A method to construct model structures

Joint work with Zhenxing Di and Liping Li

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1. Model structures
2. A correspondence on model structures
3. Some examples

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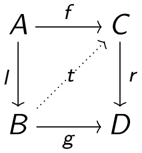
Setup

$\mathcal{E}$: *bicomplete category* — $\mathcal{A}$: *bicomplete abelian category*

- $l : A \to B$, $r : C \to D$ morphisms in E. l has the left lifting property with respect to r (or r has the right lifting property with respect to l), if for every pair of morphisms $f : A \to C$ and $g : B \to D$ such that $rf = gl$, $\exists\ t : B \to C$ s.t. $f = tl$ and $g = rt$, i.e., the next diagram commutes:



Model structures

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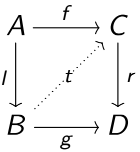

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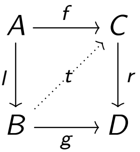

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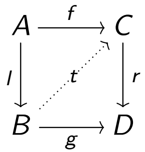

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- A pair (C, F) of classes of objects in A is called a complete cotorsion pair if $\mathsf{C}^{\perp} = \mathsf{F}$ and ${}^{\perp}\mathsf{F} = \mathsf{C}$, and for each object M in A, $\exists$ s.e.s. $0 \to M \to F \to C \to 0$ and $0 \to F' \to C' \to M \to 0$ with $C, C' \in \mathsf{C}$ and $F, F' \in \mathsf{F}$.

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$$\mathsf{C}^{\perp} = \{N \in \mathsf{A} \mid \operatorname{Ext}^1_{\mathsf{A}}(C, N) = 0 \text{ for all } C \in \mathsf{C}\},$$

$${}^{\perp}\mathsf{F} = \{M \in \mathsf{A} \mid \operatorname{Ext}^1_{\mathsf{A}}(M, F) = 0 \text{ for all } F \in \mathsf{F}\}.$$

Model structures

Theorem (Hovey, 2002)

(C, F)*: a pair of classes of objects in* A*. Then* (C, F) *is a complete cotorsion pair if and only if the pair of classes of morphisms* $(\mathrm{Mon}(\mathsf{C}), \mathrm{Epi}(\mathsf{F}))$ *is a weak factorization system in* A*.*

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$$\begin{array}{ccccc} A & \xrightarrow{i} & C & \xrightarrow{p} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & {\scriptstyle f}\downarrow \\ B & \xrightarrow{i'} & D & \xrightarrow{p'} & B \end{array}$$

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Definition (Quillen, 1967)

A model structure on a category E *is a triple of classes of morphisms* $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *satisfying:*

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- Morphisms in the classes $\mathcal{C}$, $\mathcal{W}$ and $\mathcal{F}$ are called cofibrations, weak equivalences and fibrations, respectively.
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- A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{A}$ is called abelian if $\mathcal{C}$ is the class of all monomorphisms with cofibrant cokernels and $\mathcal{F}$ is the class of all epimorphisms with fibrant kernels.

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- A model category is said to be abelian if its underlying category is abelian and the model structure is abelian.

Model structures

Theorem (Hovey, 2002)

An abelian model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *on* A *corresponds bijectively to a Hovey triple* $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ *of classes of objects.*

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- $(\mathsf{C}, \mathsf{W} \cap \mathsf{F})$ *and* $(\mathsf{C} \cap \mathsf{W}, \mathsf{F})$ *are complete cotorsion pairs.*
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- Given a Hovey triple $(\mathsf{C}, \mathsf{W}, \mathsf{F})$, $\exists$ an abelian model structure $(\mathrm{Mon}(\mathsf{C}), \mathcal{W}, \mathrm{Epi}(\mathsf{F}))$, where $\mathcal{W} = \{w \mid w = fc \text{ with } c \in \mathrm{Mon}(\mathsf{C} \cap \mathsf{W}), \mathsf{f} \in \mathrm{Epi}(\mathsf{W} \cap \mathsf{F})\}$.

Model structures

Theorem (Gillespie, 2015)

Suppose that $(\mathsf{C}, \widetilde{\mathsf{F}})$ *and* $(\widetilde{\mathsf{C}}, \mathsf{F})$ *are two complete hereditary cotorsion pairs in* A *such that they are compatible, i.e.,* $\widetilde{\mathsf{C}} \subseteq \mathsf{C}$ *(or* $\widetilde{\mathsf{F}} \subseteq \mathsf{F}$*) and* $\mathsf{C} \cap \widetilde{\mathsf{F}} = \widetilde{\mathsf{C}} \cap \mathsf{F}$*. Then* $\exists$ *an unique thick subcategory* W *for which* $(\mathsf{C}, \mathsf{W}, \mathsf{F})$ *forms a Hovey triple. Moreover, this thick subcategory can be described as follows:*

$$\begin{aligned} \mathsf{W} &= \{M \mid \exists \text{ s.e.s } M \rightarrowtail A \twoheadrightarrow B \text{ with } A \in \widetilde{\mathsf{F}} \text{ and } B \in \widetilde{\mathsf{C}}\ \} \\ &= \{M \mid \exists \text{ s.e.s } A' \rightarrowtail B' \twoheadrightarrow M \text{ with } A' \in \widetilde{\mathsf{F}} \text{ and } B' \in \widetilde{\mathsf{C}}\ \}. \end{aligned}$$

