Generalized Quantum Cluster Algebras: The Laurent Phenomenon and Upper Bounds

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Introduction

Cluster algebras were discovered by Fomin and Zelevinsky. This is a family of commutative algebras designed to serve as an algebraic framework for the theory of canonical bases. It has rapidly turned out that the combinatorics of cluster algebras also appear in many other subjects, for example in

- Poisson geometry.
- higher Teichmüller spaces.
- combinatorics.
- commutative and non commutative algebraic geometry and in particular the study of stability conditions, Calabi-Yau algebras, Donaldson-Thomas invariants in geometry.
- in the representation theory of quivers and finite-dimensional algebras.

Fomin-Zelevinsky: Two Types of Dynamics

Exploring the interplay between two types of dynamics—that of cluster variables and that of coefficients—leads to a better understanding of both phenomena. The constructions Fomin-Zelevinsky use to express this interplay are close in spirit to the notion of *cluster ensemble* introduced and studied by Fock-Goncharov as a tool in higher Teichmüller theory. The coefficient-based approach uncovers an unexpected “common source" of the two types of dynamics, expressing both the cluster variables and the coefficients in terms of a new family of *F-polynomials*. This approach also yields a new constructive way to express the (conjectural) “Langlands duality" between the two kinds of dynamics.

Two kinds of Quantization of Cluster Algebras

There are two kinds of formulations of quantum cluster algebras,

1. the one quantizing the *cluster variables* by Berenstein-Zelevinsky
2. the one quantizing the *coefficients* by Fock-Goncharov,

It is known that they are closely related to each other.

Generalized Cluster Algebras of Geometric Type

Chekhov and Shapiro introduced *generalized cluster algebras*, which naturally generalize the ordinary cluster algebras.

Let $m \geq n \in \mathbb{N}$. Suppose that $\widetilde{B} = (b_{ij})_{m \times n} = \begin{bmatrix} B \\ C \end{bmatrix}$ with $b_{ij} \in \mathbb{Z}$ and principal part B: $n \times n$ skew-symmetrizable matrix. For each $k \in \{1, \ldots, n\}$, $\exists$ $d_k \in \mathbb{N}$ such that $d_k | b_{jk}$ for all $1 \leq j \leq n$. Set

$$\mathbf{d} = (d_1, \ldots, d_n)$$

and

$$\beta_{ij} = \begin{cases} \dfrac{b_{ij}}{d_j} & \text{if } 1 \leq j \leq n, \\ \lfloor \dfrac{b_{ij}}{d_j} \rfloor & \text{if } n+1 \leq j \leq m. \end{cases}$$

Generalized Cluster Algebras of Geometric Type

Let $\mathcal{F} := \mathbb{Q}(x_1, \ldots, x_m)$ and $\mathbb{ZP} = \mathbb{Z}[x_{n+1}^{\pm 1}, \cdots, x_m^{\pm 1}]$. For each $1 \leq i \leq n$, the i-th string ρ_i is a collection of monomials $\rho_{i,r} \in \mathbb{Z}[x_{n+1}, \cdots, x_m]$, $0 \leq r \leq d_i$ satisfying that $\rho_{i,0} = \rho_{i,d_i} = 1$.

Definition (Chekhov-Shapiro 2014)

A generalized seed in $\mathcal{F}$ is a triple $(\widetilde{\mathbf{x}}, \rho, \widetilde{B})$, where

1. $\widetilde{\mathbf{x}} = \{x_1, \cdots, x_m\}$–an extended cluster, a transcendence basis of $\mathcal{F}$, the n-tuple $\mathbf{x} = \{x_1, \cdots, x_n\}$– a cluster, whose elements are called cluster variables;
2. B: the exchange matrix, $\widetilde{B}$: the extended exchange matrix;
3. n-tuple of strings $\rho = (\rho_1, \cdots, \rho_n)$: a coefficient tuple.

