Tilting and Cotilting Subcategories in Categories of Quiver Representations

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GOALS OF THE TALK

Today, we want to talk about tilting and cotilting subcatecories of the category of representations of a quiver.

Let $\mathcal{M}$ be an abelian category, $\mathcal{Q}$ a rooted quiver, and $\mathrm{Rep}(\mathcal{Q}, \mathcal{M})$ the category of $\mathcal{M}$-valued representations of $\mathcal{Q}$.

By using some recent results about cotorsion torsion triples (resp. torsion cotorssion triples), under certain assumptions, we show that if $\mathcal{T}$ is a 1-tilting (resp. 1-cotilting) subcategory of $\mathcal{M}$, then the monomorphism category $\Phi(\mathcal{T})$ (resp. the epimorphism category $\Psi(\mathcal{T})$) is a 1-tilting (resp. 1-cotilting) subcategory of $\mathrm{Rep}(\mathcal{Q}, \mathcal{M})$.

GOALS OF THE TALK

We also talk about another types of induced subcategories in $\mathrm{Rep}(\mathcal{Q}, \mathcal{M})$ and, by using nice descriptions of monomorphisms and epimorphism categories, show that if $\mathcal{T}$ is a tilting (resp. cotilting) subcategory of $\mathcal{M}$, then the epimorphism category $\Psi(\mathcal{T})$ (resp. the monomorphism category $\Phi(\mathcal{T})$) is a tilting (resp. cotilting) subcategory of $\mathrm{Rep}(\mathcal{Q}, \mathcal{M})$ for every finite acyclic quiver $\mathcal{Q}$.

This result is a generalization of a lemma due to Zhang about induced cotilting modules in 2011.

We finally extend Zhang's reciprocity of the monomorphism operator and the left perpendicular operator for cotilting modules to cotilting subcategories. The results give us a systematical method to create new tilting and cotilting subcategories.

Rooted Quiver

A quiver $Q = (\mathrm{V}, \mathrm{E}, s, t)$ is a directed graph with vertex set V and arrow set E. An arrow a of a quiver from a vertex v to another vertex w is denoted by $a : v \to w$, we write $v = s(a)$ the initial vertex and $w = t(a)$ the terminal vertex. We usually denote the quiver $Q = (\mathrm{V}, \mathrm{E}, s, t)$ briefly by $Q = (\mathrm{V}, \mathrm{E})$ or even simply by Q.

A quiver Q is said to be finite if both V and E are finite sets.

Rooted Quiver

A path p of length $l \geqslant 1$ with source a and target b (from a to b) is a sequence of arrows $\alpha_l \cdots \alpha_2 \alpha_1$, where $\alpha_i \in$ E, for all $1 \leq i \leq l$, and $s(\alpha_1) = a$, $s(\alpha_i) = t(\alpha_{i-1})$ and $t(\alpha_l) = b$.

The trivial path at vertex v is denoted by ϵ_v.

A cycle is a path p of length $l \geqslant 1$ such that $s(p) = t(p)$.

A quiver Q is called acyclic if it contains no cycles.

A quiver Q is right (resp. left) rooted if and only if there exists no path of the form $\bullet \longrightarrow \bullet \longrightarrow \cdots$ (resp. $\cdots \longrightarrow \bullet \longrightarrow \bullet$) in Q.

Note that the notion of rooted quivers in fact is a generalization of the notion of finite acyclic quivers.

A quiver $\mathcal{Q}$ can be considered as a category whose objects are the vertices of $\mathcal{Q}$ and morphisms are all paths in $\mathcal{Q}$.
Assume that $\mathcal{M}$ is an abelian category. An $\mathcal{M}$-valued representation X of $\mathcal{Q}$ is a covariant functor $X : \mathcal{Q} \to \mathcal{M}$.

Such a representation is determined by giving an object $X_v := X(v) \in \mathcal{M}$ to each vertex v of $\mathcal{Q}$ and a morphism $X_a := X(a) : X_v \to X_w$ in $\mathcal{M}$ to each arrow $a : v \to w$ of $\mathcal{Q}$. For the trivial path ϵ_v, the morphism $X(\epsilon_v)$ is the identity on $X(v)$. A morphism between two representations X and Y is a natural transformation.

The $\mathcal{M}$-valued representations of quiver Q form a category, denoted by $[Q, \mathcal{M}]$ or $\mathrm{Rep}(Q, \mathcal{M})$. Note that since $\mathcal{M}$ is an abelian category, then so is $\mathrm{Rep}(Q, \mathcal{M})$ [JP10, Theorem 1].

[JP10] M. Jardim and D. M. Prata, Representations of quivers on abelian categories and monads on projective varieties, Sao Paulo J. Math. Sci. **4**(3) (2010), 399-423. DOI 10.11606/issn.2316-9028.v4i3p399-423. MR 2856193.

The Evaluation Functor

Let $\mathcal{M}$ be an abelian category. For any quiver $Q = (\mathrm{V}, \mathrm{E})$ and for every $v \in \mathrm{V}$, there is an evaluation functor $e_v : \mathrm{Rep}(Q, \mathcal{M}) \to \mathcal{M}$, which maps every $\mathcal{M}$-valued representation X of Q to its value at vertex v, i.e. $e_v(X) := X(v) = X_v$.

This, in fact, is a spatial case of the following fact about categories.

