Shifted quantum affine algebras and cluster algebras

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ICRA 21

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Plan

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1. **Motivations and overview of main results**

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2. **Shifted quantum affine $\mathfrak{sl}_2$**

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2. **Shifted quantum affine $\mathfrak{sl}_2$**
3. **General case**

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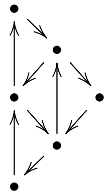

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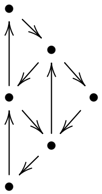

- Fomin-Zelevinsky : $\mathscr{A} \cong \mathbb{C}[N]$, where

$$N = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in SL(4, \mathbb{C}) \right\}$$

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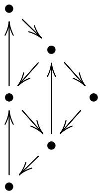

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- Hernandez-L : $\mathscr{A} \cong K_0(\mathscr{C}_Q)$, where
  $\mathscr{C}_Q$ is a subcategory of $\mathscr{C} := \{\text{finite-dim.} U_q(\widehat{\mathfrak{sl}}_4)\text{-modules}\}$

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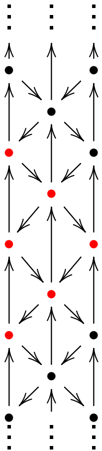
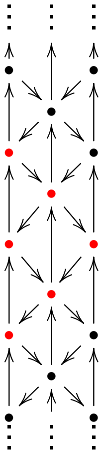


$\mathscr{A}_\infty =$ 

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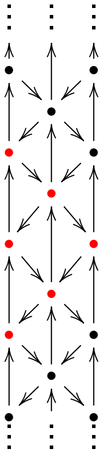
$\mathscr{A}_\infty =$ 

- Hernandez-L : $\mathscr{A}_\infty$ contains cluster variables equal to the q-characters of all Kirillov-Reshetikhin modules of $\mathscr{C}_\mathbb{Z}$
- Most cluster variables are not q-characters of objects of $\mathscr{C}_\mathbb{Z}$

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Q3: What is the representation-theoretical meaning of the remaining ones ?

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- If $\mathfrak{g} \neq \mathfrak{sl}_2$, every one-step mutation has degree > 0, so it is not the q-character of an object in $\mathscr{C}_\mathbb{Z}$.

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Fourth question

Q4: What about infinite-dimensional modules and category $\mathscr{O}_{\mathbb{Z}}^{\mathrm{sh}}$?

Can we find a cluster algebra structure on $K_0(\mathscr{O}_{\mathbb{Z}}^{\mathrm{sh}})$?

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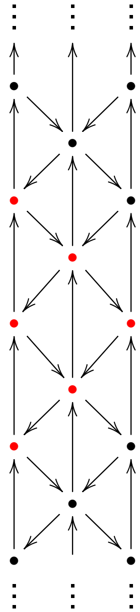
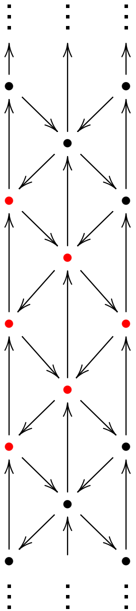
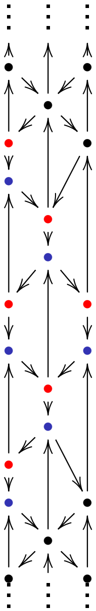


$\mathscr{A}_\infty =$ 

$\mathscr{B}_\infty :=$ 

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There is an explicit injective homomorphism $F : \mathscr{B}_\infty \to K_0\left(\mathscr{O}_{\mathbb{Z}}^{\mathrm{sh}}\right)$, such that $\overline{F(\mathscr{B}_\infty)} = K_0\left(\mathscr{O}_{\mathbb{Z}}^{\mathrm{sh}}\right)$

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Let $\mathfrak{g}$ be of type A, D, E.
$K_0\left(\mathscr{O}_{\mathbb{Z}}^{\mathrm{sh}}\right)$ contains a cluster subalgebra $\mathscr{B}$ isomorphic to the coordinate ring of the open double Bruhat cell $\mathbb{C}[G^{w_0,w_0}]$.

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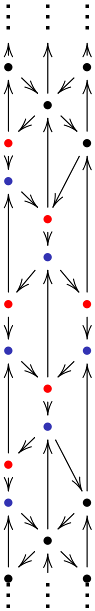

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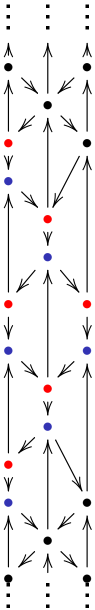

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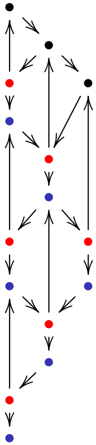