Generalized Cluster Algebras of Geometric Type

Definition (Seed Mutation)

Let $(\widetilde{\mathbf{x}}, \rho, \widetilde{B})$ be a generalized seed in $\mathcal{F}$. For each $1 \leq k \leq n$, the mutation in direction k is $\mu_k(\widetilde{\mathbf{x}}, \rho, \widetilde{B}) := (\widetilde{\mathbf{x}}', \rho', \widetilde{B}')$, where

1. $\widetilde{B}' = (b'_{ij})$ with
$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|) & \text{otherwise;} \end{cases}$$

2. $\widetilde{\mathbf{x}}' = (\widetilde{\mathbf{x}} \setminus \{x_k\}) \cup x'_k$, with
$$x'_i = \begin{cases} x_i & \text{if } i \neq k, \\ x_k^{-1}\left(\sum_{s=0}^{d_k} \rho_{k,s} \prod_{j=1}^{m} x_j^{\beta_{jk}s+[-b_{jk}]_+}\right) & \text{otherwise;} \end{cases}$$

3. $\rho' = (\rho'_1, \cdots, \rho'_n)$, with
$$\rho'_{i,s} = \begin{cases} \rho_{i,d_i-s} & \text{if } i = k, \\ \rho_{i,s} & \text{otherwise.} \end{cases}$$

Generalized Cluster Algebras of Geometric Type

Definition (Chekhov-Shapiro 2014)

Let $\mathcal{S}$ be a set consisting of all generalized seeds in $\mathcal{F}$ which are mutation-equivalent to the initial seed $(\widetilde{\mathbf{x}}, \rho, \widetilde{B})$. The generalized cluster algebra $\mathcal{A}(\widetilde{\mathbf{x}}, \rho, \widetilde{B})$ is the $\mathbb{Z}[x_{n+1}^{\pm 1}, \ldots, x_m^{\pm 1}]$-subalgebra of $\mathcal{F}$ generated by all cluster variables from all seeds in $\mathcal{S}$.

Theorem

Each cluster variable in $\mathcal{A}(\widetilde{\mathbf{x}}, \rho, \widetilde{B})$ is a Laurent polynomial in the initial cluster variables.

Cluster Alg's and Generalized Cluster Alg's

In generalized cluster algebras, the celebrated *binomial* exchange relation for cluster variables of ordinary cluster algebras

$$x_k' x_k = \prod_{j=1}^{m} x_j^{[-b_{jk}]_+} + \prod_{j=1}^{m} x_j^{[b_{jk}]_+}$$

is replaced by the *polynomial* one of arbitrary degree $d_k \geq 1$,

$$x_k' x_k = \left(\prod_{j=1}^{m} x_j^{[-\beta_{jk}]_+} \right)^{d_k} \sum_{s=0}^{d_k} \rho_{k,s} w_k^s, \quad w_k = \prod_{j=1}^{m} x_j^{\beta_{jk}},$$

where $\beta_{jk} = \lfloor b_{jk}/d_k \rfloor$ are as above.

Upper Bounds and Lower Bounds

Definition

For a generalized seed $(\widetilde{\mathbf{x}}, \rho, \widetilde{B})$, the upper bound is defined by

$$\mathcal{U}(\widetilde{\mathbf{x}}, \rho, \widetilde{B}) := \mathbb{ZP}[x_{i_1}^{\pm 1}, \ldots, x_{i_n}^{\pm 1}] \cap \bigcap_{k=1}^{n} \mathbb{ZP}[x_{i_1}^{\pm 1}, \ldots, x_{i_{k-1}}^{\pm 1}, (x'_{i_k})^{\pm 1}, x_{i_{k+1}}^{\pm 1}, \ldots, x_{i_n}^{\pm 1}]$$

and the lower bound by

$$\mathcal{L}(\widetilde{\mathbf{x}}, \rho, \widetilde{B}) := \mathbb{ZP}[x_{i_1}, x'_{i_1}, \ldots, x_{i_n}, x'_{i_n}].$$

For each $i \in [1, n]$, let

$$P_i := x_i x'_i = \sum_{r=0}^{d_i} \rho_{i,r} \prod_{j=1}^{m} x_j^{r[\beta_{ji}]_+ + (d_i - r)[-\beta_{ji}]_+}.$$

Upper Bounds and Lower Bounds

The seed $(\widetilde{\mathbf{x}}, \rho, \widetilde{B})$ is acyclic if and only if there exists a permutation $\sigma \in S_n$ such that $b_{\sigma(l),\sigma(k)} \geq 0$ for $1 \leq k < l \leq n$.

