Homotopy equivalences over rings with finite Gorenstein weak global dimension

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Outline

Gorenstein projective and Gorenstein flat modules

• Let R be a ring. Denote by R-Mod the category of left R-modules and by $\mathcal{GP}$ (resp. $\mathcal{GF}$ and $\mathcal{GI}$) its subcategory consisting of all Gorenstein projective (resp. Gorenstein flat and Gorenstein injective) modules.

• A left R-module M is called **Gorenstein projective** (resp. **Gorenstein flat**) if there exists a totally acyclic complex $P^\bullet$ of projective left R-modules(resp. an **F**-totally acyclic complex $P^\bullet$ of flat left R-modules)such that $M = \mathrm{Z}^0(P^\bullet)$.

• A complex $P^\bullet$ of projective left R-modules is called **totally acyclic** if $\mathrm{Hom}_R(P^\bullet, Q)$ is acyclic for any projective left R-module Q; A complex $P^\bullet$ of flat left R-modules is called **F-totally acyclic** if $I \otimes_R P^\bullet$ is acyclic for any injective right R-module I.

Gorenstein projective and Gorenstein flat modules

- It is an open problem whether or not $\mathcal{GP} \subseteq \mathcal{GF}$ for any ring R.

Proposition (W E) Let R be a **left virtually Gorenstein ring** (that is, R satisfies $\mathcal{GP}^{\perp} = {}^{\perp}\mathcal{GI}$). Then $\mathcal{GP} \subseteq \mathcal{GF}$.

Gorenstein projective dimension and Gorenstein flat dimension

- Let R be a ring and M a left R-module. As usual, the notations $\mathrm{Gpd}_R(M)$ and $\mathrm{Gfd}_R(M)$ stand for the Gorenstein projective and Gorenstein flat dimensions of M, respectively. That is, the $\mathcal{GP}$-resolution and $\mathcal{GF}$-resolution dimensions of M, respectively.

- It is an open problem whether or not $\mathcal{GP} \subseteq \mathcal{GF}$ for any ring R. So we do not know that

$$\mathrm{Gfd}_R(M) \leq \mathrm{Gpd}_R(M)$$

hold in general.

Gorenstein (weak) global dimension of rings

If we take the supremum from all modules, we can get two invariants of R:

$$\mathrm{Ggldim}(R) \overset{df}{=} \sup\{\mathrm{Gpd}_R(M) \mid M \text{ is a left } R\text{-module}\}$$

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• They are a refinement of $\mathrm{gldim}(R)$ and $\mathrm{wgldim}(R)$ respectively.
• It is know that $\mathrm{wgldim}(R) \leq \mathrm{gldim}(R)$, the key is $\mathcal{P} \subseteq \mathcal{F}$.
• Although we do not know whether or not $\mathcal{GP} \subseteq GF$, using different methods, we (CELTW; WYSZ) obtain an inequality as follows.

$$\mathrm{Gwgldim}(R) \leq \mathrm{Ggldim}(R).$$

Gorenstein (weak) global dimension of rings, continued

- Note that the inequality

$$\mathrm{Gwgldim}(R) \leq \mathrm{Ggldim}(R)$$

can be strict and even can form an extreme.

- For example, let $R = F_\alpha$ be the free Boolean ring on $\aleph_\alpha$ generators with α an infinite cardinality. Then one has $\mathrm{Gwgldim}(R) = 0$ and $\mathrm{Ggldim}(R) = \infty$.

Three different generalizations of Gorenstein rings

A ring R is called *Gorenstein* (resp. *Ding-Chen*) if it is a two-sided Noetherian (resp. two-sided coherent) ring with finite self-injective (self-FP-injective) dimension on both sides; A ring R is called *left Gorenstein* if $\mathrm{Ggldim}(R) < \infty$.

Three different generalizations of Gorenstein rings

• We summarize that generalizations of Gorenstein rings go in three different directions:

$$\{Gorenstein\ rings\} \xrightarrow{\subsetneq} \{two-sided\ Noetherian\ rings\ with\ a\ duality\ complex\}$$

$$\{Gorenstein\ rings\} \xrightarrow{\subsetneq} \{left\ Gorenstein\ rings\} \xrightarrow{\subsetneq} \{R\ with\ \mathrm{Gwgldim}(R) < \infty\}$$

$$\{Gorenstein\ rings\} \xrightarrow{\subsetneq} \{Ding-Chen\ rings\} \xrightarrow{\subsetneq} \{R\ with\ \mathrm{Gwgldim}(R) < \infty\}.$$

In summary, the second and the third ones all admit finite Gorenstein weak global dimension, and hence are left and right virtually Gorenstein.

