Characterizations of tame algebras with separating families of almost cyclic coherent components

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A is of infinite representation type $\xrightarrow{\text{Coelho-Marcos-Merklen-Skowroński}}$ $(\text{rad}_A^\infty)^2 \neq 0$

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Examples

Postprojective components, preinjective components, connecting components of tilted algebras, tubes over tame tilted, tubular and canonical algebras.

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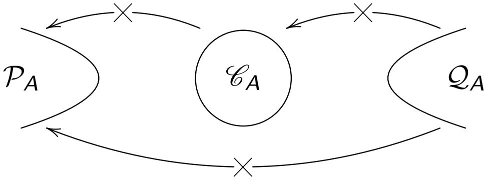

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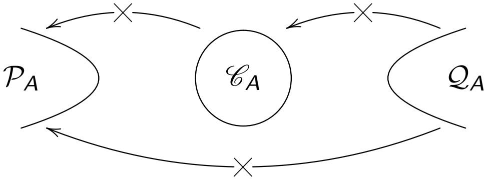

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$\mathcal{P}_A$ and $\mathcal{Q}_A$ are uniquely determined by $\mathscr{C}_A$

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Note that the stable tubes, ray tubes and coray tubes of Γ_A are special types of coherent almost cyclic components.

Separating family of almost cyclic coherent components

Theorem (M.–Skowroński)

Let A be an algebra with a separating family $\mathscr{C}_A$ of almost cyclic coherent components in Γ_A, and $\operatorname{ind} A = \mathcal{P}_A \cup \mathscr{C}_A \cup \mathcal{Q}_A$. *Then*

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5. *There is a unique quotient algebra A_l of A which is a quasitilted algebra of canonical type having a separating family $\mathcal{T}_{A_l}$ of coray tubes such that* $\operatorname{ind} A_l = \mathcal{P}_{A_l} \cup \mathcal{T}_{A_l} \cup \mathcal{Q}_{A_l}$, $\mathcal{P}_{A_l} = \mathcal{P}_A$.

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6. *There is a unique quotient algebra A_r of A which is a quasitilted algebra of canonical type having a separating family $\mathcal{T}_{A_r}$ of ray tubes such that $\operatorname{ind} A_r = \mathcal{P}_{A_r} \cup \mathcal{T}_{A_r} \cup \mathcal{Q}_{A_r}$, $\mathcal{Q}_{A_r} = \mathcal{Q}_A$.*

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7. *A is tame if and only if A_l and A_r are tame.*

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Remark (Bongartz)

Although R is not uniquely determined by I, the numbers $r_{i,j}$ do not depend on the set R.

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- Tits quadratic form of A is the integer quadratic form $q_A : \mathbb{Z}^n \to \mathbb{Z}$, $n = |Q_0|$ defined for $x = (x_i) \in \mathbb{Z}^n$ by the formula

$$q_A(x) = \sum_{i \in Q_0} x_i^2 - \sum_{(i \to j) \in Q_1} x_i x_j + \sum_{i,j \in Q_0} r_{i,j} x_i x_j$$

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A is tame $\xrightarrow{\text{de la Peña}}$ q_A is weakly nonnegative
($q_A(x) \geq 0$ for all vectors x with all coordinates nonnegative)

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Lemma (de la Peña–Skowroński)

Let A be a tame algebra with only finitely many indecomposable sincere directing modules. Let X be an indecomposable sincere directing A-module, then $m(X) \leq 2$.

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Let A be a tame algebra with only finitely many indecomposable sincere directing modules. Let X be an indecomposable sincere directing A-module, then $m(X) \leq 2$.

Proposition (de la Peña–Skowroński)

Let A be any tame algebra. Assume X is a directing module with $m(X) \geq 3$. Then $B(X)$ is a representation-infinite tilted algebra of Euclidean type and X is a postprojective or preinjective $B(X)$-module.

Lemma

Let A be an algebra with a separating family $\mathscr{C}_A$ of almost cyclic coherent components in Γ_A and X be an indecomposable module in $\operatorname{mod} A$*. Then* $\operatorname{Ext}^2_A(X,X) = 0$.

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- If $X \in \mathcal{Q}_A$, then $\mathrm{id}_A X \leq 1$, and $\mathrm{Ext}_A^2(X, X) = 0$.
- Let $X \in \mathscr{C}_A$. Consider the projective cover $\pi : P(X) \to X$ of X in mod A. Then

$$0 \to \mathrm{Ker}\,\pi \to P(X) \to X \to 0 \qquad (\Omega(X) = \mathrm{Ker}\,\pi)$$

$$\cdots \to \mathrm{Ext}_A^1(X, X) \to \mathrm{Ext}_A^1(P(X), X) \to \mathrm{Ext}_A^1(\Omega(X), X) \to \mathrm{Ext}_A^2(X, X) \to \mathrm{Ext}_A^2(P(X), X) \to \cdots$$

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- Finally, by the Auslander-Reiten formula, we obtain

$$\mathrm{Ext}_A^1(\Omega(X),X) \cong D\overline{\mathrm{Hom}}_A(X,\tau_A\Omega(X)) \cong D\overline{\mathrm{Hom}}_A(X,\tau_A X_2) = 0.$$

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Let A be an algebra with a separating family $\mathscr{C}_A$ of almost cyclic coherent components in Γ_A. Assume that for every indecomposable A-module X we have $q_A(\mathbf{dim}\, X) \in \{0, 1\}$. Then A is tame.

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- We know that A is a triangular algebra, so the Tits form q_A and the Euler form χ_A are well defined.
- Because any quasitilted algebra Λ is of global dimension at most two, applying result of Bongartz, we deduce that q_Λ and χ_Λ coincide.
- Therefore, for every Λ-module Y we have the equality

$$q_\Lambda(\mathbf{dim}\, Y) = \dim_K \operatorname{End}_\Lambda(Y) - \dim_K \operatorname{Ext}^1_\Lambda(Y, Y) + \dim_K \operatorname{Ext}^2_\Lambda(Y, Y).$$

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- Moreover, $\operatorname{End}_B(X) = \operatorname{End}_A(X)$ and $\dim_K \operatorname{Ext}^1_A(X,X) \geq \dim_K \operatorname{Ext}^1_B(X,X)$.

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- Moreover, $\operatorname{End}_B(X) = \operatorname{End}_A(X)$ and $\dim_K \operatorname{Ext}^1_A(X,X) \geq \dim_K \operatorname{Ext}^1_B(X,X)$.
- Finally,

$$q_A(\mathbf{dim}\, X) = \dim_K \operatorname{End}_A(X) - \dim_K \operatorname{Ext}^1_A(X,X) < 0,$$

because by the above Lemma $\operatorname{Ext}^2_A(X,X) = 0$ for any $X \in \operatorname{ind} A$.

Corollary

Let A be a wild algebra with a separating family of almost cyclic coherent components in Γ_A. Then there is an indecomposable A-module X such that $q_A(\mathbf{dim}\, X) < 0$.

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Let A be an algebra with a separating family of almost cyclic coherent components in Γ_A. The following statements are equivalent:

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5. For every $X \in \operatorname{ind} A$ with $m(X) \geq 2$, we have $q_A(\mathbf{dim}\, X) \in \{0, 1\}$.
6. The form q_A is weakly nonnegative and for every $X \in \operatorname{ind} A$ with $q_A(\mathbf{dim}\, X) \geq 2$, we have $m(X) = 1$.