Theorem (B-Chen-Ding-Xu)

If the generalized seed $(\widetilde{\mathbf{x}}, \rho, \widetilde{B})$ is acyclic, then the standard monomials in $x_1, x_1', \ldots, x_n, x_n'$ are $\mathbb{ZP}$-linearly independent in $\mathcal{L}(\widetilde{\mathbf{x}}, \rho, \widetilde{B})$.

The generalized seed $(\widetilde{\mathbf{x}}, \rho, \widetilde{B})$ is called coprime if P_i and P_j are coprime for any two different $i, j \in [1, n]$.

Theorem (B-Chen-Ding-Xu)

If the generalized seed $(\widetilde{\mathbf{x}}, \rho, \widetilde{B})$ is coprime and acyclic, then

$$\mathcal{L}(\widetilde{\mathbf{x}}, \rho, \widetilde{B}) = \mathcal{U}(\widetilde{\mathbf{x}}, \rho, \widetilde{B}).$$

A Quantum Analogue of Generalized Cluster Alg's

Definition

Let $\widetilde{B} = (b_{ij})$ be an $m \times n$ integer matrix with $m \geq n$ and $\Lambda = (\lambda_{ij})$ be an $m \times m$ skew-symmetric integer matrix. The pair $(\Lambda, \widetilde{B})$ is said to be compatible if we have

$$-\Lambda\widetilde{B} = \begin{bmatrix} D \\ 0 \end{bmatrix}$$

where $D = \mathrm{diag}\{\widetilde{d}_1, \ldots, \widetilde{d}_n\}$ is an $n \times n$ diagonal matrix with positive integers diagonal entries $\widetilde{d}_i$, $1 \leq i \leq n$.

A Quantum Analogue of Generalized Cluster Alg's

Remark

If the pair $(\Lambda, \widetilde{B})$ is compatible, then the matrix $\widetilde{B} = \begin{bmatrix} B \\ C \end{bmatrix}$ has full rank and the product matrix DB is skew-symmetric.

Let $\varepsilon = \{\pm 1\}$. For each $1 \leq k \leq n$ and each sign ε, the matrix $\widetilde{B}' = \mu_k(\widetilde{B}) = E_\varepsilon \widetilde{B} F_\varepsilon$ where

(1) $E_\varepsilon = (e_1, e_2, \ldots, e_{k-1}, e'_k, e_{k+1}, \ldots, e_m)$ is the square matrix of degree m with $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{Z}^m$ for $i \neq k$, and $e'_k = ([-\varepsilon b_{1k}]_+, \ldots, [-\varepsilon b_{(k-1)k}]_+, -1, [-\varepsilon b_{(k+1)k}]_+, \ldots, [-\varepsilon b_{mk}]_+)^T$;

(2) $F_\varepsilon = (f_1, f_2, \ldots, f_{k-1}, f'_k, f_{k+1}, \ldots, f_n)^T$ is the square matrix of degree n with $f_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n$ for $j \neq k$, and $f'_k = ([\varepsilon b_{k1}]_+, \ldots, [\varepsilon b_{k(k-1)}]_+, -1, [\varepsilon b_{k(k+1)}]_+, \ldots, [\varepsilon b_{kn}]_+)$.

A Quantum Analogue of Generalized Cluster Alg's

For a compatible pair $(\Lambda, \widetilde{B})$, we denote

$$\Lambda' := E_\varepsilon^T \Lambda E_\varepsilon.$$

It is easy to see that Λ' is a skew-symmetric matrix. The new pair (Λ', B') is also compatible and Λ' is independent of the choice of the sign ε. We write $(\Lambda', \widetilde{B}') = \mu_k(\Lambda, \widetilde{B})$ and say that $(\Lambda', \widetilde{B}')$ is the mutation of $(\Lambda, \widetilde{B})$ in direction k. It follows that

$$\mu_k(\mu_k(\Lambda, \widetilde{B})) = (\Lambda, \widetilde{B}),$$

i.e., μ_k is an involution.
The skew-symmetric matrix $\Lambda = (\lambda_{ij})$ gives the skew-symmetric bilinear form on the lattice $\mathbb{Z}^m$ through the mapping

$$\Lambda : \mathbb{Z}^m \times \mathbb{Z}^m \longrightarrow \mathbb{Z}$$

which sends (c, d) to $c^T \Lambda d$ for any $c, d \in \mathbb{Z}^m$.

