(De)Coloring in operad theory with applications to homotopy theory of relative Rota-Baxter algerbas

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Part I: Hochschild cohomology

Let $\mathbf{k}$ be a field of characteristic zero. Let (A, μ) be an associative algebra and M be an A-bimodule.

Definition (Hochschild 1945)

The **Hochschild cochain complex** *of A over M is* $(C^\bullet_{\mathrm{Alg}}(A, M), \partial^\bullet_{\mathrm{Alg}})$ *with*

- $C^n_{\mathrm{Alg}}(A, M) = \mathrm{Hom}(A^{\otimes n}, M), n \geq 0,$
- $\partial^n_{\mathrm{Alg}}(f)(a_1 \otimes \cdots \otimes a_{n+1})$

$$= (-1)^{n-1} a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) + f(a_1 \otimes \cdots \otimes a_n) a_{n+1} + \sum_{i=1}^{n} (-1)^{n-i+1} f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}).$$

In particular, we write $C^\bullet_{\mathrm{Alg}}(A) := C^\bullet_{\mathrm{Alg}}(A, A)$.

Definition (Hochschild cohomology)

For $n \geq 0$, the n-th **Hochschild cohomology group** *of A is defined to be*

$$\mathrm{H}^n_{\mathrm{Alg}}(A) = \mathrm{H}^n(C^\bullet_{\mathrm{Alg}}(A)).$$

Part I: Gerstenhaber Lie bracket

For a $\mathbb{Z}$-graded vector space V, define sV as $(sV)_p = V_{p-1}, p \in \mathbb{Z}$.

Theorem (Gerstenhaber 1963)

For an associative algebra A, $sC^\bullet_{\mathrm{Alg}}(A)$ together with differential $-\partial^\bullet_{\mathrm{Alg}}$ and the Gerstenhaber Lie bracket $[\ ,\ ]_G$ is a **dg Lie algebra** *(called Gerstenhaber dg Lie algebra).*

Gerstenhaber Lie bracket: $[f, g]_G = f\bar{\circ}g - (-1)^{(n-1)(m-1)}g\bar{\circ}f$ with

$$f\bar{\circ}g = \sum_{i=1}^{n}(-1)^{(i-1)(m-1)}f(\mathrm{Id}^{\otimes i-1}\otimes g\otimes \mathrm{Id}^{\otimes n-i}).$$

M. Gerstenhaber, *The cohomology structure of an associative ring*. Ann. Math. (2) **78** (1963) 267-288.

Part I: Differential graded Lie algebras

Definition (Differential graded Lie algebra)

A **differential graded Lie algebra** *(aka dgla) is a triple* (L, l_1, l_2)*, where* $L = \bigoplus_{i \in \mathbb{Z}} L_i$ *is a graded space,* $l_1 : L \to L$ *and* $l_2 : L \otimes L \to L$ *such that*

(i) (L, l_1) *is a cochain complex,*

(ii) (L, l_2) *is a graded Lie algebra,*

(iii) l_1 *is a derivation with respect to* $l_2 = [\ ,\ ]$*. i.e.*
$l_1[a, b] = [l_1(a), b] + (-1)^{|a|}[a, l_1(b)]$.

Part I: Differential graded Lie algebras

Definition (Maurer-Cartan elements)

Let L be a dgla. An element $\alpha \in L_{-1}$ is a **Maurer-Cartan element** *if*

$$l_1(\alpha) - \frac{1}{2} l_2(\alpha \otimes \alpha) = 0.$$

Proposition (Twisting procedure)

Let L be a dgla. Given a Maurer-Cartan element $\alpha \in L_{-1}$, it produces a **new dgla** $L^\alpha = (L, l_1^\alpha, l_2^\alpha)$ *by imposing*

$$l_1^\alpha(x) = l_1(x) - l_2(\alpha \otimes x) \quad \text{and} \quad l_2^\alpha(x \otimes y) = l_2(x \otimes y).$$

Part I: Algebraic deformation theory of associative algebras

Theorem

(i) *Even if V is only a (ungraded) vector space, the Hochschild cochain complex (with zero differential)*

$$\mathfrak{C}_{Alg}(V) = \prod_{n=0}^{\infty} \mathrm{Hom}((sV)^{\otimes n}, sV)$$

is still a **graded Lie algebra** *(called Gerstenhaber Lie algebra) endowed with the Gerstenhaber Lie bracket.*

