The maximality of weakly separated collections under boundary maps

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Introduction

In 2019, Nick Early introduced a polyhedral subdivision of hypersimplex $\Delta_{k,n}$ by moving blades to its vertices. He proved that this subdivision is positroidial if and only if the vertices correspond to a weakly seperated collection.

Question: How to characterize this positroid subdivision? And how to characterize the flips of this subdivision as the maximal weakly speared collection do cluster mutations?

1. Weakly Separated Collections
2. Nick Early's Construction
3. Main results
4. Sketch of proof

Weakly Separated Collections

Definition (Leclerc, Zelevinsky 1998)

Given $I, J \in \binom{[n]}{k}$. The subsets I, J are **weakly separated** if there does not exist four elements a, b, c, d with $a, c \in I \backslash J$ and $b, d \in J \backslash I$ have

$$a < b < c < d$$

or its cyclic rotations. Subsets $\{J_1, J_2, \cdots, J_m\}$ are pairwise weakly separated, then $W = \{J_1, J_2, \cdots, J_m\}$ is called a **weakly separated collection**.

Weakly Separated Collections

Definition (Mutations)

A **mutation**(flip) of a maximal weakly separated collection is a replacement

$$W \longleftrightarrow (W \backslash \{Lac\} \cup \{Lbd\})$$

where W is a maximal weakly separated collection in $\binom{[n]}{k}$ and $\{Lab, Lbc, Lcd, Lad, Lac\} \subset W$ with $a, b.c.d$ in cyclic order in $[n] \backslash L$.

Weakly Separated Collections

Theorem (OPS2011, DKK2010)

Every maximal weakly separated collection of $\binom{[n]}{k}$ has cardinality $k(n-k)+1$. Any two maximal weakly separated collections are linked by a sequence of mutations

Hypersimplex

Definition (Hypersimplex)

$\Delta_{k,n} := \{x \in [0,1]^n \mid \sum_{i=1}^{n} x_i = k,\ 2 \leq k \leq n-2\}$ is a polytope with vertices $\{e_J := \sum_{j \in J} e_j \mid J \in \binom{[n]}{k}\}$. For any $L \subset [n]$ with $|L| \leq k-2$, there is a natural isomorphism

$$\partial_L(\Delta_{k,n}) := (\bigcap_{i \in L} \{x_i = 1\}) \cap \Delta_{k,n} \cong \Delta_{k-|L|,n-|L|},$$

where ∂_L is called a **boundary map** on hypersimplex $\Delta_{k,n}$

Positroid polytopes

Definition (Postnikov 2006)

For $\mathcal{M} \subset \binom{[n]}{k}$, let

$$S_{\mathcal{M}} := \{V \in Gr_{k,n}^{\geq 0}(\mathbb{R}) \mid p_I(V) > 0 \; if \; and \; only \; if \; I \in \mathcal{M}\}.$$

If $S_{\mathcal{M}} \neq \emptyset$, then $\mathcal{M}$ is called a **positroid** and $S_{\mathcal{M}}$ a **positroid cell**. $P_{\mathcal{M}} := Conv\{e_J \mid J \in \mathcal{M}\}$ is the **positroid polytope** related to $\mathcal{M}$.

Blades

Denote by $[S_1^{(s_1)}, \cdots, S_l^{(s_l)}]$ the polyhedral cone in $\mathcal{H}_{k,n} := \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = k\}$ formulated by the following facet inequalities

$$x_{S_1} \geq s_1$$
$$x_{S_1 \cup S_2} \geq s_1 + s_2$$
$$\vdots$$
$$x_{S_1 \cup \cdots \cup S_{l-1}} \geq s_1 + \cdots s_{l-1}.$$

where $\{S_1, \cdots, S_l\}$ is an ordered set partition of $[n]$ with $0 \leq s_i \leq |S_i| - 1$.

Blades

Cones $[S_1^{(s_1)}, S_2^{(s_2)} \cdots, S_l^{(s_l)}]$, $[S_2^{(s_2)}, S_3^{(s_3)} \cdots, S_1^{(s_1)}] \cdots$ $[S_l^{(s_l)}, S_1^{(s_1)} \cdots, S_{l-1}^{(s_{l-1})}]$ form a complete simplicial fan in $\mathcal{H}_{k,n}$

Definition

The **blade** $\left\langle S_1^{(s_1)}, \cdots, S_l^{(s_l)} \right\rangle$ is the union of codimension 1 faces of this fan, that is

$$\left\langle S_1^{(s_1)}, \cdots, S_l^{(s_l)} \right\rangle = \bigcup_{j=1}^{l} \partial[S_j^{(s_j)}, S_{j+1}^{(s_{j+1})} \cdots, S_{j-1}^{(s_{j-1})}].$$

Besides, if $s_j = 0$ for some $j = 1, \cdots, l$ then the superscript (s_j) will be omitted.

