoplus_{i=0}^{5} \Bbbk e_i$ with $|e_0| = |e_1| = |e_2| = 0$, $|e_3| = |e_4| = |e_5| = -1$ and a multiplication structure given by

$\cdot$	e_0	e_1	e_2	e_3	e_4	e_5
e_0	e_0	e_1	e_2	e_3	e_4	e_5
e_1	e_1	e_2	0	e_4	e_5	0
e_2	e_2	0	0	e_5	0	0
e_3	e_3	e_4	e_5	0	0	0
e_4	e_4	e_5	0	0	0	0
e_5	e_5	0	0	0	0	0

It is a symmetric Frobenius graded algebra.

Let $\mathscr{A}$ be a connected cochain DG algebra.

- The cohomology graded algebra of $\mathscr{A}$ is the algebra
$$H(\mathscr{A}) = \bigoplus_{i\in\mathbb{Z}} \frac{\ker(\partial^i_{\mathscr{A}})}{\operatorname{im}(\partial^{i-1}_{\mathscr{A}})}.$$
- $\forall z \in \ker(\partial^i_{\mathscr{A}})$, we denote by $\lceil z \rceil$ the cohomology class in $H(\mathscr{A})$ represented by z.

Proposition 3.2 [Mao-Yang-Ye, (2019)]

If the trivial DG algebra $(H(\mathscr{A}), 0)$ is Calabi-Yau DG algebra, then $\mathscr{A}$ is a Calabi-Yau DG algebra.

Proposition 3.3 [Mao-He, (2017)]

Let $\mathscr{A}$ be a connected cochain DG algebra. Then $\mathscr{A}$ is Koszul and Calabi-Yau if $H(\mathscr{A}) = \frac{\Bbbk\langle \lceil z_1\rceil, \lceil z_2\rceil\rangle}{(\lceil z_1\rceil\lceil z_2\rceil + \lceil z_2\rceil\lceil z_1\rceil)}$, $z_1, z_2 \in \ker(\partial^1_{\mathscr{A}})$. And $\mathscr{A}$ is not Calabi-Yau but Koszul, homologically smooth and Gorenstein if $H(\mathscr{A}) = \Bbbk[\lceil z_1\rceil, \lceil z_2\rceil]$, $z_1, z_2 \in \ker(\partial^1_{\mathscr{A}})$.

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If the trivial DG algebra $(H(\mathscr{A}), 0)$ is Calabi-Yau DG algebra, then $\mathscr{A}$ is a Calabi-Yau DG algebra.

Proposition 3.3 **[Mao-He, (2017)]**

Let $\mathscr{A}$ be a connected cochain DG algebra. Then $\mathscr{A}$ is Koszul and Calabi-Yau if $H(\mathscr{A}) = \frac{\Bbbk\langle \lceil z_1\rceil, \lceil z_2\rceil\rangle}{(\lceil z_1\rceil\lceil z_2\rceil + \lceil z_2\rceil\lceil z_1\rceil)}$, $z_1, z_2 \in \ker(\partial^1_{\mathscr{A}})$. And $\mathscr{A}$ is not Calabi-Yau but Koszul, homologically smooth and Gorenstein if $H(\mathscr{A}) = \Bbbk[\lceil z_1\rceil, \lceil z_2\rceil]$, $z_1, z_2 \in \ker(\partial^1_{\mathscr{A}})$.

Let $\mathscr{A}$ be a connected cochain DG algebra.

- The cohomology graded algebra of $\mathscr{A}$ is the algebra
$$H(\mathscr{A})=\bigoplus_{i\in\mathbb{Z}}\frac{\ker(\partial^i_{\mathscr{A}})}{\operatorname{im}(\partial^{i-1}_{\mathscr{A}})}.$$
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Let $\mathcal{G}$ and $\mathcal{D}$ be the category of connected graded algebras and the category of connected cochain DG algebras, respectively.