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A model structure on a category E *is a triple of classes of morphisms* $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *satisfying:*

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- $\mathcal{W}$ *is closed under retracts and satisfies the (2-3) property*

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Definition (Joyal, 2008)

A class $\mathcal{C}$ of morphisms in E is said to satisfy left cancellation property if $\beta\alpha \in \mathcal{C}$ and $\beta \in \mathcal{C} \Rightarrow \alpha \in \mathcal{C}$. A class $\mathcal{F}$ of morphisms in E is said to satisfy right cancellation property if $\beta\alpha \in \mathcal{F}$ and $\alpha \in \mathcal{F}$ $\Rightarrow \beta \in \mathcal{F}$.

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Theorem (Di-Li-Liang, 2024)

(C, F) *is a hereditary and complete cotorsion pair if and only if* $(\mathrm{Mon}(\mathsf{C}), \mathrm{Epi}(\mathsf{F}))$ *is a weak factorization system such that* $\mathrm{Mon}(\mathsf{C})$ *satisfies left cancellation property and* $\mathrm{Epi}(\mathsf{F})$ *satisfies right cancellation property.*

- Two cotorsion pairs $(\mathsf{C}, \widetilde{\mathsf{F}})$ and $(\widetilde{\mathsf{C}}, \mathsf{F})$ in A are called compatible if the following conditions holds:

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Theorem (Di-Li-Liang, 2024)

$(\mathsf{C}, \widetilde{\mathsf{F}})$ and $(\widetilde{\mathsf{C}}, \mathsf{F})$ are two compatible and complete cotorsion pairs if and only if $(\mathrm{Mon}(\mathsf{C}), \mathrm{Epi}(\widetilde{\mathsf{F}}))$ and $(\mathrm{Mon}(\widetilde{\mathsf{C}}), \mathrm{Epi}(\mathsf{F}))$ are two compatible weak factorization systems.

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Theorem (Gillespie, 2015)

$(\mathsf{C}, \widetilde{\mathsf{F}})$ and $(\widetilde{\mathsf{C}}, \mathsf{F})$ are two compatible hereditary complete cotorsion pairs in A. Then $\exists$ an unique $\mathsf{W}_{\widetilde{\mathsf{C}},\widetilde{\mathsf{F}}}$ for which $(\mathsf{C}, \mathsf{W}_{\widetilde{\mathsf{C}},\widetilde{\mathsf{F}}}, \mathsf{F})$ forms a Hovey triple. Moreover,

$$\begin{aligned}\mathsf{W}_{\widetilde{\mathsf{C}},\widetilde{\mathsf{F}}} &= \{M \mid \exists \text{ s.e.s } M \rightarrowtail A \twoheadrightarrow B \text{ with } A \in \widetilde{\mathsf{F}} \text{ and } B \in \widetilde{\mathsf{C}}\ \} \\ &= \{M \mid \exists \text{ s.e.s } A' \rightarrowtail B' \twoheadrightarrow M \text{ with } A' \in \widetilde{\mathsf{F}} \text{ and } B' \in \widetilde{\mathsf{C}}\ \}.\end{aligned}$$

A correspondence on model structures

Theorem (Gillespie, 2015)

$(\mathsf{C}, \widetilde{\mathsf{F}})$ *and* $(\widetilde{\mathsf{C}}, \mathsf{F})$ *are two compatible hereditary complete cotorsion pairs in* A*. Then* $\exists$ *an unique* $\mathsf{W}_{\widetilde{\mathsf{C}},\widetilde{\mathsf{F}}}$ *for which* $(\mathsf{C}, \mathsf{W}_{\widetilde{\mathsf{C}},\widetilde{\mathsf{F}}}, \mathsf{F})$ *forms a Hovey triple. Moreover,*

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Theorem (Di-Li-Liang, 2024)

$(\mathcal{C}, \widetilde{\mathcal{F}})$ *and* $(\widetilde{\mathcal{C}}, \mathcal{F})$ *are two compatible weak factorization systems on* E *such that* $(\widetilde{\mathcal{C}}, \mathcal{F})$ *satisfies the Frobenius property, and* $\mathcal{C}$ *and* $\widetilde{\mathcal{C}}$ *satisfy the left cancellation property. Then* $\exists$ *an unique* $\mathcal{W}_{\widetilde{\mathcal{C}},\widetilde{\mathcal{F}}}$ *for which* $(\mathcal{C}, \mathcal{W}_{\widetilde{\mathcal{C}},\widetilde{\mathcal{F}}}, \mathcal{F})$ *forms a model structure on* E*. Moreover,*

$$\mathcal{W}_{\widetilde{\mathcal{C}},\widetilde{\mathcal{F}}} = \{\alpha \mid \alpha = fc \text{ with } c \in \widetilde{\mathcal{C}} \text{ and } f \in \widetilde{\mathcal{F}}\ \}$$

A correspondence on model structures

Definition (Gambino and Sattler, 2017)

A weak factorization system $(\mathcal{C}, \mathcal{F})$ is said to satisfy the Frobenius property if the morphisms in $\mathcal{C}$ are preserved under pullback along the morphisms in $\mathcal{F}$.