Given two categories C and D and an object A in C,
We have the following functor that is called the evaluation functor.

$$e_A : (C, D) \longrightarrow D$$

$$F \longrightarrow F(A)$$

$$\alpha \longrightarrow \alpha_A$$

The Right Adjoint of the evaluation functor

It is proved that the evaluation functor e^v possesses a right and also a left adjoint g_v and f_v (e^v_ρ and e^v_λ in some references), respectively.

If $\mathcal{M}$ has small products, i.e. if it satisfies Ab3*, by [HJ19, Theorem 3.7(b)] (see also [EH99, Theorem 4.1]), e_v possesses a right adjoint $g_v : \mathcal{M} \to \mathrm{Rep}(Q, \mathcal{M})$ that is defined as follows:
For each object M in $\mathcal{M}$ and for each vertex $w \in \mathrm{V}$,

$$g_v(M)_w := \prod_{Q(w,v)} M.$$

If there are no paths in Q from w to v, then this product is empty and hence $g_v(M)_w$ is the zero (or terminal) object in $\mathcal{M}$.

[HJ19] H. Holm and P. Jørgensen, Cotorsion pairs in categories of quiver representations, Kyoto J. Math. **59**(3) (2019), 575-606. DOI 10.1215/21562261-2018-0018. MR 3990178.

The Right Adjoint of the evaluation functor

For an arrow $a : w \to w'$ in Q, every path $p' \in Q(w', v)$ yields a path $p'a \in Q(w, v)$ and the morphism $g_v(M)_a$ is defined as the unique one that makes the following diagram commutative for every $p' \in Q(w', v)$:

$$\begin{array}{ccccc} g_v(M)_w & & \overset{g_v(M)_a}{\dashrightarrow} & & g_v(M)_{w'} \\ {\scriptstyle \pi_{p'a}}\downarrow & & & & \downarrow{\scriptstyle \pi_{p'}} \\ M_{p'a} & = & M & = & M_{p'} \end{array}$$

Here the vertical morphisms π_* are the canonical projections. If $\mathcal{M}$ is a module category, then the morphism $g_v(M)_a$ maps any $(m_p)_{p \in Q(w,v)} \in \prod_{Q(w,v)} M$ to $(m'_{p'})_{p' \in Q(w',v)} \in \prod_{Q(w',v)} M$, where $m'_{p'} = m_{p'a}$.

The Left Adjoint of the evaluation functor

The left adjoint of e^v is also defined similarly. If $\mathcal{M}$ has small coproducts, i.e. if it satisfies Ab3, by [HJ19, Theorem 3.7(a)] (see also [EOT04, Propositions 3.1 and 3.2]), e_v possesses a left adjoint $f_v : \mathcal{M} \to \mathrm{Rep}(\mathcal{Q}, \mathcal{M})$ that is defined as follows:
For each object M in $\mathcal{M}$ and each $w \in$ V,

$$f_v(M)_w := \coprod_{\mathcal{Q}(v,w)} M.$$

If there are no paths in $\mathcal{Q}$ from v to w, then this coproduct is empty and hence $f_v(M)_w$ is the zero (or initial) object in $\mathcal{M}$.

[EOT04] E. Enochs, L. Oyonarte, and B. Torrecillas, Flat covers and flat representations of quivers, Comm. Algebra **32**(4) (2004), 1319-1338. DOI 10.1081/AGB-120028784. MR 2039513.

The Left Adjoint of the evaluation functor

For an arrow $a : w \to w'$ in Q, each path $p \in Q(v, w)$ yields a path $ap \in Q(v, w')$, and the morphism $f_v(M)_a$ is defined as the unique one in $\mathcal{M}$ that makes the following diagram commutative for every $p \in Q(v, w)$: Here the

$$\begin{array}{ccccc} M_p & = & M & = & M_{ap} \\ \downarrow{\scriptstyle \varepsilon_p} & & & & \downarrow{\scriptstyle \varepsilon_{ap}} \\ f_v(M)_w & & \xrightarrow[f_v(M)_a]{\quad\quad} & & f_v(M)_{w'} \end{array}$$

vertical morphisms ε_* are the canonical injections. If $\mathcal{M}$ is a module category, then the morphism $f_v(M)_a$ maps any $(m_p)_{p \in Q(w,v)} \in \coprod_{Q(v,w)} M$ to $(m'_{p'})_{p' \in Q(v,w')} \in \coprod_{Q(v,w')} M$, where

$$m'_{p'} = \begin{cases} m_p & \text{if } p' = ap, \\ 0 & \text{otherwise.} \end{cases}$$

★ Mitchell in [Mi68, Section 1, Page 342], using Kan construction, described the left adjoints of the evaluation functors and applied them to find a projective generator for $(\mathbf{X}, \mathscr{A})$, where $\mathbf{X}$ is a finite poset and $\mathscr{A}$ is an abelian category.

★ In [EH99] Enochs and Herzog, for a a module category $\mathcal{M}$, described the right adjoints of the evaluation functors in $\mathrm{Rep}(\mathcal{Q}, \mathcal{M})$, and used them to construct injective representations of some quivers.

[EH99] E. Enochs and I. Herzog, A homotopy of quiver morphisms with applications to representations, Canad. J. Math. **51(2)** (1999), 294-308.

[Mi68] B. Mitchell, On the dimension of objects and categories II, J. Algebra **9** (1968), 341-368.