A Quantum Analogue of Generalized Cluster Alg's

Let q be a formal variable and let $\mathbb{Z}[q^{\pm\frac{1}{2}}] \subset \mathbb{Q}(q^{\frac{1}{2}})$.

Definition

The quantum torus $\mathcal{T} = \mathcal{T}(\Lambda)$ is the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra with a distinguished $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis $\{X(c) | c \in \mathbb{Z}^m\}$ and the multiplication is given by

$$X(c)X(d) = q^{\frac{1}{2}\Lambda(c,d)}X(c+d)$$

for any $c, d \in \mathbb{Z}^m$.

$X(c)X(d) = q^{\Lambda(c,d)}X(d)X(c)$, $X(0) = 1$ and $X(-c) = X(c)^{-1}$.
Set $X(e_i) = X_i$ for $1 \leq i \leq m$, then

$$X(c) = q^{\frac{1}{2}\sum\limits_{l<k} c_k c_l \lambda_{kl}} X_1^{c_1} X_2^{c_2} \ldots X_m^{c_m}$$

for each $c = (c_1, \ldots, c_m) \in \mathbb{Z}^m$.

A Quantum Analogue of Generalized Cluster Alg's

Let $\mathcal{F}$ be the skew-field of fractions of the quantum torus $\mathcal{T}$. Denote by $\mathbf{h} = (\mathbf{h}_1, \ldots, \mathbf{h}_n)$ with the i-th string $\mathbf{h}_i = \{h_{i,0}(q^{\frac{1}{2}}), \ldots, h_{i,d_i}(q^{\frac{1}{2}})\}$ for $1 \leq i \leq n$, where $h_{i,0}(q^{\frac{1}{2}}) = h_{i,d_i}(q^{\frac{1}{2}}) = 1$ and $h_{i,r}(q^{\frac{1}{2}})$ are Laurent polynomials in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ satisfying $h_{i,r}(q^{\frac{1}{2}}) = h_{i,d_i-r}(q^{\frac{1}{2}})$.

Definition

Let $\widetilde{B}$, $\mathbf{h}$, Λ be described as above. The quadruple $(X, \mathbf{h}, \Lambda, \widetilde{B})$ is called a quantum seed if $(\Lambda, \widetilde{B})$ is a compatible pair.

Notations

- $d_j \in \mathbb{Z}_{>0}$ and $d_j | b_{ij}$ for $i \in [1, m]$.
- $\beta_{ij} := \dfrac{b_{ij}}{d_j} \in \mathbb{Z}$.
- b^i: the i-th column of $\widetilde{B}$.
- $\beta^i := \frac{1}{d_i} b^i$.

Definition

Let $(X, \mathbf{h}, \Lambda, \widetilde{B})$ be a quantum seed. For any $1 \leq k \leq n$, the new quadruple $\mu_k(X, \mathbf{h}, \Lambda, \widetilde{B}) := (X', \mathbf{h}', \Lambda', \widetilde{B}')$ obtained from $(X, \mathbf{h}, \Lambda, \widetilde{B})$ in direction k is defined by

$$X'(e_i) = \begin{cases} X(e_i), & \text{if } i \neq k, \\ \sum_{r=0}^{d_k} h_{k,r}(q^{\frac{1}{2}}) X(r[\beta^k]_+ + (d_k - r)[-\beta^k]_+ - e_k); & \text{if } i = k \end{cases}$$

and

$$\mathbf{h}' = \mu_k(\mathbf{h}) = \mathbf{h}, \;\; \Lambda' = \mu_k(\Lambda), \;\; \widetilde{B}' = \mu_k(\widetilde{B}).$$

