Projective Modules and Cohomology for Integral Basic Algebras

(Joint work with David J. Benson)

Kay Jin Lim

Nanyang Technological University (NTU)

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Notation

- $\mathcal{O}$ is a local PID with the maximal ideal (π), $\mathbb{K}$ is the field of fractions of $\mathcal{O}$, and $\mathbb{F}$ is the residue field $\mathcal{O}/(\pi)$ of characteristic $p < \infty$.
- For an $\mathcal{O}$-torsion free $\mathcal{O}$-algebra A, let

  $$\hat{A} = \mathbb{K} \otimes_{\mathcal{O}} A, \qquad \bar{A} = \mathbb{F} \otimes_{\mathcal{O}} A, \qquad J_A = A \cap \mathsf{Rad}(\hat{A}),$$

  and, for an $\mathcal{O}$-free A-module M, let

  $$\hat{M} = \mathbb{K} \otimes_{\mathcal{O}} M, \qquad \bar{M} = \mathbb{F} \otimes_{\mathcal{O}} M$$

  be the $\hat{A}$- and $\bar{A}$-modules respectively.
- For simplicity, we may assume all $\mathcal{O}$-modules are finitely generated.

Motivation

Example 1

Let G be a finite group. Choose a suitable p-modular system $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ and suppose that $p \nmid |G|$. The representation theory of $\mathbb{K}G$ and $\mathbb{F}G$ are the same.

Example 2

Let $\mathscr{D}$ be the descent algebra of type $\mathbb{A}_5$ (a subalgebra of $\mathcal{O}\mathfrak{S}_6$). The algebras $\hat{\mathscr{D}}$ and $\bar{\mathscr{D}}$ are basic. The simple $\hat{\mathscr{D}}$-modules (respectively, $\bar{\mathscr{D}}$-modules) are parametrised by the set consisting of (respectively, p-regular) partitions of 6. When $p \geqslant 7$, we have the following radical layers of the PIMs

$$\hat{P}_{(2,2,1,1)} = \begin{bmatrix} \hat{S}_{(2,2,1,1)} \\ \hat{S}_{(3,2,1)} \\ \hat{S}_{(5,1)} \oplus \hat{S}_{4,2} \oplus \hat{S}_{3,3} \\ \hat{S}_{(6)} \end{bmatrix}, \quad \bar{P}_{(2,2,1,1)} = \begin{bmatrix} \bar{S}_{(2,2,1,1)} \\ \bar{S}_{(3,2,1)} \\ \bar{S}_{(5,1)} \oplus \bar{S}_{4,2} \oplus \bar{S}_{3,3} \\ \bar{S}_{(6)} \end{bmatrix}.$$

Integral Basic Algebra

Definition 1

Let $Q = (Q_0, Q_1)$ be a quiver. The quiver algebra $\mathcal{O}Q$ is the $\mathcal{O}$-free algebra with a formal basis consisting of all paths in Q and, for any two paths γ and ξ, the multiplication is defined as

$$\gamma \cdot \xi = \begin{cases} \gamma\xi & \text{if } \mathsf{t}(\xi) = \mathsf{h}(\gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Let J_Q be the (two-sided) ideal of $\mathcal{O}Q$ consisting of paths of length at least one.

The Hypotheses (D $\Rightarrow$ C $\Rightarrow$ B $\Rightarrow$ A)

$$\mathbb{F} = \mathcal{O}/(\pi) \qquad \bar{A} = \mathbb{F} \otimes_{\mathcal{O}} A \qquad J_A = A \cap \mathsf{Rad}(\hat{A}) \qquad \hat{S} = \mathbb{K} \otimes_{\mathcal{O}} S$$

Hypothesis A

If M and S are finitely generated $\mathcal{O}$-free A-modules with $\hat{S}$ simple then $\mathrm{Ext}_A^t(M, S)$ *is $\mathcal{O}$-free for all $t \geqslant 0$.*

Hypothesis B

Suppose that the algebra A has finite rank over $\mathcal{O}$, we have $\mathrm{Rad}(A) = \pi A + J_A$ *and orthogonal idempotent decompositions of the identity in $\bar{A}$ lift to A.*

Hypothesis C

There exist a finite quiver Q, an ideal I of $\mathcal{O}Q$, and $n \geqslant 2$ such that $J_Q^n \subseteq I \subseteq J_Q^2$ and $A \cong \mathcal{O}Q/I$ is $\mathcal{O}$-free of finite rank.
(Equivalently, $\hat{A}$ is basic and A, A/J_A and J_A/J_A^2 are all $\mathcal{O}$-free.)

