Cores and Weights of Multipartitions and Blocks of Ariki-Koike Algebras

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Summer 2024

Throughout $e \in \mathbb{Z}_{\geq 2}$.

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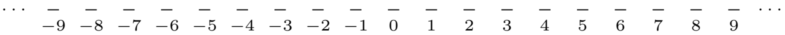
We obtain the **e-abacus display** of S by cutting up its ∞-abacus display into sections $[ie, ie+e-1]$ ($i \in \mathbb{Z}$), and putting the section $[ie, ie+e-1]$ directly on top of $[(i+1)e, (i+1)e+e-1]$.

Example

$\cdots$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\_$	$\cdots$
	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	

Example



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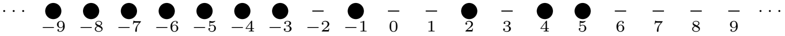

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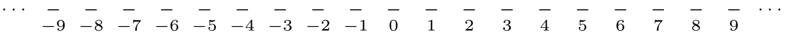

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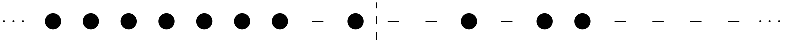

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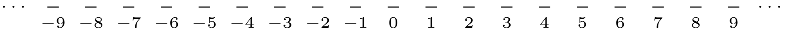

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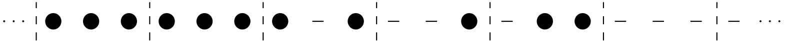
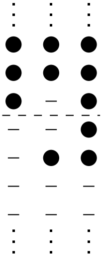


3-abacus display of B



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Conversely, given a β-set $B = \{b_1 > b_2 > \cdots\}$, it is associated to the unique partition $\lambda = (\lambda_1, \lambda_2, \dots)$ where

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Indeed, $\beta_{\mathfrak{s}(B)}(\lambda) = B$.

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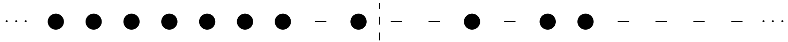

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e-Core and e-Weight of a Partition

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$\mathrm{core}_e(\lambda)$ and $\mathrm{wt}_e(\lambda)$ are independent of the charge s.

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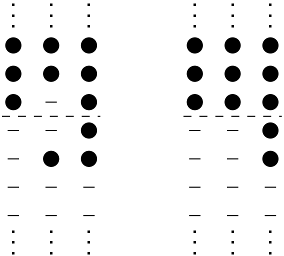

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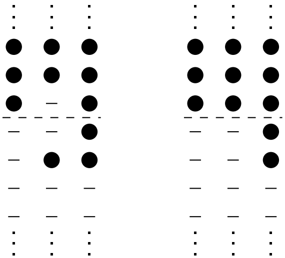

$\beta_2(\lambda)$ $\qquad$ $\beta_2(\mathsf{core}_3(\lambda))$

$\mathsf{core}_3(\lambda) = (4, 2, 0, 0, 0, \dots) = (4, 2)$, and $\mathsf{wt}_3(\lambda) = 2$.

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- Stack the ∞-abacus displays of $\beta_{s_i}(\lambda^{(i)})$ on top of each other, with the display of $\beta_{s_1}(\lambda^{(1)})$ at the bottom and that of $\beta_{s_\ell}(\lambda^{(\ell)})$ at the top.

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- Cut up this stacked ∞-abaci into sections with positions $[ie, ie+e-1]$ ($i \in \mathbb{Z}$), and put the section with positions $[ie, ie+e-1]$ on top of that with positions $[(i+1)e, (i+1)e+e-1]$.

Example

$$\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) = ((1), (1,1), (2)),\ \mathbf{s} = (s_1, s_2, s_3) = (-1, 2, 1).$$

$\beta_{s_3}(\lambda^{(3)}) = \{2, -1, -2, -3 \dots\}$	$\cdots$	●	●	● ¦	–	–	●	–	–	–	$\cdots$
$\beta_{s_2}(\lambda^{(2)}) = \{2, 1, -1, -2, \dots\}$	$\cdots$	●	●	● ¦	–	●	●	–	–	–	$\cdots$
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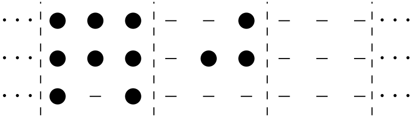
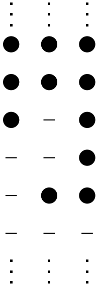