Projective and Injective Objects

It will be also useful to note that since the pair (e_v, g_v) (resp. (f_v, e_v)) is an adjoint pair for every $v \in$ V, if E (resp. P) is an injective (resp. projective) obeject of $\mathcal{M}$, then the functor $\mathrm{Rep}(\mathcal{Q}, \mathcal{M})(-, g_v(E)) \simeq \mathcal{M}(e_v(-), E)$ (resp. $\mathrm{Rep}(\mathcal{Q}, \mathcal{M})(f_v(P), -) \simeq \mathcal{M}(P, e_v(-))$) is exact and so $g_v(E)$ (resp. $f_v(P)$) is an $\mathcal{M}$-valued injective (resp. projective) representation of $\mathcal{Q}$ for every $v \in$ V.

In the following, for every full subcategory $\mathcal{T}$ of $\mathcal{M}$, $f_*(\mathcal{T}) := \{f_v(T) | v \in$ V and $T \in \mathcal{T}\}$ and $g_*(\mathcal{T}) := \{g_v(T) | v \in$ V and $T \in \mathcal{T}\}$. Also, for every vertex $v \in$ V, $f_v(\mathcal{T}) := \{f_v(T) | T \in \mathcal{T}\}$ and $g_v(\mathcal{T}) := \{g_v(T) | T \in \mathcal{T}\}$.

The Stalk Functors

Let $\mathcal{M}$ be an abelian category. For any quiver $Q = (\mathrm{V}, \mathrm{E}, s, t)$ and for every $v \in \mathrm{V}$, there is a stalk functor $\mathrm{s}_v : \mathcal{M} \to \mathrm{Rep}(Q, \mathcal{M})$, which maps an object $M \in \mathcal{M}$ to the stalk representation $\mathrm{s}_v(M)$ given by $\mathrm{s}_v(M)_w = 0$ for $w \neq v$ and $\mathrm{s}_v(M)_v = M$.

For every arrow a of Q, the morphism $\mathrm{s}_v(M)_a$ is zero. For every full subcategory $\mathcal{T}$ of $\mathcal{M}$, we set $\mathrm{s}_*(\mathcal{T}) := \{\mathrm{s}_v(T) | v \in \mathrm{V} \text{ and } T \in \mathcal{T}\}$. Also, for every vertex $v \in \mathrm{V}$, $\mathrm{s}_v(\mathcal{T}) := \{\mathrm{s}_v(T) | T \in \mathcal{T}\}$.

Under some assumption, it is proved in [HJ19, Theorem 4.5] that s_v has a left adjoint c_v and a right adjoint k_v, respectively. Let us be more precise.

If $\mathcal{M}$ satisfies Ab3, for each representation $X \in \operatorname{Rep}(\mathcal{Q}, \mathcal{M})$, we denote by φ_v^X the unique morphism in $\mathcal{M}$ that makes the following diagram commutative for every arrow a of $\mathcal{Q}$:

$$\begin{array}{ccc} X_{s(a)} & & \\ \downarrow{\scriptstyle \varepsilon_a} & \searrow^{X_a} & \\ \coprod_{a \in \mathrm{E}^{*\to v}} X_{s(a)} & \xrightarrow[\varphi_v^X]{} & X_v \end{array}$$

Here, ε_a denotes the canonical injection. The assignment $X \mapsto \varphi_v^X$ is a functor from $\operatorname{Rep}(\mathcal{Q}, \mathcal{M})$ to the category of morphisms in $\mathcal{M}$ and so there is a functor $\mathrm{c}_v : \operatorname{Rep}(\mathcal{Q}, \mathcal{M}) \to \mathcal{M}$ given by $X \mapsto \operatorname{Coker} \varphi_v^X$, where $\operatorname{Coker} \varphi_v^X$ is the cokernel of φ_v^X. It is proved in [HJ19, Theorem 4.5(a)] that the functor c_v is the left adjoint of s_v.

The Right Adjoint of the stalk functor

Dually, if $\mathcal{M}$ satisfies Ab3*, for each representation $X \in \operatorname{Rep}(Q, \mathcal{M})$, we denote by ψ_v^X the unique morphism in $\mathcal{M}$ that makes the following diagram commutative for every arrow a of Q: Here, π_a denotes the canonical pro-

$$
\begin{array}{ccc}
X_v & \overset{\psi_v^X}{\dashrightarrow} & \prod_{a \in \mathrm{E}^{v \to *}} X_{t(a)} \\
 & \searrow^{X_a} & \downarrow{\pi_a} \\
 & & X_{t(a)}
\end{array}
$$

jection. This construction yields a functor $\mathrm{k}_v : \operatorname{Rep}(Q, \mathcal{M}) \to \mathcal{M}$ given by $X \mapsto \operatorname{Ker} \psi_v^X$, where $\operatorname{Ker} \psi_v^X$ is the kernel of ψ_v^X. It is proved in [HJ19, Theorem 4.5(b)] that the functor k_v is the right adjoint of s_v.

