On Utumi rings

(joint work with Cosmin Roman, Nguyen Khanh Tung, and Xiaoxiang Zhang)

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* $Q(R)$ is a maximal right ring of quotients of a ring R.

* $E(R)$ is an injective hull of a ring R.

* $\mathbf{r}_R(I) = \{s \in R \mid Is = 0\}$ for $I \leq R$.

* $\mathbf{l}_R(J) = \{t \in R \mid tJ = 0\}$ for $J \leq R$.

* For $x \in R$, $x^{-1}K = \{r \in R \mid xr \in K\} \leq R_R$ for $K \leq R$.

* I is a right essential ideal of R: $I \leq^{\text{ess}} R_R$,
$\forall 0 \neq x \in R$, $\exists r \in R$ such that $0 \neq xr \in I$.

* J is a right dense ideal of R: $J \leq^{\text{den}} R$,
$\forall x, 0 \neq y \in R$, $\exists r \in R$ such that $xr \in J$ and $0 \neq yr$.

* Note that $R \leq^{\text{den}} Q(R) \leq^{\text{ess}} E(R)$.

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Let R be a commutative ring and let S be the set of elements which are not zero divisors in R; then S is a multiplicatively closed set.

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$as^{-1}bt^{-1} \overset{?}{\in} RS^{-1}$

Definition

A set S is called right permutable (or right Ore)
if $aS \cap sR \neq \emptyset$ for any $a \in R$ and $s \in S$.
A set S is called right reversible
if $s'a = 0$ for some $s' \in S$ implies that $as = 0$ for some $s \in S$.

A multiplicative set $S \subseteq R$ is called a right denominator set
if S is both right permutable and right reversible.

Definition

Let S be the multiplicative set of all regular elements.
We say that R is a right Ore ring iff S is right permutable iff RS^{-1} exists.

In this case, we speak of RS^{-1} as the (total) classical right ring of quotients of R, and denote it by $Q^r_{c\ell}(R)$.
Similarly, $Q^\ell_{c\ell}(R) = S^{-1}R$ is called the (total) classical left ring of quotients of R

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As the injective hull of a module M is a minimal injective module including M, the next result shows that the rational hull of a module M is a minimal rationally complete module including M.

Theorem (Lee)

The following are equivalent for right R-modules M, F and $M \subseteq F$:

- *F is maximal dense over M;*
- *F is rationally complete, and is dense over M;*
- *F is minimal rationally complete, and is essential over M.*

Note that a right R-module F is exactly the rational hull of a module M if F satisfies any one of the above equivalent conditions.
It is denoted by $\widetilde{E}(M)$.

Thus, $\widetilde{E}(R) = \{x \in E(R) \mid \vartheta(R) = 0 \text{ with } \vartheta \in \mathrm{End}_R(E(R)) \Rightarrow \vartheta(x) = 0\}$.
We say $\widetilde{E}(R) = Q(R)$ the maximal right ring of quotients of a ring R.

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Several characterizations for the maximal right ring of quotients of a ring are provided.

Theorem (Lee)

Let R be a ring and $H = End_R(E(R))$.
Then the following statements hold true:

(i) $Q(R) = \mathbf{r}_{E(R)}(\mathbf{l}_H(R))$.

(ii) *([4, Exercises 5])*
$Q(R) = \{x \in E(R) \mid \vartheta|_R = 1_R \text{ with } \vartheta \in H \Rightarrow \vartheta(x) = x\}$.

(iii) *([3, Proposition 8.16])*
$Q(R) = \{x \in E(R) \mid \forall y \in E(R) \setminus \{0\},\ y \cdot x^{-1}R \neq 0\}$.

(iv) $Q(R) = \{x \in E(R) \mid x^{-1}R \leq^{den} R\}$.

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Example

Consider $R = \{(a_n) \in \prod_{n=1}^{\infty} \mathbb{Z} \mid a_n \text{ is eventually constant } \}$.
(i) Total right ring of quotients: $\{(a_n) \in \prod_{n=1}^{\infty} \mathbb{Q} \mid a_n \text{ is even. constant}\}$.
(ii) Classical right ring of quotients: $\{(a_n) \in \prod_{n=1}^{\infty} \mathbb{Q} \mid a_n \text{ is even. constant}\}$.
(iii) Maximal right ring of quotients: $\prod_{n=1}^{\infty} \mathbb{Q}$.

Definition

A ring R is called right nonsingular if $\mathbf{r}_R(t) \leq^{\text{ess}} R_R \Longrightarrow t = 0$.
Note that $0 \in Rt \subseteq \mathbf{l}_R(\mathbf{r}_R(t))$ but $\mathbf{l}_R(\mathbf{r}_R(t)) = 0$ if R is right nonsingular.
A ring R is called right cononsingular
if for any right ideal I, $\mathbf{l}_R(I) = 0 \Longrightarrow I \leq^{\text{ess}} R_R$.
The left version of (co)nonsingularity is defined similarly.

Definition

A ring R is called right Utumi if R is a right nonsingular, right cononsingular ring. The notion of the left-sided is defined similarly.
A ring R is called Utumi if R is right and left Utumi.