$\mathsf{U}_3(\boldsymbol{\lambda}; \mathbf{s})$



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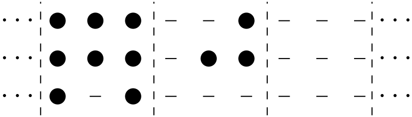
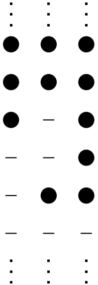


$$\mathsf{U}_3(\boldsymbol{\lambda}; \mathbf{s}) = (4,4,3,1)$$



e-Core and e-Weight of a Multipartition

Definition

Let $\boldsymbol{\lambda}$ be an ℓ-multipartition and let s be an ℓ-multicharge. We define the e-core and the e-weight of $(\boldsymbol{\lambda};\mathrm{s})$ to be those of $\mathsf{U}_e(\boldsymbol{\lambda};\mathrm{s})$; i.e.

$$\mathsf{core}_e(\boldsymbol{\lambda};\mathrm{s}) = \mathsf{core}_e(\mathsf{U}_e(\boldsymbol{\lambda};\mathrm{s}));$$
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Example

$$\mathsf{core}_3(((1),(1,1),(2));(-1,2,1)) = \mathsf{core}_3(4,4,3,1) = (4,2)$$
$$\mathsf{wt}_3(((1),(1,1),(2));(-1,2,1)) = \mathsf{wt}_3(4,4,3,1) \quad = 2.$$

The extended affine Weyl group $\widehat{\mathbf{W}}_\ell$

Given any nonempty set X, the symmetric group $\mathfrak{S}_\ell$ on ℓ letters has a natural right place permutation action on X^ℓ via

$$(x_1, \dots, x_\ell)^\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(\ell)}) \qquad (\sigma \in \mathfrak{S}_\ell).$$

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This right action gives rise to the extended affine Weyl group $\widehat{\mathbf{W}}_\ell = \mathbb{Z}^\ell \rtimes \mathfrak{S}_\ell$, which has a natural right action on the pairs of ℓ-multipartitions and their respective associated ℓ-multicharges via

$$(\boldsymbol{\lambda}; \mathbf{s})^{\mathbf{t}\sigma} = (\boldsymbol{\lambda}^\sigma; (\mathbf{s} + e\mathbf{t})^\sigma) \qquad (\mathbf{t} \in \mathbb{Z}^\ell, \sigma \in \mathfrak{S}_\ell).$$

Let $\overline{\mathscr{A}_e^\ell} = \{(s_1, \dots, s_\ell) \in \mathbb{Z}^\ell : s_1 \leq s_2 \leq \cdots \leq s_\ell \leq s_1 + e\}$.

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Theorem (Li-T.)

Let $\boldsymbol{\lambda}$ be an ℓ-multipartition and $\mathbf{s}$ be an ℓ-multicharge. Let $(\boldsymbol{\mu}; \mathbf{t}) \in (\boldsymbol{\lambda}; \mathbf{s})^{\widehat{\mathbf{W}}_\ell}$, the $\widehat{\mathbf{W}}_\ell$-orbit of $(\boldsymbol{\lambda}; \mathbf{s})$.

1. $\mathrm{core}_e(\boldsymbol{\mu}; \mathbf{t}) = \mathrm{core}_e(\boldsymbol{\lambda}; \mathbf{s})$.
2. $\mathrm{wt}_e(\boldsymbol{\mu}; \mathbf{t}) = \min(\mathrm{wt}_e((\boldsymbol{\lambda}; \mathbf{s})^{\widehat{\mathbf{W}}_\ell}))$ *if and only if* $\mathbf{t} \in \overline{\mathscr{A}_e^\ell}$.

Ariki-Koike Algebras

Let $\mathbf{r} = (r_1, \ldots, r_\ell) \in \mathbb{Z}^\ell$ and $n \in \mathbb{Z}^+$.
Let $\mathbb{F}$ be a field of characteristic p ($p = 0$ or prime), and $q \in \mathbb{F}$ with either $q = 1_\mathbb{F}$ or q is a primitive e-th root of $1_\mathbb{F}$.