We can now define the following subcategories of $\mathrm{Rep}(\mathcal{Q},\mathcal{M})$ for every subcategory $\mathcal{T}$ of $\mathcal{M}$:

$$\Phi(\mathcal{T}) := \{X \in \mathrm{Rep}(\mathcal{Q},\mathcal{M}) \mid \varphi_v^X \text{ is monic and } X_v, \mathrm{c}_v(X) \in \mathcal{T};\ \forall\, v \in \mathrm{V}\},$$

$$\Psi(\mathcal{T}) := \{X \in \mathrm{Rep}(\mathcal{Q},\mathcal{M}) \mid \psi_v^X \text{ is epic and } X_v, \mathrm{k}_v(X) \in \mathcal{T};\ \forall\, v \in \mathrm{V}\}.$$

These categories are nothing but the separated monomorphism and epimorphism categories in the sense of Zhang and Xiong [ZX19, Sections 2 and 6] that satisfy some local conditions; see also [Zh11]. For this reason, although we follow notations from [HJ19], we will call $\Phi(\mathcal{T}) := \mathrm{Mon}(\mathcal{Q},\mathcal{T})$ (resp. $\Psi(\mathcal{T}) := \mathrm{Epi}(\mathcal{Q},\mathcal{T})$) the monomorphism category (resp. the epimorphism category) associated to $\mathcal{T}$ following some recent work on representation theory of algebras.

Note that the definition of the monomorphism (resp. epimorphism) category $\Phi(\mathcal{T})$ (resp. $\Psi(\mathcal{T})$) in [HJ19, Section 7] is slightly different from its definition in this talk.

Indeed, any representation X in $\Phi(\mathcal{T})$ (resp. $\Psi(\mathcal{T})$) as defined in [HJ19, Section 7] does not need to satisfy $X_v \in \mathcal{T}$ for all $v \in \mathrm{V}$. But if Q is left (resp. right) rooted and $\mathcal{T}$ is closed under coproduct (resp. product), then by [HJ19, Proposition 7.2(a)] (resp. [HJ19, Proposition 7.2(b)]) this seeming difference is not real.

[Zh11] P. Zhang, Monomorphism categories, cotilting theory, and Gorenstein-projective modules, J. Algebra **339** (2011), 181-202. DOI 10.1016/j.jalgebra.2011.05.01. MR 2811319.

[ZX19] P. Zhang and B. L. Xiong, Separated monic representations II: Frobenius subcategories and RSS equivalences, Trans. Am. Math. Soc. 372 (2) (2019), 981-1021. DOI 10.1090/tran/7622. MR 3968793.

Definition (k-Tilting Subcategories)

Let k be a non-negative integer and $\mathcal{M}$ be an abelian category with enough projectives. An additively closed full subcategory $\mathcal{T}$ of $\mathcal{M}$, i.e., one which is closed under taking finite direct sums and summands, is called weak k-tilting if

(i) *For each positive integer i, $\mathrm{Ext}^i_{\mathcal{M}}(\mathcal{T}, \mathcal{T}) = 0$, i.e. $\mathrm{Ext}^i_{\mathcal{M}}(T_1, T_2) = 0$ for all $T_1, T_2 \in \mathcal{T}$;*

(ii) *The projective dimension of any object of $\mathcal{T}$ is at most k;*

(iii) *For any projective P in $\mathcal{M}$, there is an exact sequence $0 \to P \to T^0 \to \cdots \to T^k \to 0$, where $T^i \in \mathcal{T}$ for every $i = 0, \cdots, k$.*

A weak k-tilting subcategory of $\mathcal{M}$ is called k-tilting if it is additionally contravariantly finite in $\mathcal{M}$. The subcategory $\mathcal{T}$ is called a tilting subcategory if there is a non-negative integer k such that $\mathcal{T}$ is a k-tilting subcategory.

Cotorsion Torsion Triple

Recently, Bauer, Botnan, Oppermann, and Steen introduced the notion of cotorsion torsion triples, studied their relationship with tilting subcategories and proved the following nice result, see [BBOS20, Subsections 2.1 and 2.2]. Recall that a cotorsion torsion triple in an abelian category $\mathcal{M}$ is a triple of subcategories $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ such that the pair $(\mathcal{C}, \mathcal{T})$ is a (hereditary) complete cotorsion pair and the pair $(\mathcal{T}, \mathcal{F})$ is a torsion pair, see [BBOS20, Definition 2.9].

[BBOS20] U. Bauer, M. B. Botnan, S. Oppermann, and J. Steen, Cotorsion torsion triples and the representation theory of filtered hierarchical clustering, Adv. Math. **369** (2020), 107171. DOI 10.1016/j.aim.2020.107171. MR 4091895.

BBOS' Correspondence I

Theorem ([BBOS20, Theorem 2.29]) Let $\mathcal{M}$ be an abelian category with enough projectives. Then there are mutually inverse bijections between the collection of 1-tilting subcategories and the collection of cotorsion torsion triples as follows:

$$\{\text{1-tilting subcategories}\} \longleftrightarrow \{\text{cotorsion torsion triples}\}$$

$$\mathcal{T} \longmapsto ({}^{\perp_1}(\operatorname{Fac}\mathcal{T}), \operatorname{Fac}\mathcal{T}, \mathcal{T}^{\perp_0})$$

$$\mathcal{C} \cap \mathcal{T} \longleftarrow\!\shortmid (\mathcal{C}, \mathcal{T}, \mathcal{F})$$

where $\operatorname{Fac}\mathcal{T}$ is the full subcategory of $\mathcal{M}$ consisting of factor objects of objects in $\mathcal{T}$.