(ii) *There is a 1-1 correspondence*

$$\left\{ \begin{array}{c} \text{Maurer-Cartan elements} \\ \text{in } \mathfrak{C}_{Alg}(V) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Associative algebra} \\ \text{structures on } V \end{array} \right\}.$$

(iii) *Let μ be an associative algebra structure on V, the twisted dgla $\mathfrak{C}_{Alg}(V)^{\mu}$ is exactly the Gerstenhaber dg Lie algebra $sC^{\bullet}_{\mathrm{Alg}}(V)$.*

Part I: A_∞-algebras

Definition (Homotopy associative algebras)

Let V be a graded vector space. A **homotopy associative algebra structure** *on graded space V is defined to be a Maurer-Cartan element in the graded Lie algebra*

$$\overline{\mathfrak{C}_{Alg}}(V) = \prod_{n=1}^{\infty} \mathrm{Hom}((sV)^{\otimes n}, sV).$$

Equivalently, this definition coincides with the following.

Definition (Stasheff 1963)

An A_∞**-algebra structure** *on a graded space V consists of $\{m_n : V^{\otimes n} \to V\}_{n\geqslant 1}$ with $|m_n| = n-2$ and the* **Stasheff identities***:*

$$\sum_{i+j+k=n, i,k\geq 0, j\geq 1} (-1)^{i+jk} m_{i+1+k} \circ (\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes k}) = 0, \forall n \geqslant 1.$$

Stasheff identities:

$$\sum_{i+j+k=n, i,k\geq 0, j\geq 1} (-1)^{i+jk} m_{i+1+k} \circ (\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes k}) = 0, \forall n \geqslant 1.$$

- $n = 1$, $m_1 \circ m_1 = 0$ with $|m_1| = -1$, i.e. m_1 is a differential;
- $n = 2$, $m_1 \circ m_2 = m_2 \circ (\mathrm{id} \otimes m_1 + m_1 \otimes \mathrm{id})$, i.e. m_1 is a derivation with respect to m_2;
- $n = 3$, $m_2 \circ (m_2 \otimes \mathrm{id}) - m_2 \circ (\mathrm{id} \otimes m_2) =$ $-(m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathrm{id}^{\otimes 2} + \mathrm{id} \otimes m_1 \otimes \mathrm{id} + m_1 \otimes \mathrm{id}^{\otimes 2})) = -\partial(m_3)$, i.e. m_2 is associative up to homotopy.

J. D. Stasheff, *Homotopy associativity of H -spaces. I, II*. Trans. Amer. Math. Soc. **108** (1963), 275-292; ibid. 108 1963 293-312.

Part I: Operad of A_∞-algebras

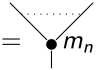
- Stasheff identities:

$$\sum_{i+j+k=n,i,k\geq 0,j\geq 1} (-1)^{i+jk} m_{i+1+k} \circ (\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes k}) = 0, \forall n \geqslant 1.$$

We rewrite the Stasheff identities as follows: for $n \geqslant 2$,

$$\partial(m_n) := m_1 \circ m_n - (-1)^{n-2} m_n \circ m_1 = \sum_{j=2}^{n-1} \sum_{i=1}^{n-j+1} \pm m_{n-j+1} \circ_i m_j.$$

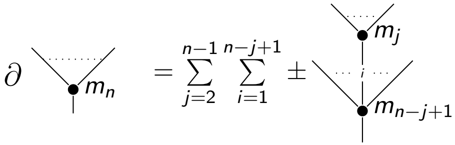
In operads, we present $m_n : A^{\otimes n} \to A$ as the following tree (with n leaves):

$m_n =$ 

Part I: Operad of A_∞-algebras

$$\partial(m_n) := m_1 \circ m_n - (-1)^{n-2} m_n \circ m_1 = \sum_{j=2}^{n-1} \sum_{i=1}^{n-j+1} \pm m_{n-j+1} \circ_i m_j.$$

Then the **Stasheff identities** can be presented by the following.