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The Ariki-Koike algebra $\mathscr{H}_n = \mathscr{H}_{\mathbb{F},q,\mathbf{r}}(n)$ is the unital $\mathbb{F}$-algebra generated by $\{T_0, T_1, \ldots, T_{n-1}\}$ subject to:

$$\begin{aligned}
(T_0 - q^{r_1})(T_0 - q^{r_2}) \cdots (T_0 - q^{r_\ell}) &= 0; \\
(T_i - q)(T_i + 1) &= 0 && (1 \le i \le n-1); \\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0; \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && (1 \le i \le n-2); \\
T_i T_j &= T_j T_i && (|i-j| \ge 2).
\end{aligned}$$

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T_0T_1T_0T_1 &= T_1T_0T_1T_0; \\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && (1 \le i \le n-2); \\
T_iT_j &= T_jT_i && (|i-j| \ge 2).
\end{aligned}$$

When $\ell = 1$, $\mathscr{H}_n$ is the Iwahori-Hecke algebra of type A.
When $\ell = 2$, $\mathscr{H}_n$ is the Iwahori-Hecke algebra of type B.

Blocks of $\mathscr{H}_n$

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The residue of $(a, b, j) \in [\boldsymbol{\lambda}]$ is the residue class of $b - a + r_j$ modulo e, and (a, b, j) is called an i-**node** if its residue equals i.

Theorem (Lyle-Mathas 07)

Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be ℓ-multipartitions of n. The Specht modules $S^{\boldsymbol{\lambda}}$ and $S^{\boldsymbol{\mu}}$ lie in the same block of $\mathscr{H}_n$ if and only if $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same number of i-nodes for all $i \in \mathbb{Z}/e\mathbb{Z}$.

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Theorem (James)

($\ell = 1$) The partitions λ and μ have the same number of i-nodes for all $i \in \mathbb{Z}/e\mathbb{Z}$ if and only if λ and μ have the same e-core and the same e-weight.

Weights of Multipartitions

Definition (Fayers 06)

Let $\boldsymbol{\lambda}$ be an ℓ-multipartition of n. Define

$$\mathrm{wt}_{\mathscr{H}}(\boldsymbol{\lambda}) = \sum_{j=1}^{\ell} c_{\overline{r_j}}(\boldsymbol{\lambda}) + \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (c_i(\boldsymbol{\lambda}) - c_{i+1}(\boldsymbol{\lambda}))^2,$$

where $c_i(\boldsymbol{\lambda})$ is the number of i-nodes in $[\boldsymbol{\lambda}]$, and $\overline{r_j}$ is the residue class of r_j modulo e.

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Theorem (Fayers 06)

$\mathrm{wt}_{\mathscr{H}}(\boldsymbol{\lambda})$ *is a block invariant,* $\mathrm{wt}_{\mathscr{H}}(\boldsymbol{\lambda}) \in \mathbb{Z}_{\geq 0}$, *and when* $\ell = 1$, $\mathrm{wt}_{\mathscr{H}}(\lambda) = \mathrm{wt}_e(\lambda)$.

Theorem (Jacon-Lecouvey 21)

If $\mathbf{r} \in \overline{\mathscr{A}_e^{\ell}}$, *then*

$$\mathsf{wt}_{\mathscr{H}}(\boldsymbol{\lambda}) = \mathsf{wt}_e(\mathsf{U}_e(\boldsymbol{\lambda}; \mathbf{r})) \ (= \mathsf{wt}_e(\boldsymbol{\lambda}; \mathbf{r})).$$

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Corollary (Li-T.)

$$\mathsf{wt}_{\mathscr{H}}(\boldsymbol{\lambda}) = \min(\mathsf{wt}_e((\boldsymbol{\lambda}; \mathbf{r})^{\widehat{\mathbf{W}}_\ell})).$$

Cores of Multipartitions

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Corollary (Nakayama's 'Conjecture' for Ariki-Koike algebras)

Two Specht modules $S^{\boldsymbol{\lambda}}$ and $S^{\boldsymbol{\mu}}$ (possibly of different algebras) lie in the same block if and only if $\mathrm{core}_e(\boldsymbol{\lambda};\mathbf{r}) = \mathrm{core}_e(\boldsymbol{\mu};\mathbf{r})$ and $\min(\mathrm{wt}_e((\boldsymbol{\lambda};\mathbf{r})^{\widehat{\mathbf{W}}_\ell})) = \min(\mathrm{wt}_e((\boldsymbol{\mu};\mathbf{r})^{\widehat{\mathbf{W}}_\ell}))$.