Dually, an additively closed full subcategory $\mathcal{T}$ of $\mathcal{M}$ is called a weak cotilting subcategory if and only if it is weakly tilting in $\mathcal{M}^{\mathrm{op}}$. In other words:

Definition (k-Cotilting Subcategories)

Let k be a non-negative integer and $\mathcal{M}$ be an abelian category with enough injectives. An additively closed full subcategory $\mathcal{T}$ of $\mathcal{M}$ is called weak k-cotilting subcategory if

(i) *For each positive integer i, $\mathrm{Ext}^i_{\mathcal{M}}(\mathcal{T},\mathcal{T}) = 0$, i.e. $\mathrm{Ext}^i_{\mathcal{M}}(T_1, T_2) = 0$ for all $T_1, T_2 \in \mathcal{T}$;*

(ii) *The injective dimension of any object of $\mathcal{T}$ is at most k;*

(iii) *For any injective I in $\mathcal{M}$, there is an exact sequence $0 \to T_k \to \cdots \to T_0 \to I \to 0$, where $T_i \in \mathcal{T}$, for every $i = 0, \cdots, k$.*

A weak k-cotilting subcategory of $\mathcal{M}$ is called k-cotilting if it is additionally covariantly finite in $\mathcal{M}$. The subcategory $\mathcal{T}$ is called a cotilting subcategory if there is a non-negative integer k such that $\mathcal{T}$ is a k-cotilting subcategory.

As a dual version of the notion of cotorsion torsion triple, Bauer, Botnan, Oppermann, and Steen introduced the notion of torsion cotorsion triples, studied their relationship with weakly cotilting subcategories and proved the following nice result, see [BBOS20, Subsection 2.3].

Recall that a torsion cotorsion triple in an abelian category $\mathcal{M}$ is a triple of subcategories $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ such that the pair $(\mathcal{T}, \mathcal{F})$ is a torsion pair and the pair $(\mathcal{F}, \mathcal{D})$ is a (hereditary) complete cotorsion pair, see [BBOS20, Page 29 before Theorem 2.33].

Theorem ([BBOS20, Theorem 2.34]) Let $\mathcal{M}$ be an abelian category with enough injectives. Then there are mutually inverse bijections between the collection of 1-cotilting subcategories and the collection of torsion cotorsion triples as follows:

$$\{\text{1-cotilting subcategories}\} \longleftrightarrow \{\text{torsion cotorsion triples}\}$$

$$\mathcal{T} \longmapsto ({}^{\perp_0}\mathcal{T}, \operatorname{Sub}\mathcal{T}, (\operatorname{Sub}\mathcal{T})^{\perp_1})$$

$$\mathcal{F} \cap \mathcal{D} \longleftarrow (\mathcal{T}, \mathcal{F}, \mathcal{D})$$

where $\operatorname{Sub}\mathcal{T}$ is the full subcategory of $\mathcal{M}$ consisting of subobjects of objects in $\mathcal{T}$.

We start with the following lemma that may be well-known for the specialist. It is convenient to recall that an abelian category satisfies Ab4 if it satisfies Ab3, i.e. if it has small coproducts, and any coproduct of monomorphisms is a monomorphism. The axioms Ab3* and Ab4* are dual to Ab3 and Ab4.

Lemma

Let $\mathcal{X}$ and $\mathcal{Y}$ be two subcategories of an abelian category $\mathcal{M}$ that satisfies Ab3 and Ab3 and $\mathcal{Q} = (\mathrm{V}, \mathrm{E})$ be a quiver. The pair $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{M}$ if and only if the induced pair $(\mathrm{Rep}(\mathcal{Q}, \mathcal{X}), \mathrm{Rep}(\mathcal{Q}, \mathcal{Y}))$ is a torsion pair in $\mathrm{Rep}(\mathcal{Q}, \mathcal{M})$.*

Induced 1-cotilting and 1-tilting subcategories

Theorem

Let $\mathcal{M}$ be an abelian category that satisfies Ab4 and Ab4 and which has enough projectives and injectives. If $\mathcal{Q} = (\mathrm{V}, \mathrm{E}, s, t)$ is a left (resp. right) rooted quiver and $\mathcal{T}$ is a 1-tilting (resp. 1-cotilting) subcategory of $\mathcal{M}$, then the monomorphism category $\Phi(\mathcal{T})$ (resp. the epimorphism category $\Psi(\mathcal{T})$) is a 1-tilting (resp. 1-cotilting) subcategory of $\mathrm{Rep}(\mathcal{Q}, \mathcal{M})$.*

An Example

Let $\mathcal{Q} = (\mathrm{V}, \mathrm{E})$ be a finite acyclic quiver. If $\mathcal{M}$ is an abelian category with enough projectives, by [BBOS20, Proposition 3.9], the subcategory

$$\mathbb{T} := \mathrm{add}\; g_*(\mathrm{Proj}\; \mathcal{M}) = \mathrm{add}\; \{g_v(P) | v \in \mathrm{V} \text{ and } P \in \mathrm{Proj}\; \mathcal{M}\}$$

is a 1-tilting subcategory of $\mathrm{Rep}(\mathcal{Q}, \mathcal{M})$ and so we have a 1-tilting subcategory that is not, in general, of the form $\Phi(\mathcal{T})$ for some tilting subcategory $\mathcal{T}$ of $\mathcal{M}$. For instance, if $\mathcal{Q}$ is the line quiver $\overrightarrow{A_2} = \bullet \longrightarrow \bullet$, then for a nonzero projective object P, the representation $P \longrightarrow 0$ is in $\mathbb{T}$ but it is not in $\Phi(\mathcal{T})$ for every subcategory $\mathcal{T}$ of $\mathcal{M}$.