A Quantum Analogue of Generalized Cluster Alg's

Proposition

The quadruple $(X', \mathbf{h}', \Lambda', \widetilde{B}')$ *is a quantum seed.*

Proposition

For each $1 \leq k \leq n$, *the mutation* μ_k *is an involution, i.e.,*

$$\mu_k(\mu_k(X, \mathbf{h}, \Lambda, \widetilde{B})) = (X, \mathbf{h}, \Lambda, \widetilde{B}).$$

Definition

The generalized quantum cluster algebra $\mathcal{A}(X, \mathbf{h}, \Lambda, \widetilde{B})$ associated with the initial seed $(X, \mathbf{h}, \Lambda, \widetilde{B})$, is the $\mathbb{Z}[q^{\pm\frac{1}{2}}][X_{n+1}^{\pm 1}, \ldots, X_m^{\pm 1}]$-subalgebra of $\mathcal{F}$ generated by the cluster variables from the seeds which are mutation-equivalent to $(X, \mathbf{h}, \Lambda, \widetilde{B})$.

A Quantum Analogue of Generalized Cluster Alg's

Remark

We have that

(1) if $d_k = 1$ for all $1 \leq k \leq n$, then the generalized quantum cluster algebra $\mathcal{A}(X, \mathbf{h}, \Lambda, \widetilde{B})$ is exactly the quantum cluster algebra introduced by Berenstein and Zelevinsky;

(2) if $q = 1$, then the generalized quantum cluster algebra $\mathcal{A}(X, \mathbf{h}, \Lambda, \widetilde{B})$ is a class of generalized cluster algebras.

A Quantum Analogue of Generalized Cluster Alg's

Example (Type B_2)

Let $\mathcal{A}(1,2)$ denote the generalized quantum cluster algebra associated with the compatible pair $(\Lambda, \widetilde{B})$, where $\mathbf{d} = (2,1)$,

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B = \widetilde{B} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}.$$

The generalized quantum cluster algebra $\mathcal{A}(1,2)$ is the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathcal{F}$ generated by $\{X_i \mid i \in \mathbb{Z}\}$, where the cluster variables X_i are given by the following exchange relations

$$X_{k-1}X_{k+1} = \begin{cases} 1 + q^{\frac{1}{2}}X_k & \text{if } k \text{ is odd,} \\ 1 + q^{\frac{1}{2}}h(q^{\frac{1}{2}})X_k + qX_k^2 & \text{if } k \text{ is even,} \end{cases}$$

for any $h(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$.

A Quantum Analogue of Generalized Cluster Alg's

Example (Type B_2)

For for $a_1, a_2 \in \mathbb{Z}$, denote by $X(a_1, a_2) := q^{-\frac{a_1 a_2}{2}} X_1^{a_1} X_2^{a_2}$. We can compute all cluster variables as follows:

$$
\begin{aligned}
X_3 &= X(-1,-2) + X(-1,0) + h(q^{\frac{1}{2}})X(-1,1);\\
X_4 &= X(0,-1) + X(-1,1) + h(q^{\frac{1}{2}})X(-1,0) + X(-1,-1);\\
X_5 &= X(1,-2) + (q^{-\frac{1}{2}} + q^{\frac{1}{2}})X(0,-2) + X(-1,-2) + X(-1,0)\\
&\quad + h(q^{\frac{1}{2}})X(-1,-1) + h(q^{\frac{1}{2}})X(0,-1);\\
X_6 &= X(0,-1) + X(1,-1);\\
X_7 &= X_1;\\
X_8 &= X_2.
\end{aligned}
$$

A Quantum Analogue of Generalized Cluster Alg's

Example (Type $A_2^{(2)}$)

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B = \widetilde{B} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}.$$

$$\mathbf{d} = (4, 1)$$

$$X_{k-1}X_{k+1} = \begin{cases} q^{\frac{1}{2}}X_k + 1, & k \text{ odd}, \\ q^2X_k^4 + q^{\frac{3}{2}}h_1X_k^3 + qh_2X_k^2 + q^{\frac{1}{2}}h_1X_k + 1, & k \text{ even}, \end{cases}$$

for $h_1(q^{\frac{1}{2}}), h_2(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$.