Blades

The **standrad blade** $\beta = \langle 1, 2, \cdots, n \rangle$ is characterized as follows

$$\beta = \langle 1, 2, \cdots, n \rangle = \bigcup_{j=1}^{n} \partial [j, j+1, \cdots, j-1].$$

Let $\beta_J := \langle 1, 2, \cdots, n \rangle_{e_J}$ be the translation of β from origin to the vertex e_J of $\Delta_{k,n}$.

Nick Early's Construction

Theorem (Nick Early 2019)

Given a collection of vertices $\{e_{I_1}, e_{I_2}, \cdots, e_{I_m}\}$ of $\Delta_{k,n}$, the blade arrangements

$$\langle 1, 2, \cdots, n \rangle_{e_{I_1}}, \cdots, \langle 1, 2, \cdots, n \rangle_{e_{I_m}}$$

induce a positroid subdivision of $\Delta_{k,n}$ if and only if I_1, I_2, $\cdots$, I_m are weakly seperated collections.

Boundary map

Proposition (Nick Early 2019)

The subdivision on the boundary $\partial_j(\Delta_{k,n})$ induced by $\langle 1, 2, \cdots, n \rangle_{e_J}$ is characterized as follows

$$\partial_j(\langle 1, 2, \cdots, n \rangle_{e_J} \cap \Delta_{k,n}) = \left\langle 1, 2, \cdots, \widehat{j}, \cdots n \right\rangle_{e_{J'}} \cap \partial_j(\Delta_{k,n})$$

where $J = \{j_1, \cdots, j_k\}$ and $J' = J \backslash \{j_{a+1}\}$ if j satisfies $j_a < j \leq j_{a+1}$ (the indices are cyclic).

Boundary map

Definition

Denote by $\partial_j(J) := J'$ the **boundary map** ∂_j on k-set J. If W is a collection of k-sets, let $\partial_j(W) := \{\partial_j(J) \mid J \in W\}$.

Example: For $k = 5, n = 8$ and $J = \{1, 3, 4, 5, 7\}$, then

$$\partial_2(\langle 1, 2, \cdots, 8 \rangle_{e_J} \cap \Delta_{5,8}) = \langle 1, 3, 4, 5, 6, 7, 8 \rangle_{e_{1457}} \cap \partial_2(\Delta_{5,8})$$

Therefore, $\partial_2(\{1, 3, 4, 5, 7\}) = \{1, 4, 5, 7\}$.

Boundary map

Question(Nick Early2019)

If W is a maximal weakly separated collection in $\binom{[n]}{k}$, is $\partial_j(W)$ a maximal weakly separated collection in $\binom{[n]\backslash\{j\}}{k-1}$?

Main results

Theorem

Let W be a maximal w-collection in $\binom{[n]}{k}$, then $\partial_j(W)$ is also a maximal w-collection in $\binom{[n]\backslash\{j\}}{k-1}$ for any $j \in [n]$

Corollary

The translated blades $\{\beta_J | J \in W\}$ corresponding to a maximal weakly separated collection W induce a finest regular positroid subdivison of $\Delta_{k,n}$.

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Plabic graphs

Theorem(Oh,Postnikov,Speyer,2011)

Let $\mathcal{F}(G)$ denotes the collection of labels of a reduced biparite plabic graph G of type (k, n). Then there is a one-to-one correspondence between maximal weakly separated collections $\{W\}$ to $\{\mathcal{F}(G)\}$.

Let W be a maximal w-collection in $\binom{[n]}{k}$, and $\Sigma_0(W)$ be the biparite reduced plabic graph obtained from the duality of plabic tiling.

Operations on plabic graphs

Definition

Without loss of generality, we take $j = n$. The plabic graph $\partial_n(\Sigma_0(W))$ is obtained from $\Sigma_0(W)$ through the following steps

- Delete the vertex labeled $n - k$ on the boundary and the unique edge adjacent to it, but preserve the internal vertex of this edge.
- Delete all the edge with vertices $\mathcal{B}$ and $\mathcal{W}$ such that $\mathcal{B} \backslash \mathcal{W} = M_2^n(\mathcal{B})$.