An Example

Dually, If $\mathcal{M}$ is an abelian category with enough injectives, by [BBOS20, Theorem 3.12], the subcategory

$$\mathbb{C} := \text{add } f_*(\text{Inj } \mathcal{M}) = \text{add } \{f_v(I) | v \in \text{V and } I \in \text{Inj } \mathcal{M}\}$$

is a 1-cotilting subcategory of $\text{Rep}(\mathcal{Q}, \mathcal{M})$ and so we have a cotilting subcategory that is not, in general, of the form $\Psi(\mathcal{C})$ for some cotilting subcategory $\mathcal{C}$ of $\mathcal{M}$.

For instance, if $\mathcal{Q}$ is the line quiver $\overrightarrow{A_2} = \bullet \longrightarrow \bullet$, then for a nonzero injective object I, the representation $0 \longrightarrow I$ is in $\mathbb{C}$ but it is not in $\Psi(\mathcal{C})$ for every subcategory $\mathcal{C}$ of $\mathcal{M}$.

An Example

In the above example, the subcategory $\mathbb{T}$ has a nice description and we can show that $\mathbb{T} = \Psi(\text{Proj } \mathcal{M})$. Indeed, by [BBOS20, Corollary 3.10], the pair $(\text{Rep}(\mathcal{Q}, \text{Proj } \mathcal{M}), \text{Fac } \mathbb{T})$ is a cotorsion pair and $\mathbb{T} = \text{Rep}(\mathcal{Q}, \text{Proj } \mathcal{M}) \cap \text{Fac } \mathbb{T}$. On the other hand, by [HJ19, Theorem B], the pair $(\text{Rep}(\mathcal{Q}, \text{Proj } \mathcal{M}), \Psi(\mathcal{M}))$ is a cotorsion pair and one can easily show that

$$\Psi(\text{Proj } \mathcal{M}) = \text{Rep}(\mathcal{Q}, \text{Proj } \mathcal{M}) \cap \Psi(\mathcal{M})$$

as the finite product of projective objects is projective. Hence, $\mathbb{T} = \Psi(\text{Proj } \mathcal{M})$. Dually, one can easily show that $\mathbb{C} = \Phi(\text{Inj } \mathcal{M})$.

Subcategories $\mathrm{Proj}\ \mathcal{M}$ and $\mathrm{Inj}\ \mathcal{M}$ are self orthogonal and in the following we gave a generalization of the above mentioned results to self orthogonal subcategories of $\mathcal{M}$.

Theorem

Let $\mathcal{Q} = (\mathrm{V}, \mathrm{E}, s, t)$ be a quiver, $\mathcal{M}$ an abelian category, and $\mathcal{T}$ an additively closed subcategory of $\mathcal{M}$ such that $\mathrm{Ext}^1_{\mathcal{M}}(\mathcal{T}, \mathcal{T}) = 0$. Then, the following statements hold:

(*i*) *If $\mathcal{Q}$ is left rooted, $\mathcal{M}$ satisfies Ab4, and $\mathcal{T}$ is closed under small coproducts, then $\Phi(\mathcal{T}) = \mathrm{Add}\ f_*(\mathcal{T})$;*

(*ii*) *If $\mathcal{Q}$ is right rooted, $\mathcal{M}$ satisfies Ab4*, and $\mathcal{T}$ is closed under small products, then $\Psi(\mathcal{T}) = \mathrm{Prod}\ g_*(\mathcal{T})$.*

Let k be a non-negative integer, Γ an Artin algebra, and Q the line quiver with at least one arrow. By combining theorem above and [Zh11, Lemma 3.7], if T is a k-cotilting Γ-module, then $\Phi(\text{add } T)$ is a $(k+1)$-cotilting subcategory of $\text{Rep}(Q, \Gamma\text{-mod})$, where Γ-mod is the category of all finitely generated left Γ-modules.

In the following we give a generalization of Zhang's lemma in categorical sense by following Bauer, Botnan, Oppermann, and Steen in [BBOS20, Section 3].

We need the following lemmas which are of independent interest.

Lemma

Let $Q = (\mathrm{V}, \mathrm{E}, s, t)$ be a finite acyclic quiver and $\mathcal{M}$ an abelian category with enough projectives and injectives. Then if $\mathcal{T}$ is a weak 1-cotilting (resp. 1-tilting) subcategory of $\mathcal{M}$, then $\Phi(\mathcal{T})$ (resp. $\Psi(\mathcal{T})$) is a weak 2-cotiltiing (resp. 2-tiltiing) subcategory of $\mathrm{Rep}(Q, \mathcal{M})$.

As a generalization of [Zh11, Lemma 1.2], we have the following lemma due to Bauer, Botnan, Oppermann, and Steen. For the convenience of the reader we give the proof.

Lemma ([BBOS20, Lemma 3.8])

Let $Q = (\mathrm{V}, \mathrm{E}, s, t)$ be a finite acyclic quiver and $\mathcal{M}$ an abelian category. Then for every vertex $w \in \mathrm{V}$, the functor f_w (resp. g_w) has a left (resp. right) adjoint that we denote it by f'_w (resp. g'_w).