A correspondence on model structures

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Theorem (Di-Li-Liang, 2024)

Let (C, F) *be a complete cotorsion pair in* A. *Then* $(\mathrm{Mon}(\mathsf{C}), \mathrm{Epi}(\mathsf{F}))$ *is a weak factorization system satisfying the Frobenius property.*

Some examples

Theorem (Di-Li-Liang, 2024)

$(\mathcal{C}, \widetilde{\mathcal{F}})$ and $(\widetilde{\mathcal{C}}, \mathcal{F})$ are two compatible weak factorization systems on E such that $(\widetilde{\mathcal{C}}, \mathcal{F})$ satisfies the Frobenius property, and $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ satisfy the left cancellation property. Then $\exists$ an unique $\mathcal{W}_{\widetilde{\mathcal{C}},\widetilde{\mathcal{F}}}$ for which $(\mathcal{C}, \mathcal{W}_{\widetilde{\mathcal{C}},\widetilde{\mathcal{F}}}, \mathcal{F})$ forms a model structure on E.

Some examples

Theorem (Di-Li-Liang, 2024)

$(\mathcal{C}, \widetilde{\mathcal{F}})$ and $(\widetilde{\mathcal{C}}, \mathcal{F})$ are two compatible weak factorization systems on E such that $(\widetilde{\mathcal{C}}, \mathcal{F})$ satisfies the Frobenius property, and $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ satisfy the left cancellation property. Then $\exists$ an unique $\mathcal{W}_{\widetilde{\mathcal{C}}, \widetilde{\mathcal{F}}}$ for which $(\mathcal{C}, \mathcal{W}_{\widetilde{\mathcal{C}}, \widetilde{\mathcal{F}}}, \mathcal{F})$ forms a model structure on E.

Example

Let $(\mathsf{C}, \widetilde{\mathsf{F}})$ and $(\widetilde{\mathsf{C}}, \mathsf{F})$ be compatible complete hereditary cotorsion pairs in an abelian category A. Then

$$(\mathrm{Mon}(\mathsf{C}), \mathrm{Epi}(\widetilde{\mathsf{F}})) \text{ and } (\mathrm{Mon}(\widetilde{\mathsf{C}}), \mathrm{Epi}(\mathsf{F}))$$

are compatible weak factorization systems such that $(\mathrm{Mon}(\widetilde{\mathsf{C}}), \mathrm{Epi}(\mathsf{F}))$ satisfies the Frobenius property and $\mathrm{Mon}(\mathsf{C})$ and $\mathrm{Mon}(\widetilde{\mathsf{C}})$ satisfy the left cancellation property.

Some examples

Example

Consider the category $\mathrm{Ch}_{\geqslant 0}(R)$, which is a bicomplete abelian category. Let

- $\mathcal{C}$: the monomorphisms $f : X \to Y$ in $\mathrm{Ch}_{\geqslant 0}(R)$ such that each cokernel of $f_k : X_k \to Y_k$ is a projective R-module for $k \geq 0$;
- $\mathcal{W}$: the quasi-isomorphisms in $\mathrm{Ch}_{\geqslant 0}(R)$;
- $\mathcal{F}$: the morphisms $f : X \to Y$ in $\mathrm{Ch}_{\geqslant 0}(R)$ such that all $f_k : X_k \to Y_k$ are epimorphisms for $k > 0$.

Then $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are compatible weak factorization systems such that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ satisfies the Frobenius property and $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$ satisfy the left cancellation property.

Some examples

Example

Let sSet denote the category of simplicial sets, defined as usual to be the category of presheaves over the simplex category $\triangle$. Let

- $\mathcal{C}$: monomorphisms, i.e., morphisms $f : X \to Y$ in sSet such that $f_k : X_k \to Y_k$ is an injection of sets for each $k \in \mathbb{N}$;
- $\mathcal{W}$: morphisms $f : X \to Y$ in sSet whose geometric realization $|f|$ is a weak homotopy equivalence of topological spaces;
- $\mathcal{F}$: the Kan fibrations, i.e., morphisms $f : X \to Y$ in sSet that have the right lifting property with respect to all horn inclusions.

Then $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are compatible weak factorization systems such that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ satisfies the Frobenius property and $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$ satisfy the left cancellation property.

A method to construct model structures

Joint work with Zhenxing Di and Liping Li

Li Liang

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Zhenxing Di, Liping Li and Li Liang, Compatible weak factorization systems and model structures, arXiv: 2405.00312.