Operations on plabic graphs

- Relabel the boundary vertices $n, 1, 2, \cdots, n-k-1$ by $1, 2, \cdots, n-k$ clockwise.
- Delete all the single points and bivalent points, use moves (M2) to get a biparite plabic graph.

Then we can prove the following proposition

Proposition

Let W be a maximal weakly separated collection, then there is an isomorphism $\partial_n(\Sigma_0(W)) \cong \Sigma_0(\partial_n(W))$ and $\partial_n(\Sigma_0(W))$ is a reduced biaparite graph of type $(k-1, n-1)$.

An example

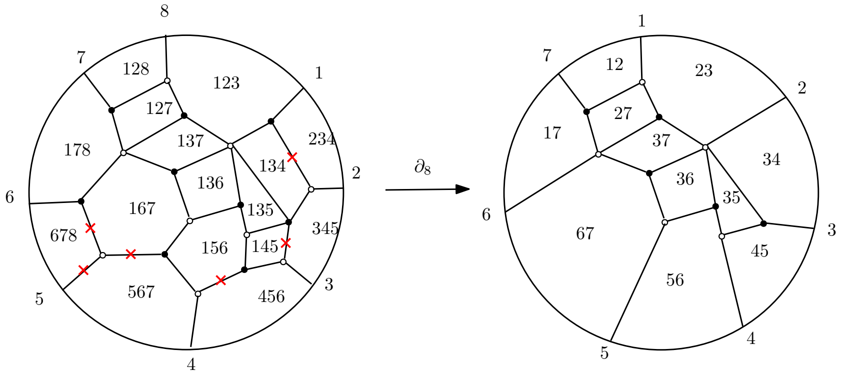


Figure: An example of producing reduced plabic graph $\partial_8(\Sigma_0(W))$

What can we get?

- Every positroid subdivision induced by a maximal weakly separated collection corresponds to a maximal cone of $\mathrm{Trop}^{+}\mathrm{Gr}_{\mathrm{k,n}}$.
- This subdivision is a positroid tiling researched by [Lukowski-Parisi-Williams 2020] where every positroid tile can be parameterized by a tree plabic graph.
- There are exact $\binom{n-2}{k-1}$ tree plabic graphs to parameterize such a subdivision.

Cluster mutations

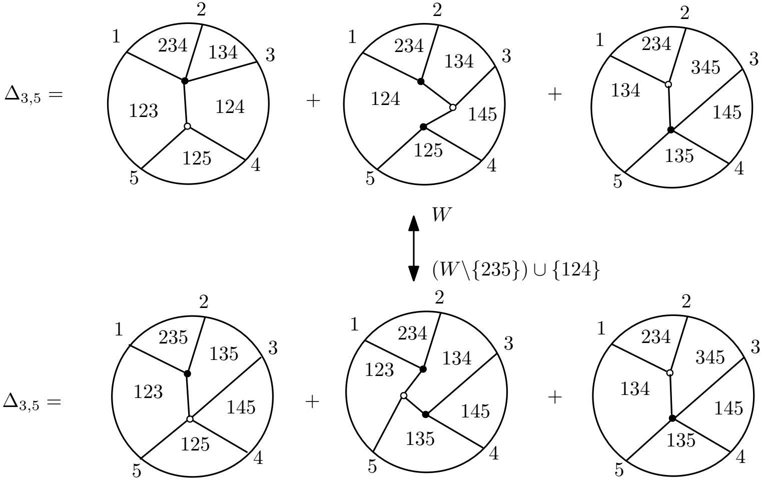
Example

Take $k = 3$, $n = 5$, $W = \{123, 234, 345, 145, 125, 235, 245\}$. Consider the mutation

$$W \longleftrightarrow (W \backslash \{235\}) \cup \{124\}$$

How does the positroid subdivision change under this mutation?

An example



$\Delta_{3,5} =$
1 2 3 4 5
234 134 123 124 125
+
1 2 3 4 5
234 134 124 145 125
+
1 2 3 4 5
234 345 134 145 135
W
$(W \backslash \{235\}) \cup \{124\}$
$\Delta_{3,5} =$
1 2 3 4 5
235 135 123 145 125
+
1 2 3 4 5
234 134 123 145 135
+
1 2 3 4 5
234 345 134 145 135

Thank you

Main References

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