Definition (Operad of A_∞-algebras)

The **dg operad of $\mathbf{A}_\infty$-algebra**, *denoted by $\mathcal{A}ss_\infty$, is the free graded operad generated by $m_n, n \geq 2$ with $|m_n| = n - 2$ endowed with the above differential.*

Part I: Minimal models

Theorem (Ginzburg-Kapranov 94)

The operad $\mathcal{A}ss$ *is* **Koszul** *and*

$$\mathcal{A}ss_\infty \cong \Omega(\mathcal{A}ss^{\text{i}}),$$

where $\mathcal{A}ss^{\text{i}}$ *is its* **Koszul dual cooperad**. *The dg operad* $\mathcal{A}ss_\infty$ *is called the* **minimal model** *of* $\mathcal{A}ss$.

V. Ginzburg, M. Kapranov, *Koszul duality for operads.* Duke Math. J. **76** (1994), no. 1, 203-272.

Part I: Minimal models

Theorem

Let V be a vector space. Then

$$\mathrm{Hom}(\mathcal{A}ss^{\mathsf{i}}, \mathrm{End}_V) = \prod_{n=1}^{\infty} \mathrm{Hom}(\mathcal{A}ss^{\mathsf{i}}(n), \mathrm{End}_V(n))$$

has a graded Lie algebra structure, which is exactly the **Gerstenhaber Lie algebra**.

M. Kontsevich and Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Conférence Moshé Flato 1999, Vol. I (Dijon), Math. Phys. Stud. 21 (2000), 255-307.

Part II: Strategy

Four steps:

formal deformations

$\Downarrow$

deformation complex

$\Downarrow$

L_∞−structure $\dashleftarrow$ derived bracket

(Chengming Bai, Li Guo, Yunhe Sheng et al.)

$\Updownarrow$

minimal model

Part II: Rota-Baxter Lie algebras

Definition (Rota-Baxter operators)

Let $(\mathfrak{g}, \ell = [-,-])$ *be a Lie algebra. A linear map* $T : \mathfrak{g} \to \mathfrak{g}$ *is said to be a* **Rota-Baxter operator of weight** λ *if it satisfies*

$$[T(a), T(b)] = T([a, T(b)] + [T(a), b] + \lambda[a, b]),$$

or equivalently,

$$\ell \circ (T \otimes T) = T \circ \ell \circ (\mathrm{Id}_{\mathfrak{g}} \otimes T + T \otimes \mathrm{Id}_{\mathfrak{g}} + \lambda \cdot \mathrm{Id}_{\mathfrak{g}} \otimes \mathrm{Id}_{\mathfrak{g}}). \qquad (1)$$

The triple $(\mathfrak{g}, \ell, T)$ *is called a Rota-Baxter Lie algebra.*

Remark

The operad of Rota-Baxter Lie algebras is **NOT** *Koszul as relation* (1) *contains terms of degree 3:*

$$\ell \circ (T \otimes T),\ T \circ \ell \circ (\mathrm{Id}_{\mathfrak{g}} \otimes T) \text{ and } T \circ \ell \circ (T \otimes \mathrm{Id}_{\mathfrak{g}}).$$

Part II: Rota-Baxter Lie modules

Definition (Rota-Baxter Lie modules)

Let $(\mathfrak{g}, \ell, T)$ *be a Rota-Baxter Lie algebra with weight* λ*. A* **Rota-Baxter Lie module** *over* $\mathfrak{g}$ *is a vector space* M *such that*

(i) M *is a representation for the Lie algebra* $(\mathfrak{g}, \ell)$*;*

(ii) *there is a linear map* $T_M : M \to M$ *such that*

$$T(a)T_M(m) = T_M\big(aT_M(m) + T(a)m + \lambda am\big)$$

for $a \in \mathfrak{g}$*,* $m \in M$*.*

Part II: Descendent properties

$$[T(a), T(b)] = T(\underline{[a, T(b)] + [T(a), b] + \lambda[a, b]})$$

Proposition

(i) *Let* $(\mathfrak{g}, \ell, T)$ *be a Rota-Baxter Lie algebra with weight* λ. *Define*

$$[a, b]_\star := [a, T(b)] + [T(a), b] + \lambda[a, b]$$

for any $a, b \in \mathfrak{g}$. *Then* $\mathfrak{g}_\star = (\mathfrak{g}, [-, -]_\star, T)$ *is a Rota-Baxter Lie algebra.*