Iyengar-Krause's homotopy equivalence

• As usual, denote by K(R-GProj) (resp. K(R-GInj)) the homotopy category of Gorenstein projective (resp. Gorenstein injective) left R-modules. Its homotopy subcategory consisting of projective (resp. injective) left R-modules is denoted by K(R-Proj) (resp. K(R-Inj)).

• We also denote by K_{ac}(R-Proj (resp. K_{ac}(R-Inj) the homotopy subcategory of K(R-Proj) (resp. K(R-Inj) consisting of exact complexes of projective (resp. injective) left R-modules.

Iyengar-Krause's homotopy equivalence

Theorem A Let R be a commutative Noetherian ring with a duality complex D, then there is a triangle equivalence

$$\mathrm{K}(R\text{-Proj}) \simeq \mathrm{K}(R\text{-Inj})$$

Asadollahi, Hafezi and Salarian (2014) proved that

Let R be a commutative Noetherian ring with a duality complex D, then there is a triangle equivalence $\mathrm{K}(R\text{-GProj}) \simeq \mathrm{K}(R\text{-GInj})$ which restricts to a triangle equivalence $\mathrm{K}(R\text{-Proj}) \simeq \mathrm{K}(R\text{-Inj})$.

Chen's homotopy equivalence

Using the tools of balanced pairs, Chen (2010) obtained a parallel result.

Theorem B Let R be a left Gorenstein ring (that is, $\mathrm{Ggldim}(R) < \infty$). Then there is a triangle equivalence $\mathrm{K}(R\text{-GProj}) \simeq \mathrm{K}(R\text{-GInj})$ which restricts to a triangle equivalence $\mathrm{K}(R\text{-Proj}) \simeq \mathrm{K}(R\text{-Inj})$.

For commutative Gorenstein rings, Chen's equivalence extends Iyengar-Krause's homotopy equivalence.

The question

Question Does the same result hold for any Ding-Chen ring?

A general result

Using the tools of torsion pairs (or stable t-structures) of triangulated categories,
Theorem 0 Let R be a left virtually Gorenstein ring (i.e., $\mathcal{GP}^{\perp} = {}^{\perp}\mathcal{GI}$). Then we have a triangle equivalence

$$\mathrm{K}(\mathsf{dgGProj}) \simeq \mathrm{K}(\mathsf{dgGInj})$$

Affirmative answer to the question

Recall that a ring R is Ding-Chen if and only if R is a two-sided coherent ring with $\mathrm{Gwgldim}(R) < \infty$.

Theorem C Let R be a ring with $\mathrm{Gwgldim}(R) < \infty$. Then there is a triangle equivalence $\mathrm{K}(R\text{-GProj}) \simeq \mathrm{K}(R\text{-GInj})$ which restricts to a triangle equivalence $\mathrm{K}(R\text{-Proj}) \simeq \mathrm{K}(R\text{-Inj})$.

The use of finiteness of Gorenstein weak global dimension

• $\mathsf{Gwgldim}(R) < \infty$ implies that R is left (and right) virtually Gorenstein, so Theorem 0 applies to obtain

$$\mathrm{K}(\mathsf{dgGProj}) \simeq \mathrm{K}(\mathsf{dgGInj}).$$

• $\mathsf{Gwgldim}(R) < \infty$ also implies that

$$\mathrm{K}(\mathsf{dgGProj}) = \mathrm{K}(R\text{-}\mathsf{GProj}) \textit{ and } \mathrm{K}(\mathsf{dgGInj}) = \mathrm{K}(R\text{-}\mathsf{GInj}).$$

• the restrictness is easy.

Compare Theorem B and Theorem C

• Applying Theorem C, one can obtain a triangle equivalence $\mathrm{K}_{\mathrm{ac}}(R\text{-Proj}) \simeq \mathrm{K}_{\mathrm{ac}}(R\text{-Inj})$.
• Using symmetry of Gwgldim(R), one can also obtain the triangle equivalences for right R-modules from Theorem C.

An application of Theorem C

- Let R be a Ding-Chen ring. Then we have a triangle equivalence:

$$\mathrm{K}(R\text{-FlatCot}) \simeq \mathrm{K}(R\text{-FInjFProj})$$

Thank you!