Sheaves of modules on ringed sites

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Layout

1. **Grothendieck topologies and sheaves**

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2. **A torsion theoretic interpretation**

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2. A torsion theoretic interpretation
3. Applications in group representation theory

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- A **sieve** S on $x \in \mathrm{Ob}(\mathcal{C})$ is a subfunctor of $\mathcal{C}(-, x)$.
- Equivalently, S can be viewed as a set of morphisms ending at x satisfying:

$$\forall (f : y \to x) \in S, \forall (g : z \to y) \Rightarrow f \circ g \in S;$$

that is, a **right ideal** of the morphism set.

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- The pair $(\mathcal{C}, J)$ is called a **Grothendieck site**.

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- **Subcategory topology**: Given a full subcategory $\mathcal{D}$ of $\mathcal{C}$, for object y in $\mathcal{C}$, define

$$S_y = \bigsqcup_{x \in \mathrm{Ob}(\mathcal{D})} \mathcal{C}(x, y) \circ \mathcal{C}(-, x)$$

and $J(y) = \{S \subseteq \mathcal{C}(-, y) \mid S \supseteq S_y\}$.

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Theorem (DLL, 2023)

Let $\mathcal{C}$ be a directed category. Then every Grothendieck topology on it is a subcategory topology if and only if $\mathcal{C}$ is an artinian EI category.

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Motivation

The above definitions are hard to check in practice for representation theorists. Want to obtain a **more homological** (rather than categorical) interpretation.

J-torsion theory

- Given an $\mathcal{O}$-module V, $x \in \mathrm{Ob}(\mathcal{C})$, an element $v \in V$ is called ***J*-torsion** if $v \cdot f = 0$, $\forall f \in \mathcal{C}(-, x)$.

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Proposition

The full subcategories of J-torsion $\mathcal{O}$-modules and J-torsion free $\mathcal{O}$-modules form a hereditary torsion pair.

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Moreover, one has

$$\mathrm{Sh}(\mathcal{C}, J, \mathcal{O}) \simeq \mathcal{O}\text{-Mod}/\mathcal{O}\text{-Mod}^{\mathrm{tor}}.$$

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Theorem (Artin)

There is an equivalence $\mathrm{Sh}(\mathcal{O}, J_{at}) \simeq BG$ *where* J_{at} *is the atomic topology and* BG *is the category of discrete* G*-sets.*

- We obtain a bijective correspondence between structure sheaves $\mathcal{A}$ over $(\mathcal{O}, J_{at})$ and commutative rings A on which G acts as automorphisms continuously. Moreover, $\mathrm{Sh}(\mathcal{O}, J_{at}, \mathcal{A}) \simeq A\sharp G\text{-Mod}^{\mathrm{dis}}$.

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Theorem (LPY, 2024)

Every infinite set can be equipped with a homogeneous linear order.

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- $G = \mathbb{Z}$, $\mathcal{O}$ the orbit category of finite cosets, $k = \bar{k}$, then $\mathrm{Sh}(\mathcal{C}, J_{at}, \underline{k}) \simeq \prod_\xi k\text{-Mod}$, the Cartesian product of categories k-Mod indexed by all primitive roots of unit.

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- $\mathcal{C}$: finitely generated $\mathbb{Z}/p^n\mathbb{Z}$-modules and surjective homomorphisms, then $\mathrm{sh}(\mathcal{C},\, J_{at},\, \underline{k}) \simeq \mathcal{C}^{\mathrm{op}}\text{-fdmod}$.

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- $\mathcal{C}$: finite abelian p-groups and conjugacy classes of surjective homomorphisms, then $\mathrm{sh}(\mathcal{C}, J_{at}, \underline{k}) \simeq \mathcal{C}^{\mathrm{op}}\text{-fdmod}$.

Thanks

Any questions?