Lemma

Let $Q = (\mathrm{V}, \mathrm{E}, s, t)$ be a finite acyclic quiver, $\mathcal{M}$ an abelian category, and $\mathcal{T}$ a self orthogonal additively closed subcategory of $\mathcal{M}$. Then if $\mathcal{T}$ is a contravariantly (resp. covariantly) finite subcategory of $\mathcal{M}$, then $\Phi(\mathcal{T})$ and $\Psi(\mathcal{T})$ are contravariantly (resp. covariantly) finite subcategories of $\mathrm{Rep}(Q, \mathcal{M})$.

Now, by combining Lemmas above, we have the following theorem:

Theorem

Let $Q = (\mathrm{V}, \mathrm{E}, s, t)$ be a finite acyclic quiver and $\mathcal{M}$ an abelian category with enough projectives and injectives. If $\mathcal{T}$ is a 1-cotilting (resp. 1-tilting) subcategory of $\mathcal{M}$, then the monomorphism category $\Phi(\mathcal{T})$ (resp. the epimorphism category $\Psi(\mathcal{T})$) is a 2-cotilting (resp. 2-tilting) subcategory of $\mathrm{Rep}(Q, \mathcal{M})$. □

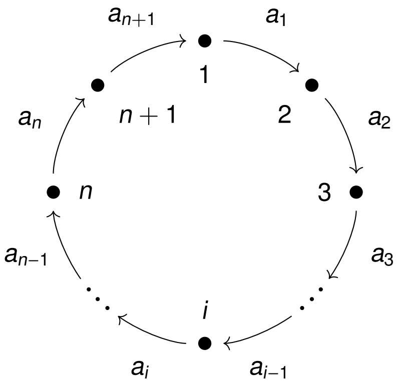
In fact, based on the proof of theorem above , we have the following result that can be considered as a generalization of [Zh11, Lemma 3.7], [BBOS20, Proposition 3.9], and their dual.

Theorem

Let $Q = (\mathrm{V}, \mathrm{E}, s, t)$ be a finite acyclic quiver and $\mathcal{M}$ an abelian category with enough projectives and injectives. If $\mathcal{T}$ is a tilting (resp. cotilting) subcategory of $\mathcal{M}$, then the monomorphism category $\Phi(\mathcal{T})$ and the epimorphism category $\Psi(\mathcal{T})$ are tilting (resp. cotilting) subcategories of $\mathrm{Rep}(Q, \mathcal{M})$. □

An Example

Let K be a field and $\Gamma = KQ/I$ an algebra with the quiver Q:



and the admissible ideal I of KQ generated by paths $a_{i+1}a_i$ for every $1 \leq i \leq n$. It is well-known that Γ is an n-Auslander algebra.

Indeed, the global and dominant dimensions of Γ are $n+1$, and $0 \to \Gamma \to (\bigoplus_{i=2}^{n+1} I(i)) \oplus I(2) \to I(2) \to I(n+1) \to \cdots \to I(2) \to I(1) \to 0$ is the minimal injective resolution of Γ.

Also, by some results in the literature, it is not difficult to show that for every integer $0 \leq k \leq n+1$, $\mathcal{T}_k = \text{proj}^{\leq k}(\Gamma) \cap \text{inj}^{\leq n+1-k}(\Gamma)$ is a k-tilting and $(n+1-k)$-cotilting subcategory of Γ-mod. Hence, the following statements hold:

(1) For every left rooted quiver $\mathcal{Q}$, the monomorphism category $\Phi(\mathcal{T}_1)$ is a 1-tilting subcategory of Rep$(\mathcal{Q}, \Gamma\text{-mod})$;

(2) For every right rooted quiver $\mathcal{Q}$, the epimorphism category $\Psi(\mathcal{T}_n)$ is a 1-cotilting subcategory of Rep$(\mathcal{Q}, \Gamma\text{-mod})$;

(3) For every finite acyclic quiver $\mathcal{Q}$, the epimorphism category $\Psi(\mathcal{T}_k)$ (resp. the monomorphism category $\Phi(\mathcal{T}_k)$) is a $(k+1)$-tilting and $(n+1-k)$-cotilting (resp. k-tilting and $(n+2-k)$-cotilting subcategory of Rep$(\mathcal{Q}, \Gamma\text{-mod})$.

An Example

Let now Q be the line quiver

$$\overrightarrow{A_3} = 1 \xrightarrow{a} 2 \xrightarrow{b} 3.$$

The tilting subcategory $\mathcal{T}_1 = \mathrm{proj}^{\leq 1}(\Gamma) \cap \mathrm{inj}^{\leq n}(\Gamma)$ of Γ-mod induces the tilting subcategory

$$\mathbb{T}_1 := \mathrm{add}\ \mathrm{s}_2(\mathcal{T}_1) \bigcup f_2(\mathcal{T}_1) \bigcup f_1(\mathcal{T}_1)$$

in $\mathrm{Rep}(\overrightarrow{A_3}, \Gamma\text{-mod})$ that clearly is not of the form $\Psi(\mathcal{C})$ or $\Phi(\mathcal{C})$ for every subcategory $\mathcal{C}$ of $\mathcal{M}$.
The cotilting subcategory $\mathcal{T}_n = \mathrm{proj}^{\leq n}(\Gamma) \cap \mathrm{inj}^{\leq 1}(\Gamma)$ of Γ-mod also induces the cotilting subcategory

$$\mathbb{T}_n := \mathrm{add}\ \mathrm{s}_2(\mathcal{T}_n) \bigcup g_2(\mathcal{T}_n) \bigcup g_3(\mathcal{T}_n)$$

in $\mathrm{Rep}(\overrightarrow{A_3}, \Gamma\text{-mod})$ that clearly is not of the form $\Psi(\mathcal{C})$ or $\Phi(\mathcal{C})$ for every subcategory $\mathcal{C}$ of $\mathcal{M}$.