Remark 3.4

- $\Theta : \mathcal{G} \to \mathcal{D}$: the functor s.t. $\forall A \in \mathcal{G}$, $\Theta(A)$ is a trivial DG algebra with $\Theta(A)^{\#} = A$.
- Θ preserves Koszul, Gorenstein and homologically smooth properties
- Θ doesn't necessarily preserve Calabi-Yauness
- the multiplication of the opposite algebra

graded context	DG context
$a * b = ba$	$a * b = (-1)^{\|a\| \cdot \|b\|} ba$

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Proposition 3.5 Let $\mathscr{A}$ be a connected cochain DG algebra s.t.

$$H(\mathscr{A}) = \Bbbk\frac{k\langle\lceil x\rceil,\lceil y\rceil\rangle}{(f_1,f_2)}, x, y \in Z^1(\mathscr{A}),$$
$$f_1 = a\lceil x\rceil\lceil y\rceil^2 + b\lceil y\rceil\lceil x\rceil\lceil y\rceil + a\lceil y\rceil^2\lceil x\rceil + c\lceil x\rceil^3,$$
$$f_2 = a\lceil y\rceil\lceil x\rceil^2 + b\lceil x\rceil\lceil y\rceil\lceil x\rceil + a\lceil x\rceil^2\lceil y\rceil + c\lceil y\rceil^3,$$

where $(a:b:c) \in \mathbb{P}^2_k - \mathfrak{D}$ and

$$\mathfrak{D} := \{(0:0:1),(0:1:0)\} \sqcup \{(a:b:c) | a^2 = b^2 = c^2\}.$$

Then $\mathscr{A}$ is a homologically smooth and Gorenstein DG algebra. However, it is neither Koszul nor Calabi-Yau. arXiv:2407.14805

$H(\mathscr{A})$ is a cubic Artin-Schelter algebra of type A

$$H(R\operatorname{Hom}_{\mathscr{A}}(\Bbbk,\Bbbk)) \cong \bigoplus_{i=0}^{5} \Bbbk e_i,\ |e_i| = \begin{cases} 0, & i = 0,1,2 \\ -1, & i = 3,4,5 \end{cases}$$

$\cdot$	e_0	e_1	e_2	e_3	e_4	e_5
e_0	e_0	e_1	e_2	e_3	e_4	e_5
e_1	e_1	0	0	0	$-e_5$	0
e_2	e_2	0	0	$-e_5$	0	0
e_3	e_3	0	e_5	0	0	0
e_4	e_4	e_5	0	0	0	0
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It is Frobenius but not symmetric.

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It is Frobenius but not symmetric.

Definition 3.6 **Mao-Xie-Yang-Abla, (2019)**

A connected cochain DG algebra $\mathscr{A}$ is called DG free if

$$\mathscr{A}^{\#} = \Bbbk\langle x_1, x_2, \cdots, x_n\rangle, \text{ with } |x_i| = 1, \ \forall i \in \{1, 2, \cdots, n\}.$$

Definition 3.7 **Mao-Xie-Yang-Abla, (2019)**

Let $(M^1, M^2, \cdots, M^n)$ be an ordered n-tuple of $n \times n$ matrixes

with each $\quad M^i = (c_1^i, c_2^i, \cdots, c_n^i) = \begin{pmatrix} r_1^i \\ r_2^i \\ \vdots \\ r_n^i \end{pmatrix}, i = 1, 2, \cdots, n.$

We say that $(M^1, M^2, \cdots, M^n)$ is crisscross if

$$\sum_{k=1}^{n} [c_j^k r_k^i - c_k^i r_j^k] = (0)_{n\times n}, \ \forall i, j \in \{1, 2, \cdots, n\}.$$

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Theorem 3.8 **Mao-Xie-Yang-Abla, (2019)**

Let $\mathscr{A}$ be a DG free algebra s.t. $\mathscr{A}^{\#} = \Bbbk\langle x_1, x_2, \cdots, x_n\rangle$ with $|x_i| = 1, i = 1, 2, \cdots, n$. Then $\exists$ a crisscross ordered n-tuple $(M^1, M^2, \cdots, M^n)$ of $n \times n$ matrixes s.t. $\partial_{\mathscr{A}}$ is defined by

$$\partial_{\mathscr{A}}(x_i) = (x_1, x_2, \cdots, x_n)M^i \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Conversely, given a crisscross ordered n-tuple $(M^1, M^2, \cdots, M^n)$ of $n \times n$ matrixes, we can define a differential ∂ on $\Bbbk\langle x_1, x_2, \cdots, x_n\rangle$ by

$$\partial(x_i) = (x_1, x_2, \cdots, x_n)M^i \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \forall i \in \{1, 2, \cdots, n\}$$

such that $(\Bbbk\langle x_1, x_2, \cdots, x_n\rangle, \partial)$ is a cochain DG algebra.