Example (Type $A_2^{(2)}$)

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B = \widetilde{B} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}.$$

Let $\mathbf{d} = (2, 1)$,

$$X_{k-1}X_{k+1} = \begin{cases} q^{\frac{1}{2}}X_k + 1, & k \text{ odd}, \\ q^2X_k^4 + qh(q^{\frac{1}{2}})X_k^2 + 1, & k \text{ even}, \end{cases}$$

for $h(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$.

Quantum Laurent phenomenon

Definition

The *generalized quantum upper cluster algebra* $\widetilde{\mathcal{U}}(X, \mathbf{h}, \Lambda, \widetilde{B})$ is defined as follows

$$\widetilde{\mathcal{U}}(X, \mathbf{h}, \Lambda, \widetilde{B}) := \bigcap_{(X', \mathbf{h}', \Lambda', \widetilde{B}') \sim (X, \mathbf{h}, \Lambda, \widetilde{B})} \mathbb{Z}[q^{\pm \frac{1}{2}}][X'(e_1)^{\pm 1}, \ldots, X'(e_m)^{\pm 1}],$$

where $X'(e_k)X'(e_l) = q^{\Lambda'(e_k, e_l)} X'(e_l)X'(e_k)$ for $k, l \in [1, m]$.

Quantum Laurent phenomenon

Definition

The *generalized quantum upper cluster algebra* $\widetilde{\mathcal{U}}(X, \mathbf{h}, \Lambda, \widetilde{B})$ is defined as follows

$$\widetilde{\mathcal{U}}(X, \mathbf{h}, \Lambda, \widetilde{B}) := \bigcap_{(X', \mathbf{h}', \Lambda', \widetilde{B}') \sim (X, \mathbf{h}, \Lambda, \widetilde{B})} \mathbb{Z}[q^{\pm\frac{1}{2}}][X'(e_1)^{\pm 1}, \ldots, X'(e_m)^{\pm 1}],$$

where $X'(e_k)X'(e_l) = q^{\Lambda'(e_k, e_l)} X'(e_l)X'(e_k)$ for $k, l \in [1, m]$.

Theorem (B-Chen-Ding-Xu, Quantum Laurent phenomenon)

Let $(X, \boldsymbol{h}, \Lambda, \widetilde{B})$ be a quantum seed. The generalized quantum cluster algebra $\mathcal{A}(X, \boldsymbol{h}, \Lambda, \widetilde{B})$ is contained in $\widetilde{\mathcal{U}}(X, \boldsymbol{h}, \Lambda, \widetilde{B})$.

Upper Bounds

For each $i \in [1, n]$, let $\mathbb{L}(X, \mathbf{h}, \Lambda, \widetilde{B}) = \mathbb{Z}[q^{\pm\frac{1}{2}}][X_1^{\pm 1}, \ldots, , X_m^{\pm 1}]$. The upper bound is given by

$$\mathcal{U}(X, \mathbf{h}, \Lambda, \widetilde{B}) = \mathbb{L}(X, \mathbf{h}, \Lambda, \widetilde{B}) \cap \bigcap_{i=1}^{n} \mathbb{L}(\mu_i(X, \mathbf{h}, \Lambda, \widetilde{B})).$$

Upper Bounds

For each $i \in [1, n]$, let $\mathbb{L}(X, \mathbf{h}, \Lambda, \widetilde{B}) = \mathbb{Z}[q^{\pm\frac{1}{2}}][X_1^{\pm 1}, \ldots, , X_m^{\pm 1}]$. The upper bound is given by

$$\mathcal{U}(X, \mathbf{h}, \Lambda, \widetilde{B}) = \mathbb{L}(X, \mathbf{h}, \Lambda, \widetilde{B}) \cap \bigcap_{i=1}^{n} \mathbb{L}(\mu_i(X, \mathbf{h}, \Lambda, \widetilde{B})).$$

For $i \neq j$, two polynomials $X_i X_i'$ and $X_j X_j'$ are called coprime if there does not exist any non-invertible element c in the center of $\mathbb{Z}[q^{\pm\frac{1}{2}}][X_1, \ldots, X_n]$ such that both $X_i X_i'$ and $X_j X_j'$ are divided by c. The quantum seed $(X, \mathbf{h}, \Lambda, \widetilde{B})$ is said to be coprime if $X_i X_i'$ and $X_j X_j'$ are coprime for $i, j \in [1, n]$ and $i \neq j$.

