On Amiot's conjecture

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Cluster algebras and cluster categories

Recall that a cluster algebra (Fomin–Zelevinsky, 2002) is a commutative algebra endowed with a family of distinguished generators, the *cluster variables*, which are assembled into finite sets of fixed cardinality called the *clusters*. The clusters are constructed recursively starting from the datum of a *quiver* (=oriented graph). In additive categorification, we aim at constructing, from the quiver Q of a cluster algebra $\mathcal{A}$, a triangulated category $\mathcal{C}_Q$ and a decategorification map $\chi\colon \mathcal{C}_Q \to \mathcal{A}$ which associates elements of $\mathcal{A}$ with the objects in $\mathcal{C}_Q$ so as to establish as close a correspondence as possible between the combinatorics of the indecomposable rigid objects in $\mathcal{C}_Q$ and those of the cluster variables in $\mathcal{A}$.

For an acyclic quiver Q, the construction of the corresponding *cluster category* $\mathcal{C}_Q$ is due to Buan–Marsh–Reineke–Reiten–Todorov. As shown by work of Derksen–Weyman–Zelevinsky, in the general case of a quiver admitting oriented cycles, in addition to the quiver Q, we need to consider a (non-degenerate) *potential*, i.e. a formal linear combination W of cycles of Q. In this more general case, the construction of the corresponding cluster category $\mathcal{C}_{Q,W}$ is due to Amiot and that of the decategorification map $\chi \colon \mathcal{C}_{Q,W} \to \mathcal{A}$ to Caldero–Chapoton, Palu, Plamondon and others.

Conjecture 1.1 (Amiot, 2010)

Each 2-Calabi–Yau triangulated category with a cluster-tilting object comes from a quiver with potential.

Aim: Explain a variant of the conjecture and sketch its proof.

Plan:

1. From preprojective algebras to Amiot's conjecture
2. Van den Bergh's superpotential theorem
3. Sketch of proof

1. From preprojective algebras to Amiot's conjecture

Fix the field $k = \mathbb{C}$ for simplicity. Let Δ be an *ADE* Dynkin diagram and Q an orientation of Δ, e.g. $Q = \overrightarrow{A_3}: \ \ 1 \xrightarrow{\beta} 2 \xrightarrow{\alpha} 3$.

In 1976, Gelfand–Ponomatev defined the preprojective algebra Λ_Q of Q over k, e.g. $1 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 2 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 3$ with relation $\sum_{\gamma \in Q_1} [\gamma, \gamma^*] = 0$ or with $-\beta^*\beta = 0$, $\beta\beta^* - \alpha^*\alpha = 0$, $\alpha\alpha^* = 0$.

Remark 1.1

1) Λ_Q *is finite-dimensional and selfinjective, i.e. injective as a right module over itself. So the category* $\operatorname{mod}\Lambda_Q$ *of finite dimensional right* Λ_Q*-modules is a Frobenius category and its stable category* $\underline{\operatorname{mod}}\,\Lambda_Q$ *is triangulated (Happel, 1986).*

2) $\underline{\operatorname{mod}}\,\Lambda_Q$ *is* 2*-Calabi–Yau as a triangulated category (Crawley-Boevey, 2000), i.e. we have*

$$\operatorname{Ext}^1(L,M) \xrightarrow{\sim} D\operatorname{Ext}^1(M,L)$$

for all L *and* M *in* $\underline{\operatorname{mod}}\,\Lambda_Q$*, where* $D = \operatorname{Hom}_k(?,k)$.

Remark 1.2

3) *Λ_Q is wild except if $\Delta \in \{A_1, A_2, A_3, A_4, D_4, A_5\}$ but it is always 2-representation-finite (in the sense of Iyama, 2007), i.e.* $\underline{\mathrm{mod}}\,\Lambda_Q$ *contains a (canonical) 2-cluster-tilting object T_{can} (constructed by Geiss–Leclerc–Schröer, 2006).*

Definition 1.1

An object $T \in \underline{\mathrm{mod}}\,\Lambda_Q$ is 2-cluster-tilting if

a) *T is rigid, i.e.* $\mathrm{Ext}^1(T, T) = 0$,

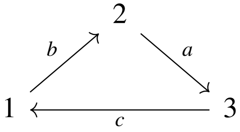
b) *T is a 2-step generator of $\underline{\mathrm{mod}}\,\Lambda_Q$, i.e. for any $M \in \underline{\mathrm{mod}}\,\Lambda_Q$, there is a triangle*

$$T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow \Sigma T_1$$

with T_0, $T_1 \in \mathrm{add}\, T$.