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Classical limit $q \to 1$:

$$x_k \to x \otimes t^k, \qquad y_k \to y \otimes t^k, \qquad h_k \to h \otimes t^k,$$
$$\frac{K - \overline{K}}{q - q^{-1}} \to h \otimes 1,\ \frac{\phi^+_a}{q-q^{-1}} \to h \otimes t^a,\ \frac{\phi^-_b}{q-q^{-1}} \to h \otimes t^b$$

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Definition

A $U_q^m(\widehat{\mathfrak{sl}}_2)$-module V such that there is $v \in V$ and $\Psi_k \in \mathbb{C}$ with

- $U_q^m(\widehat{\mathfrak{sl}}_2)v = V$
- $x_k v = 0 \quad (k \in \mathbb{Z})$
- $\phi_k^+ v = \Psi_k v, \quad (k \in \mathbb{Z}_{\geq 0})$

is called a highest ℓ-weight module.

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is called a highest ℓ-weight module.
The highest ℓ-weight is $\Psi(z) := \sum_{k \geq 0} \Psi_k z^k \in \mathbb{C}[[z]]$.

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- Conversely, for each $\Psi(z) \in \mathbb{C}(z)$ of degree m, regular at $z = 0$, there is a unique simple $L(\Psi) \in \mathscr{O}^m$ with highest ℓ-weight Ψ.

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$L(\Psi_{q^r}\Psi_{q^s}^{-1}) \in \mathcal{O}^{0}$: Kirillov-Reshetikhin modules (finite-dim.).

$$\begin{array}{c} \vdots \\ \uparrow \\ Q_6^+ \\ \uparrow \\ Q_4^+ \\ \uparrow \\ Q_2^+ \\ \uparrow \\ Q_0^+ \\ \downarrow \\ Q_{-2}^- \\ \uparrow \\ Q_{-4}^- \\ \uparrow \\ Q_{-6}^- \\ \uparrow \\ \vdots \end{array}$$

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow & & \uparrow \\
Q_6^+ & & Q_6^+ \\
\uparrow & & \uparrow \\
Q_4^+ & & Q_4^+ \\
\uparrow & & \uparrow \\
Q_2^+ & & Q_2^+ \\
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Q_0^+ & \longrightarrow & Q_0^- \\
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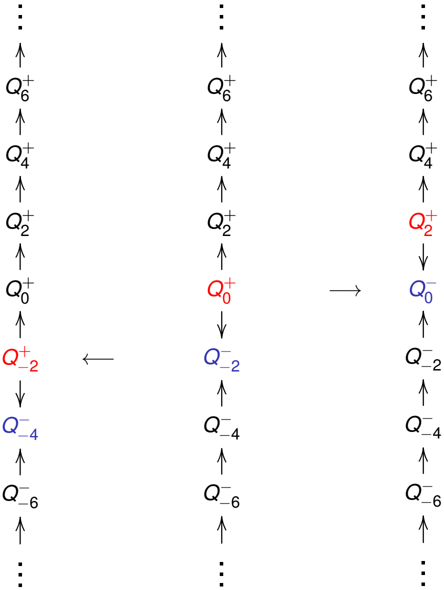
\end{array}
$$


Q_6^+
Q_4^+
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Q_{-4}^-
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Q_2^+
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In type A, D, E, one has: $M_{w,i,a} = \prod_{j:c_{ij}<0} Q_{w(\varpi_j),a}$.

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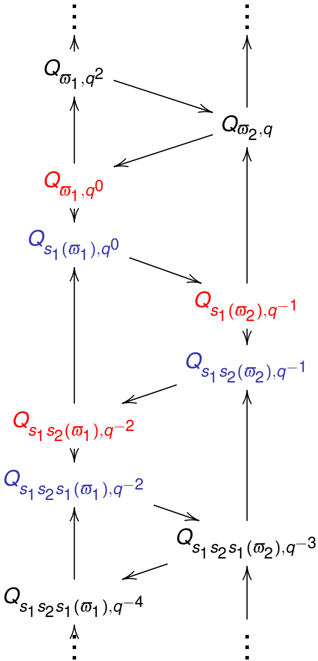
Theorem (Geiss-Hernandez-L 2023)

- There is an injective ring homomorphism $F\colon \mathscr{B}_\infty \to K_0(\mathscr{O}_{\mathbb{Z}}^{\mathrm{sh}})$ such that $\overline{F(\mathscr{B}_\infty)} = K_0(\mathscr{O}_{\mathbb{Z}}^{\mathrm{sh}})$
- For every reduced decomposition $w_0 = s_{i_1} \cdots s_{i_r}$ there is an explicit initial seed of $\mathscr{B}_\infty$ with cluster variables of the form:

$$Q_{s_{i_1} \cdots s_{i_k}(\varpi_i^\vee), q^n}, \ (0 \leq k \leq r,\ i \in I,\ n \in \mathbb{Z})$$

Type A_2

$w_0 = s_1 s_2 s_1$



Type B_2

$w_0 = s_1 s_2 s_1 s_2$