Upper Bounds

Let $(X, \mathbf{h}, \Lambda_0, \widetilde{B}_0)$ be the initial quantum seed. For simplicity, define $(X', \mathbf{h}, \Lambda_1, \widetilde{B}_1) = \mu_i(X, \mathbf{h}, \Lambda_0, \widetilde{B}_0)$, where $i \in [1, n]$.

Theorem

If the quantum seeds $(X, \mathbf{h}, \Lambda_0, \widetilde{B}_0)$ and $(X', \mathbf{h}, \Lambda_1, \widetilde{B}_1)$ are coprime, then

$$\mathcal{U}(X, \boldsymbol{h}, \Lambda_0, \widetilde{B}_0) = \mathcal{U}(X', \mathbf{h}, \Lambda_1, \widetilde{B}_1).$$

Moreover, if every quantum seed, which is mutation-equivalent to the initial seed $(X, \mathbf{h}, \Lambda_0, \widetilde{B}_0)$, is coprime, then any upper bound of $\mathcal{A}(X, \mathbf{h}, \Lambda_0, \widetilde{B}_0)$ coincides with the generalized quantum upper cluster algebra $\widetilde{\mathcal{U}}(X, \mathbf{h}, \Lambda_0, \widetilde{B}_0)$.

Upper Bounds

Remark

The above theorem implies that under the “coprimality" condition, the generalized quantum cluster algebras of geometric types have the Laurent phenomenon.

Example: Gen. Quantum Cluster Alg. of Rank 2

Consider the based quantum torus

$$\mathcal{T} = \mathbb{Z}[q^{\pm\frac{1}{2}}][X_1^{\pm 1}, X_2^{\pm 1} | X_1 X_2 = q X_2 X_1].$$

Now denoted by $\mathcal{F}$ the skew-field of fractions of the based quantum torus $\mathcal{T}$.

Let $P_1(x), P_2(x) \in \mathbb{Z}[q^{\pm\frac{1}{2}}][x]$ be the polynomials of arbitrary positive degree d_1 and d_2, respectively. Both $P_1(x)$ and $P_2(x)$ have the form

$$P(x) = 1 + q^{\frac{1}{2}} h_1(q^{\frac{1}{2}}) x + q h_2(q^{\frac{1}{2}}) x^2 + \cdots + q^{\frac{d-1}{2}} h_{d-1}(q^{\frac{1}{2}}) x^{d-1} + q^{\frac{d}{2}} x^d$$

where $h_i(x) \in \mathbb{Z}[x^{\pm 1}]$ satisfies that $h_i(q^{\frac{1}{2}}) = h_{d-i}(q^{\frac{1}{2}})$ for any $1 \leq i \leq d$. Sometimes, we use of the notations $h_0(x) = h_d(x) = 1$.

Example: Gen. Quantum Cluster Alg. of Rank 2

Inductively define $X_k \in \mathcal{F}$ for $k \in \mathbb{Z}$ by the following exchange relations

$$X_{k-1}X_{k+1} = \begin{cases} P_1(X_k) & \text{if } k \text{ is even;} \\ P_2(X_k) & \text{if } k \text{ is odd.} \end{cases}$$

Definition

The generalized quantum cluster algebra $\mathcal{A}_q(P_1, P_2)$ is defined to be the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathcal{F}$ generated by the set of all cluster variables $\{X_k\}_{k\in\mathbb{Z}}$.

Remark

When $q = 1$, $\mathcal{A}_q(P_1, P_2)$ is degenerated to the generalized cluster algebra of rank two studied by Rupel.