For example, if $Q = \overrightarrow{A_3}$, then $\underline{\mathrm{End}}_{\Lambda_Q} T$ is given by

$R =$

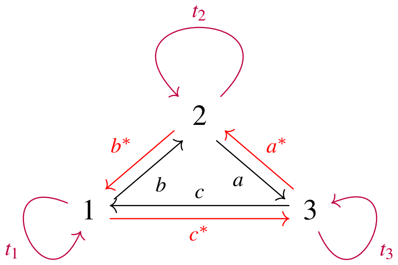


with relations $ab = 0$, $bc = 0$, $ca = 0$.

This means that $\underline{\mathrm{End}}_{\Lambda_Q} T = J_{R,W}$ is the Jacobian algebra of (R, W), where $W = abc$ gives the above relations.

The Jacobian algebra has a dg refinement, called the Ginzburg dg algebra $\Gamma_{R,W}$, i.e. the completed graded path algebra of

$\tilde{R} =$



with grading $|a^*| = |b^*| = |c^*| = -1$, $|t_i| = -2$ and differential determined by $d(a^*) = \partial_a W = bc$, $d(t_1) = cc^* - b^*b$, etc.

Theorem 1.1 (Amiot, 2009)

We have a canonical triangle equivalence $\mathcal{C}_{R,W} \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda_Q$ *taking* $\Gamma_{R,W}$ *to* T_{can}*, where* $\mathcal{C}_{R,W}$ *is the (generalized) cluster category* $\mathcal{C}_{R,W} = \mathrm{per}\,\Gamma_{R,W}/\mathrm{pvd}\,\Gamma_{R,W}$.

Here $\mathrm{per}\,\Gamma_{R,W} \subseteq \mathcal{D}(\Gamma_{R,W})$ denotes the perfect derived category, i.e. the thick subcategory generated by $\Gamma_{R,W}$ and $\mathrm{pvd}\,\Gamma_{R,W} \subseteq \mathcal{D}(\Gamma_{R,W})$ denotes the perfectly valued derived category, i.e. the full subcategory of dg modules whose homology is of finite total dimension.

Conjecture 1.2 (Amiot, 2010)

Let $\mathcal{C}$ be a Karoubian, Hom-*finite triangulated category such that*

a) *$\mathcal{C}$ is algebraic, i.e. $\mathcal{C}$ is triangle equivalent to $H^0(\mathcal{A})$ for some pretriangulated dg category $\mathcal{A}$,*

b) *$\mathcal{C}$ is* 2*-Calabi–Yau as a triangulated category,*

c) *$\mathcal{C}$ contains a cluster-tilting object T.*

Then there exists a quiver with potential (R, W) and a triangle equivalence $\mathcal{C}_{R,W} \xrightarrow{\sim} \mathcal{C}$ taking $\Gamma_{R,W}$ to T. In particular, we have an isomorphism $J_{R,W} \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}}(T)$.

Evidence:

1) It is true if $\mathrm{End}(T)$ is hereditary (Keller–Reiten, 2008).
2) It is true if $\mathcal{C} = \underline{\mathrm{mod}}\, \Lambda_Q$, $T = T_{can}$ as above.
3) It is true if $\mathcal{C} = \underline{\mathrm{cm}}\, (R^G)$, $R = k[\![x, y, z]\!]$, G is a suitable cyclic group (Amiot–Iyama–Reiten, 2011, Thanhoffer–Van den Bergh, 2015).

Incoherence in the conjecture: The Calabi–Yau structure should be given on $\mathcal{A}$, not on $H^0(\mathcal{A}) \simeq \mathcal{C}$!

Theorem 1.2 (Keller–L, 2023)

After this modification, the conjecture holds.

2. Van den Bergh's superpotential theorem (in dimension 3)

Theorem 1.3 (Van den Bergh, 2015)

Let A be a smooth connective complete dg algebra endowed with a left 3-Calabi–Yau structure. Then A is quasi-isomorphic to $\Gamma_{R,W}$ for a quiver with potential (R, W).

Remark 1.3

The converse also holds (Van den Bergh, 2011).

Terminology:

Smooth: A lies in $\operatorname{per} A^e$, where $A^e = A \otimes_k A^{op}$.