The key result of this section is a generalization of a reciprocity due to Zhang of the monomorphism operator "Φ" and the left perpendicular operator "$\perp$" and its dual version to tilting and cotilting subcategories.

Let us be more precise. Let Q be the finite line quiver with $n > 1$, Γ an

$$\overrightarrow{A_n} = 1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} n$$

Artin algebra, and $\mathcal{X}$ a full subcategory of Γ-mod. In 2011, Zhang introduced the monomorphism category $\mathcal{S}_n(\mathcal{X})$ and showed that for a cotilting Γ-module T, there is a canonical construction of a cotilting module $\mathbf{m}(T)$ over the triangular matrix algebra $\mathbf{T}_n(\Gamma)$, such that $\mathcal{S}_n({}^{\perp}T) = {}^{\perp}\mathbf{m}(T)$, where ${}^{\perp}T := \{X \in \Gamma\text{-mod} | \mathrm{Ext}^i_{\Gamma}(X, T) = 0, \ \forall\ i \geq 1\}$ (see [Zh11, Theorem 3.1]). Later, Song, Kong, and Zhang generalized this result to finite acyclic quiver with a different method [SKZ14, Theorem 3.1].

[SKZ14] K. Song, F. Kong, and P. Zhang, Monomorphism operator and perpendicular operator, Comm. Algebra **42** (2014), 3708-3723. DOI 10.1080/00927872.2013.790975. MR 3200051.

Based on our notations in this paper, Zhang's result in fact says that $\Phi(^{\perp}T) = {}^{\perp}(\coprod_{1\leq i\leq n} f_i(T))$ and so by our result, we have $\Phi(^{\perp}(\text{add } T)) = \Phi(^{\perp}T) = {}^{\perp}f_*(\{T\}) = {}^{\perp}(\text{add } f_*(\text{add } T)) = {}^{\perp}\Phi(\text{add } T)$.

Clearly, add T is a cotilting subcategory of Γ-mod and in the following we give a generalization of this result to cotilting subcategories of abelian categories by following Song, Kong, and Zhang in [SKZ14]. We also give its dual version.

We need the following lemmas which are of independent interest.
Here, for every full subcategory $\mathcal{T}$ of an abelian category $\mathcal{M}$, ${}^{\perp}\mathcal{T} := \{X \in \mathcal{M} | \text{Ext}^i_{\mathcal{M}}(X, \mathcal{T}) = 0, \ \forall\ i \geq 1\}$ and $\mathcal{T}^{\perp} := \{X \in \mathcal{M} | \text{Ext}^i_{\mathcal{M}}(\mathcal{T}, X) = 0, \ \forall\ i \geq 1\}$.

Lemma

Let $Q = (\mathrm{V}, \mathrm{E}, s, t)$ *be a finite acyclic quiver and* $\mathcal{M}$ *an abelian category with enough projectives and injectives. If* $\mathcal{T}$ *is a cotilting (resp. tilting) subcategory of* $\mathcal{M}$*, then* ${}^{\perp}\Phi(\mathcal{T}) = {}^{\perp}\mathrm{s}_*(\mathcal{T})$ *(resp.* $\Psi(\mathcal{T})^{\perp} = \mathrm{s}_*(\mathcal{T})^{\perp}$*).*

Lemma

Let $Q = (\mathrm{V}, \mathrm{E}, s, t)$ *be a finite acyclic quiver and* $\mathcal{M}$ *an abelian category with enough projectives and injectives. If* $\mathcal{T}$ *is a cotilting (resp. tilting) subcategory of* $\mathcal{M}$*, then* $\Phi({}^{\perp}\mathcal{T}) = {}^{\perp}\mathrm{s}_*(\mathcal{T})$ *(resp.* $\Psi(\mathcal{T}^{\perp}) = \mathrm{s}_*(\mathcal{T})^{\perp}$*).*

Theorem (Zhang's Reciprocity)

Let $Q = (\mathrm{V}, \mathrm{E}, s, t)$ be a finite acyclic quiver and $\mathcal{M}$ an abelian category with enough projectives and injectives. If $\mathcal{T}$ is a cotilting (resp. tilting) subcategory of $\mathcal{M}$, then $\Phi({}^{\perp}\mathcal{T}) = {}^{\perp}\Phi(\mathcal{T})$ (resp. $\Psi(\mathcal{T}^{\perp}) = \Psi(\mathcal{T})^{\perp}$). □

Having a look on the proof of theorem above shows that it is not necessary to assume that the subcategory $\mathcal{T}$ of $\mathcal{M}$ is a cotilting or tilting subcategory and like the authors in [Zh11] and [SKZ14], it is possible to rewrite the result with weaker conditions.

Thank You