Example: Gen. Quantum Cluster Alg. of Rank 2

Lemma

In $\mathcal{A}_q(P_1, P_2)$, for any $k \in \mathbb{Z}$, we have that

$$X_k X_{k+1} = q X_{k+1} X_k.$$

Lemma

In $\mathcal{A}_q(P_1, P_2)$, we have that

(1) *If $k \in \mathbb{Z}$ is even, then*

$$X_{k+1} X_{k-1} = 1 + q^{-\frac{1}{2}} h_1(q^{\frac{1}{2}}) X_k + q^{-1} h_2(q^{\frac{1}{2}}) X_k^2 + \cdots + q^{-\frac{d_1}{2}} h_{d_1}(q^{\frac{1}{2}}) X_k^{d_1};$$

(2) *if $k \in \mathbb{Z}$ is odd, then*

$$X_{k+1} X_{k-1} = 1 + q^{-\frac{1}{2}} h'_1(q^{\frac{1}{2}}) X_k + q^{-1} h'_2(q^{\frac{1}{2}}) X_k^2 + \cdots + q^{-\frac{d_2}{2}} h'_{d_2}(q^{\frac{1}{2}}) X_k^{d_2}.$$

Example: Gen. Quantum Cluster Alg. of Rank 2

Let $X \to \overline{X}$ be the $\mathbb{Z}-$linear bar-involution of the based quantum torus $\mathcal{T}$ satisfying

$$\overline{q^{\frac{r}{2}} X(a_1, a_2)} = q^{-\frac{r}{2}} X(a_1, a_2), \quad r, a_1, a_2 \in \mathbb{Z},$$

where the notation $X(a_1, a_2) := q^{-\frac{a_1 a_2}{2}} X_1^{a_1} X_2^{a_2}$.

Proposition

If the coefficients in $P(x)$ satisfies $\overline{h_i(q^{\frac{1}{2}})} = h_i(q^{\frac{1}{2}})$ for each $1 \leq i \leq d$, then all generalized quantum cluster variables of $\mathcal{A}_q(P_1, P_2)$ are invariant under the bar-involution.

Corollary

The generalized quantum cluster algebra $\mathcal{A}_q(P_1, P_2)$ is invariant under the bar-involution.

Example: Gen. Quantum Cluster Alg. of Rank 2

Definition

A standard monomial in the generalized quantum cluster variables $\{X_0, X_1, X_2, X_3\}$ is an element of the form $X_1^{a_1} X_2^{a_2} X_3^{a'_1} X_0^{a'_2}$, where all exponents are nonnegative integers with $a_1 a'_1 = 0$ and $a_2 a'_2 = 0$.

Proposition

The set of all standard monomials in the generalized quantum cluster variables $\{X_0, X_1, X_2, X_3\}$ *is a* $\mathbb{Z}[q^{\pm\frac{1}{2}}]-$*basis of* $\mathcal{A}_q(P_1, P_2)$.

Remark

The $\mathbb{Z}[q^{\pm\frac{1}{2}}]-$basis consisting of all standard monomials is not invariant under the bar-involution. How to construct various bar-invariant positive $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-bases of $\mathcal{A}_q(P_1, P_2)$ does deserve a further study.

Example: Gen. Quantum Cluster Alg. of Rank 2

Denote by $\mathcal{U}(P_1, P_2) \subset \mathcal{F}$ the $\mathbb{Z}[q^{\pm\frac{1}{2}}]-$subalgebra of $\mathcal{F}$ given by

$$\mathcal{U}(P_1, P_2) = \mathbb{Z}[q^{\pm\frac{1}{2}}][X_1^{\pm1}, X_2^{\pm1}] \cap \mathbb{Z}[q^{\pm\frac{1}{2}}][X_2^{\pm1}, X_3^{\pm1}] \cap \mathbb{Z}[q^{\pm\frac{1}{2}}][X_0^{\pm1}, X_1^{\pm1}].$$

Then we have a stronger version of the quantum Laurent phenomenon.

Theorem (Quantum Laurent phenomenon)

For generalized quantum cluster algebras of rank two, we have that

$$\mathcal{A}_q(P_1, P_2) = \bigcap_{k\in\mathbb{Z}} \mathbb{Z}[q^{\pm\frac{1}{2}}][X_k^{\pm1}, X_{k+1}^{\pm1}] = \mathcal{U}(P_1, P_2).$$

Thanks for listening!