Connective: $H^p(A) = 0$ for all $p > 0$.

Complete: Pseudo-compact and augmented over its radical quotient (holds for completed connective dg path algebras).

Left 3-Calabi–Yau structure: Class $\beta \in HN_3(A)$ whose image under

$$HN_3(A) \longrightarrow HH_3(A) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}(A^e)}(\Sigma^3 A^\vee, A)$$

is an isomorphism, where $A^\vee = \operatorname{RHom}_{A^e}(A, A^e)$.

3. Sketch of proof

The given triangulated category $\mathcal{C}$ is Karoubian and Hom-finite. A right 2-Calabi–Yau structure on a given dg enhancement $\mathcal{C}_{dg}$ is a class $\alpha \in DHC_{-2}(\mathcal{C}_{dg})$ which is non-degenerate, i.e. its image in

$$DHH_{-2}(\mathcal{C}_{dg}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\mathcal{C}_{dg}^{e})}(\mathcal{C}_{dg}, \Sigma^{-2} D\mathcal{C}_{dg})$$

is an isomorphism.

Notice: $\mathcal{C}_{dg}$ has a right 2-Calabi–Yau structure but we need a dg algebra with a left 3-Calabi–Yau structure to apply Van den Bergh's superpotential theorem.

To construct: (R, W) such that we have an exact sequence

$$0 \longrightarrow \operatorname{pvd}\Gamma_{R,W} \longrightarrow \operatorname{per}\Gamma_{R,W} \longrightarrow \mathcal{C} \longrightarrow 0\,.$$

Let $\mathcal{E}$ be a Frobenius exact category whose stable category is $\mathcal{C}$ and $\mathcal{P} \subseteq \mathcal{E}$ the full subcategory of projective-injective objects. Denote by $\mathcal{M} \subseteq \mathcal{E}$ the closure under finite direct sums and direct summands of T and the projective-injective objects in $\mathcal{E}$.

Then we have the diagram (Palu, 2010)

$$\begin{array}{ccccc} & & \mathcal{H}^b(\mathcal{P}) & = & \mathcal{H}^b(\mathcal{P}) \\ & & \downarrow & & \downarrow \\ \mathcal{H}^b_{ac}(\mathcal{M}) & \longrightarrow & \mathcal{H}^b(\mathcal{M}) & \longrightarrow & \mathcal{D}^b(\mathcal{E}) \\ \| & & \downarrow & & \downarrow \\ \mathcal{H}^b_{ac}(\mathcal{M}) & \longrightarrow & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) & \longrightarrow & \mathcal{C}\,. \end{array}$$

We define Γ to be the endomorphism algebra of the image of T in the dg enhancement of $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$. Then the bottom row becomes

$$0 \longrightarrow \operatorname{per}\Gamma^! \longrightarrow \operatorname{per}\Gamma \longrightarrow \mathcal{C} \longrightarrow 0\,.$$

We obtain an exact sequence

$$HC_{-2}(\Gamma) \longrightarrow HC_{-2}(\mathcal{C}_{dg}) \longrightarrow HC_{-3}(\Gamma^!) \longrightarrow HC_{-3}(\Gamma)$$

in cyclic homology whose leftmost and rightmost terms vanish. Then the preimage β of the given right 2-Calabi–Yau structure α under the composed isomorphism

$$HN_3(\Gamma) \xrightarrow{\sim} DHC_{-3}(\Gamma^!) \xrightarrow{\sim} DHC_{-2}(\mathcal{C}_{dg})$$

is a candidate for a left 3-Calabi–Yau structure on Γ.

Subtle point: α is non-degenerate implies that β is non-degenerate!

Then by Van den Bergh's superpotential theorem, the dg algebra Γ is isomorphic to $\Gamma_{R,W}$ for some quiver with potential (R, W).

It follows that $\mathcal{C}$ is triangle equivalent to the cluster category $\operatorname{per}\Gamma_{R,W}/\operatorname{pvd}\Gamma_{R,W}$.

Remark 1.4

All constructions and results generalize from dimension 2 to dimension $d \geq 2$.

Summary

We characterize generalized cluster categories as algebraic triangulated categories with an *algebraic* 2-CY structure and a cluster-tilting object.

References

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3. Michel Van den Bergh, *Calabi-Yau algebras and superpotentials*, Selecta Math. (N.S.) **21** (2015), no. 2, 555–603.

Thanks for your attention!