Hypothesis D

Hypothesis C holds, and for all $n \geqslant 1$, A/J_A^n is $\mathcal{O}$-free.

If A satisfies Hypothesis C, there are $\mathcal{O}$-free of rank one A-modules $S_1, \dots, S_m$ where $m = |Q_0|$ such that, for all $i = 1, \dots, m$, $\hat{S}_i$ and $\bar{S}_i$ are simple $\hat{A}$- and $\bar{A}$-modules respectively. Let $\hat{P}_i$ and $\bar{P}_i$ be their projective covers.

Main Results

Theorem 3 (Benson-L. 2024)

1. *Hypothesis D ⇒ Hypothesis C ⇒ Hypothesis B ⇒ Hypothesis A*
2. *If A satisfies Hypothesis A, then, for any finitely generated $\mathcal{O}$-free A-modules M, S with $\hat{S}$ simple and $t \geqslant 0$, we have* $\mathsf{Ext}_A^t(M, S)$ *is $\mathcal{O}$-free and*

$$\mathbb{F} \otimes_{\mathcal{O}} \mathsf{Ext}_A^t(M, S) \cong \mathsf{Ext}_{\bar{A}}^t(\bar{M}, \bar{S}),$$
$$\mathbb{K} \otimes_{\mathcal{O}} \mathsf{Ext}_A^t(M, S) \cong \mathsf{Ext}_{\hat{A}}^t(\hat{M}, \hat{S}).$$

   In particular, $\dim_{\mathbb{F}} \mathsf{Ext}_{\bar{A}}^t(\bar{M}, \bar{S}) = \dim_{\mathbb{K}} \mathsf{Ext}_{\hat{A}}^t(\hat{M}, \hat{S})$.

3. *If A satisfies Hypothesis D, then the radical layer multiplicities of $\hat{P}_i$ and $\bar{P}_i$ are equal, i.e., for all $1 \leqslant i, j \leqslant m$ and $t \geqslant 0$,*

$$(\mathsf{Rad}^t(\hat{P}_i)/\mathsf{Rad}^{t+1}(\hat{P}_i) : \hat{S}_j) = (\mathsf{Rad}^t(\bar{P}_i)/\mathsf{Rad}^{t+1}(\bar{P}_i) : \bar{S}_j).$$

Examples

Example 4

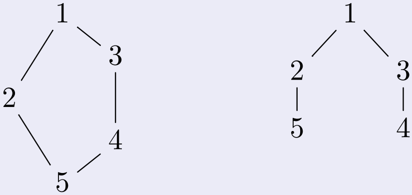
Let Q be the quiver

$$\begin{tikzcd}
 & & 2 \arrow[drr, "b"] & & \\
1 \arrow[urr, "a"] \arrow[dr, "c"'] & & & & 5 \\
 & 3 \arrow[rr, "d"'] & & 4 \arrow[ur, "e"'] &
\end{tikzcd}$$

and $A = \mathcal{O}Q/I$ with $I = (edc - \pi ba)$. The algebra A satisfies Hypothesis C, but not Hypothesis D. Let $J_A = J_Q/I$. As an $\mathcal{O}$-module, $A/J_A^3 \cong \mathbb{F} \oplus 12 \cdot \mathcal{O}$ is not $\mathcal{O}$-free.

example cont'd

The projective covers of the module 1 over $\hat{A}$ and $\bar{A}$ are given by the following diagrams respectively:



In particular, their radical lengths are different.