Example

(i) The product of fields is an Utumi ring.
(ii) Every finite matrix ring over a right and left self-injective regular ring is an Utumi ring.
(iii) Every right and left Ore domain is an Utumi ring.
(iv) Any semiprime PI-ring is an Utumi ring.

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Theorem (1951, Johnson)

Let $Q(R)$ be a maximal right ring of quotients of a ring R. Then the following conditions are equivalent:

(a) *A ring R is right nonsingular;*

(b) *$Q(R)$ is a (von Neumann) regular ring.*

Theorem (1980, Chatters-Khuri)

Let R be a ring. Then the following conditions are equivalent:

(a) *R is a right nonsingular, right extending ring;*

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It shows that for even a right and left nonsingular ring R,
the maximal right and left rings of quotients of R are different.

Example ([3, Example 7.14c] and [3, Example 13.26(4)])

Let $R = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$ be a ring with a field F.
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Consider $R = \mathsf{CFM}_{\mathbb{N}}(F)$, the column finite matrix ring over a field F.
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In 1963, Utumi established the condition of rings of which maximal any 1-sided rings of quotients of rings are two-sided.

Theorem (1963, Utumi)

Let R be a nonsingular ring.
Then $Q(R) = Q^{\ell}(R)$ if and only if R is a cononsingular ring.

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Every right Utumi ring is left nonsingular.

Using the above proposition, we extend the previous theorem.

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Let R be any one-sided nonsingular ring.
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The next example illustrates the above corollary.

Example ([3, Example 7.6(5)])

Let the ring $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{bmatrix}$. Then R is left nonsingular but not right nonsingular with a right singular ideal $\begin{bmatrix} \bar{0} & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$. Then

$$Q^r_{c\ell}(R) = Q^{\ell}_{c\ell}(R) = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_{(2)} \end{bmatrix}$$

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By the calculation, $Q(R) \cong Q^r_{c\ell}(R) = Q^{\ell}_{c\ell}(R)$.
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In contrast to the previous proposition,
a right Utumi ring does not imply left cononsingular, in general,
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Consider $R = \mathrm{CFM}_{\mathbb{N}}(\mathrm{F})$, the column finite matrix ring over a field F.
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A ring R is called directly finite if $ab = 1$ implies that $ba = 1$ for $a, b \in R$.

Theorem (Lam, Theorem 6.48)

Every right and left extending ring is directly finite.

Example

Let $R = \mathrm{CFM}_{\mathbb{N}}(F)$.
Let e_{ij} be a matrix unit, that is, (i, j)-entry is 1, otherwise 0.

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Then $AB = I$ but $BA = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots \end{bmatrix} \neq I$.

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We extend the previous theorem to the next:

Theorem

Every left cononsingular, right extending ring is directly finite.

The standard observation of Shepherdson for a directly infinite ring.

Example

Let R be not directly finite.
Then there exist $a, b \in R$ such that $ab = 1 \neq ba$.
Take $e_{ij} = b^i(1 - ba)a^j \quad (i, j \geq 1)$,
nonzero elements of R satisfying the matrix units' equations:
$e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$.

Actually, note that $R = \mathrm{CFM}_{\mathbb{N}}(F)$ is not directly finite.
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From the previous theorem and proposition, we have the natural question:

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Corollary (Goodearl, Corollary 13.23)

Every right and left continuous regular ring is a unit-regular ring.

The converse of the previous corollary does not hold true, in general.

Example

Let $R = \{(a_n) \in \prod_{n=1}^{\infty} \mathbb{Z}_2 \mid a_n \text{ is eventually constant}\}$.
Then R is a unit-regular ring but not a Baer ring because there is no idempotent $e \in R$ such that $\mathbf{r}_R(\alpha) = eR$ where $\alpha = (a_n)$ with $a_n = 1$ if $n = 2k$, otherwise $a_n = 0$.

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Then R is a unit-regular ring but not a Baer ring because there is no idempotent $e \in R$ such that $\mathbf{r}_R(\alpha) = eR$ where $\alpha = (a_n)$ with $a_n = 1$ if $n = 2k$, otherwise $a_n = 0$.

Corollary

The maximal right ring of quotients of an Utumi ring is a right and left self-injective unit-regular ring.

The converse of the above corollary does not hold true, in general.

Example (Goodearl, Exercises 2.C.3)

Let F be a field, K a proper subfield of F, and $R = \begin{bmatrix} K & F \\ 0 & F \end{bmatrix}$ a ring.
While $Q(R) = \begin{bmatrix} F & F \\ F & F \end{bmatrix}$ is a right and left self-injective unit-regular ring, R is not an Utumi ring because $Q(R) \neq Q^{\ell}(R)$.

Because for $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \in Q(R)$ where $\alpha \in F \setminus K$,
there is no $\begin{bmatrix} x & \beta \\ 0 & \gamma \end{bmatrix} \in R$ such that $0 \neq \begin{bmatrix} x & \beta \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} x & \beta \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \in R$,
i.e., R is not dense in $Q(R)$ as a left R-submodule.
Note that R is a right Utumi ring but not left Utumi.

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Thank you for your ATTENTION.

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