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Theorem 3.9 **Mao-Xie-Yang-Abla, (2019)**

Let $\mathscr{A}$ and $\mathscr{B}$ be two DG free algebras s.t.

$$\mathscr{A}^{\#} = \Bbbk\langle x_1, x_2, \cdots, x_n\rangle,\ \mathcal{B}^{\#} = \Bbbk\langle y_1, y_2, \cdots, y_n\rangle, |x_i| = |y_i| = 1.$$

Assume that $\partial_{\mathcal{A}}$ and $\partial_{\mathcal{B}}$ are defined by crisscrossed $n \times n$ matrixes $M^1, M^2, \cdots, M^n$ and $N^1, N^2, \cdots, N^n$, respectively. Then $\mathcal{A} \cong \mathcal{B}$ iff $\exists\, A = (a_{ij})_{n\times n} \in \mathrm{GL}_n(\Bbbk)$ s.t.

$$(a_{ij}E_n)_{n^2\times n^2} \begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{pmatrix} = \begin{pmatrix} A^T M^1 A \\ A^T M^2 A \\ \vdots \\ A^T M^n A \end{pmatrix}.$$

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Theorem 3.10 **Mao-Xie-Yang-Abla, (2019)**

Let $\mathscr{A}$ be a DG free algebra with 2 degree one generators. Then $\mathscr{A}$ is a Koszul Calabi-Yau DG algebra iff $\partial_{\mathscr{A}} \neq 0$.

Question 3.11

Can we generalize the result above to the cases $n \geq 3$?

Example 3.12 **arXiv:2407.14805**

Let $\mathscr{A}$ be a DG free algebra with $\mathscr{A}^{\#} = \mathbb{k}\langle x_1, x_2, x_3\rangle$, $|x_i| = 1$ and $\partial_{\mathscr{A}}(x_1) = x_1^2, \partial_{\mathscr{A}}(x_2) = x_2x_1, \partial_{\mathscr{A}}(x_3) = x_1x_3$.

The DG algebra $\mathscr{A}$ in Example 3.12 is a non-Koszul Calabi-Yau DG algebra with $H(\mathscr{A}) = \mathbb{k}[\lceil x_2x_3 \rceil]$.
Its Ext-algebra $H(R\operatorname{Hom}_{\mathscr{A}}(\mathbb{k}, \mathbb{k})) \cong \mathbb{k}[x]/(x^2)$ with $|x| = -1$.

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$\mathscr{A}$ is a Koszul homologically smooth non-Gorenstein DGA with

$$H(\mathscr{A}) = \mathbb{k}\langle\lceil x_2\rceil, \lceil x_3\rceil\rangle/(\lceil x_2\rceil\lceil x_3\rceil).$$

$$H(R\operatorname{Hom}_{\mathscr{A}}(\mathbb{k},\mathbb{k})) \cong \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & 0 & a & 0 \\ d & c & 0 & a \end{pmatrix} \ \middle|\ a,b,c,d \in \mathbb{k} \right\}$$ not Frobenius

Set $e_0 = \sum_{i=1}^{4} E_{ii}$, $e_1 = E_{21}$, $e_2 = E_{31} + E_{42}$ and $e_3 = E_{41}$.

$\cdot$	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	0	0	0
e_2	e_2	e_3	0	0
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$$H(\mathscr{A}) = \Bbbk\langle \lceil x_2\rceil, \lceil x_3\rceil\rangle/(\lceil x_2\rceil\lceil x_3\rceil).$$

$$H(R\operatorname{Hom}_{\mathscr{A}}(\Bbbk,\Bbbk)) \cong \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & 0 & a & 0 \\ d & c & 0 & a \end{pmatrix} \quad \middle| \quad a,b,c,d \in \Bbbk \right\}$$ not Frobenius

Set $e_0 = \sum\limits_{i=1}^{4} E_{ii}$, $e_1 = E_{21}$, $e_2 = E_{31} + E_{42}$ and $e_3 = E_{41}$.

$\cdot$	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	0	0	0
e_2	e_2	e_3	0	0
e_3	e_3	0	0	0

.

Example 3.13 **arXiv:2407.14805**

Let $\mathscr{A}$ be a DG free algebra with $\mathscr{A}^{\#} = \Bbbk\langle x_1, x_2, x_3\rangle$, $|x_i| = 1$ and $\partial_{\mathscr{A}}(x_1) = x_2x_3,\ \partial_{\mathscr{A}}(x_2) = 0\ \partial_{\mathscr{A}}(x_3) = 0$.