Example 5 (Descent Algebra)

Let (W, S) be a finite Coxeter system. For any subset $J \subseteq S$, let W_J be the parabolic subgroup of W generated by J and X_J be the distinguished left coset representatives consisting of minimal length elements for W/W_J. Define

$$x_J := \sum_{w \in X_J} w \in \mathcal{O}W.$$

For another subset $K \subseteq S$, let X_{JK} be the distinguished double coset representatives of (W_J, W_K) in W. L. Solomon showed that

$$x_J x_K = \sum_{L \subseteq S} a_{JK}^L x_L$$

where a_{JK}^L is the number of elements $w \in X_{JK}$ such that $w^{-1}Jw \cap K = L$. The descent algebra $\mathscr{D}$ is the $\mathcal{O}$-algebra spanned by $\{x_J : J \subseteq S\}$.

example cont'd

The algebra $\mathscr{D}$ satisfies Hypothesis A when $p \nmid |W|$, and Hypothesis D when p is large. In particular, the Ext quivers of $\hat{\mathscr{D}}$ and $\bar{\mathscr{D}}$ are identical when $p \nmid |W|$.

Their Ext quivers have previously been computed by Schocker (for $\hat{\mathscr{D}}$ of type $\mathbb{A}$) and Saliola (for both $\hat{\mathscr{D}}$ and $\bar{\mathscr{D}}$ of both types $\mathbb{A}$ and $\mathbb{B}$, and $p \nmid |W|$).

For example, the Ext quiver of the descent algebra of type $\mathbb{A}_4$ is

$$\begin{tikzcd}
p \geqslant 7 : 21^3 \arrow[r] & 31^2 \arrow[r] & 41 \arrow[r] & 5 & 32 \arrow[l] & 2^2 1 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
p = 5 : 21^3 \arrow[r] & 31^2 \arrow[r] & 41 \arrow[r] & 5 \arrow[loop above] & 32 \arrow[l] & 2^2 1 \arrow[l]
\end{tikzcd}$$

In general, the Ext quivers of $\hat{\mathscr{D}}$ of type $\mathbb{A}$ are directed. But $\hat{\mathscr{D}}$ is *not gentle*.

Example 6

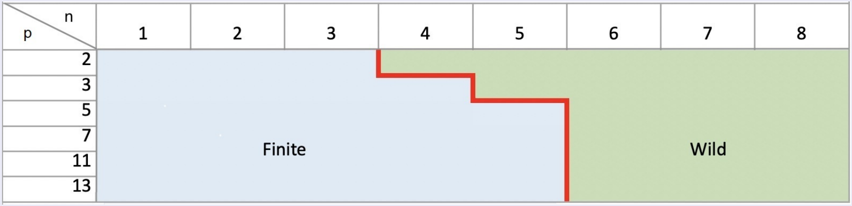
Let p be an arbitrary prime.

1. The nil-Coxeter algebra of a finite Coxeter group satisfies Hypothesis D.
2. The face algebra of hyperplane arrangements in a real space satisfies Hypothesis C.
3. The 0-Hecke algebra of a finite Coxeter group satisfies Hypothesis B.

Theorem 7

Let $\mathscr{D}_n$ *be the descent algebra of type* $\mathbb{A}_{n-1}$.

1. [Schocker 2004] *When* $p = \infty$, $\mathscr{D}_n$ *has finite type if* $n \leqslant 5$, *and wild type otherwise.*
2. [Erdmann-L. 2024] *Let* $p < \infty$. *The representation type* $\mathscr{D}_n$ *is depicted as follows:*



n p	1	2	3	4	5	6	7	8
2								
3								
5								
7								
11								
13								

Theorem 8 (Erdmann-L. 202X)

The representation type of the descent algebra of type $\mathbb{B}_n$ *is depicted as follows:*

n p	2	3	4	5	6	7	8
2	Tame	Wild					
3	Finite						
5							
7							
11							
13							

Problem(s)

1. Suppose that A satisfies Hypothesis D. Are the representation type of $\hat{A}$ and $\bar{A}$ the same, that is, $\hat{A}$ has finite type (respectively, tame or wild) if and only if $\bar{A}$ has finite type (respectively, tame or wild)?
2. $\vdots$

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