$\mathscr{A}$ is a Koszul homologically smooth non-Gorenstein DGA with

$$H(\mathscr{A}) = \Bbbk\langle\lceil x_2\rceil, \lceil x_3\rceil\rangle/(\lceil x_2\rceil\lceil x_3\rceil).$$

$$H(R\,\mathrm{Hom}_{\mathscr{A}}(\Bbbk,\Bbbk)) \cong \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & 0 & a & 0 \\ d & c & 0 & a \end{pmatrix} \quad \middle| \quad a,b,c,d \in \Bbbk \right\}$$ not Frobenius

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$\cdot$	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	0	0	0
e_2	e_2	e_3	0	0
e_3	e_3	0	0	0

.

Example 3.14 **arXiv:2407.14805**

Let $\mathcal{A}$ be a DG free algebra with $\mathscr{A}^{\#} = \Bbbk\langle x_1, x_2, x_3\rangle, |x_i| = 1$, and $\partial_{\mathscr{A}}(x_1) = x_3^2$, $\partial_{\mathscr{A}}(x_2) = x_1x_3 + x_3x_1$ $\partial_{\mathscr{A}}(x_3) = 0$.

The DG algebra $\mathscr{A}$ in Example 3.14 is a Koszul Calabi-Yau DG algebra with $H(\mathscr{A}) = \Bbbk[\lceil x_3\rceil, \lceil x_1^2 + x_2x_3 + x_3x_2\rceil]/(\lceil x_3\rceil^2)$.

$$H(R\,\mathrm{Hom}_{\mathscr{A}}(\Bbbk,\Bbbk)) \cong \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & a & 0 \\ d & c & b & a \end{pmatrix} \,\middle|\, a,b,c,d \in \Bbbk \right\} \cong \Bbbk[x]/(x^4), |x| = 0$$

is a symmetric Frobenius algebra concentrated in degree 0.

Example 3.14 arXiv:2407.14805

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- X.-F. Mao, Homological properties of homologically smooth connected cochain DG algebras, arXiv.2407.14805
- X.-F. Mao, X. Wang and M.-Y. Zhang, DG Algebra structures on the quantum affine n-space $\mathcal{O}_{-1}(k^n)$, J. Algebra 594 (2022), 389-482.
- X.-F. Mao, J.-F. Xie, Y.-N. Yang and Almire. Abla, Isomorphism problem and homological properties of DG free algebras, Comm. Algebra, 47 (10), (2019), 4031-4060.
- X.-F. Mao, Y.-N. Yang and C.-C. Ye, A sufficient condition for a connected DG algebra to be Calabi-Yau, Comm. Algebra, 47(8), (2019), 3280-3296.
- X.-F. Mao, J.-W. He, M. Liu and J.-F. Xie, Calabi-Yau properties of non-trivial Noetherian DG down-up algebras, J. Algebra Appl. 17, no.5 (2018), 1850090(45pp)
- J.-W. He and X.-F. Mao, Connected cochain DG algebras of Calabi-Yau dimension 0, Proc. Amer. Math. Soc. 145 (3), 2017, 937-953.
- X.-F. Mao and J.-W. He, A special class of Koszul Calabi-Yau DG algebras, Acta Math. Sinica, Chinese series, 60 (2017), 475-504.
- P. Jørgensen, Duality for cochain DG algebras, Sci. China Math. 56 (2013), 79–89.
- X.-F. Mao and Q.-S. Wu, Compact DG modules and Gorenstein DG algebras, Sci. china Ser. A 52 (2009), 648–676.

X.-F. Mao and Q.-S. Wu, Homological invariants for connected DG algebra, Comm. Algebra 36 (2008), 3050-3072.

J.-W. He and Q.-S. Wu, Koszul differential graded algebras and BGG correspondence, J. Algebra, 320, (2008), 2934-2962.

B. Keller, Calabi-Yau triangulated categories, pp.467-489 in “Trends in representation theory of algebras and related topics”, papers from ICRA XII, 2007, EMS Ser. Congr. Rep., EMS Zürich, 2008.

P. Jørgensen, Auslander-Reiten theory over topological spaces. Comment. Math. Helv., 79 (2004), 160-182.

P. Jørgensen, Calabi-Yau categories and Poincaré duality space, pp.399-431 in “Trends in Representation theory of algebras and related topics”, papers from ICRA XII, 2007, EMS, Ser. Congr. Rep, EMS Zürich, 2008.

Thanks for your listening!