(ii) *Let* M *be a Rota-Baxter Lie module over* $(\mathfrak{g}, \ell, T)$. *Define*

$$a \triangleright m : \quad = \quad T(a)m - T(am).$$

Then " $\triangleright$ *" make* M *into a Rota-Baxter Lie module over* $\mathfrak{g}_\star$ *which is denoted by "* ${}_\triangleright M$*".*

Part II: Cohomology theory of Rota-Baxter operators

Definition (Cohomology of Rota-Baxter operators)

Let $(\mathfrak{g}, \ell, T)$ be a Rota-Baxter Lie algebra with weight λ. Then the Chevalley-Eilenberg cochain complex of $\mathfrak{g}_\star$ over ${}_\triangleright\mathfrak{g}$

$$C^\bullet_{\mathrm{RBO}}(\mathfrak{g}) := (C^\bullet_{\mathrm{Lie}}(\mathfrak{g}_\star, {}_\triangleright\mathfrak{g}), \partial)$$

is called the **cochain complex of Rota-Baxter operator** *T.*

Its cohomology, denoted by

$$\mathrm{H}^\bullet_{\mathrm{RBO}}(\mathfrak{g})$$

is called the **cohomology of Rota-Baxter operator** *T.*

Proposition

$\mathrm{H}^\bullet_{\mathrm{RBO}}(\mathfrak{g})$ *controls the infinitesimal deformation of T.*

Part II: Cohomology theory of Rota-Baxter Lie algebras

Next, we construct a chain map $\phi^\bullet : C^\bullet_{\mathrm{Lie}}(\mathfrak{g}) \to C^\bullet_{\mathrm{RBO}}(\mathfrak{g})$ as follows:

$$\begin{array}{cccccccccc}
C^\bullet_{\mathrm{Lie}}(\mathfrak{g}): & 0 & \longrightarrow & \mathrm{Hom}(k,\mathfrak{g}) & \xrightarrow{d^0} & \mathrm{Hom}(\mathfrak{g},\mathfrak{g}) & \cdots & \mathrm{Hom}(\wedge^n\mathfrak{g},\mathfrak{g}) & \xrightarrow{d^n} & \mathrm{Hom}(\wedge^{n+1}\mathfrak{g},\mathfrak{g}) \cdots \\
\downarrow \phi^\bullet & & & \downarrow \phi^0 & & \downarrow \phi^1 & & \downarrow \phi^n & & \downarrow \phi^{n+1} \\
C^\bullet_{\mathrm{RBO}}(\mathfrak{g}): & 0 & \longrightarrow & \mathrm{Hom}(k,\mathfrak{g}) & \xrightarrow{\partial^0} & \mathrm{Hom}(\mathfrak{g},\mathfrak{g}) & \cdots & \mathrm{Hom}(\wedge^n\mathfrak{g},\mathfrak{g}) & \xrightarrow{\partial^n} & \mathrm{Hom}(\wedge^{n+1}\mathfrak{g},\mathfrak{g}) \cdots
\end{array}$$

- For $n = 0$, define $\phi_0 = \mathrm{Id}_{\mathrm{Hom}(k,\mathfrak{g})}$;
- for $n \geqslant 1$, $f \in C^n_{\mathrm{Lie}}(\mathfrak{g})$, define $\phi^n(f)$ as

$$\begin{aligned}
&\phi^n(f)(a_1 \otimes \ldots \otimes a_n) \\
=&f(T(a_1)\otimes \ldots \otimes T(a_n)) \\
-&\sum_{k=0}^{n-1} \sum_{1\leqslant i_1 < \cdots < i_k \leqslant n} \lambda^{n-k-1} T \circ f(a_1 \otimes \ldots \otimes T(a_{i_1}) \otimes \ldots \otimes T(a_{i_k}) \otimes \ldots \otimes a_n).
\end{aligned}$$

Part II: Cohomology theory of Rota-Baxter Lie algebras

Definition (Cohomology of Rota-Baxter Lie algebras)

Let $(\mathfrak{g}, \ell, T)$ *be a Rota-Baxter Lie algebra with weight. The negative shift of the mapping cone of the chain map* $\phi : C^\bullet_{\mathrm{Lie}}(\mathfrak{g}) \to C^\bullet_{\mathrm{RBO}}(\mathfrak{g})$*, denoted by*

$$(C^\bullet_{\mathrm{RBLA}}(\mathfrak{g}), \delta),$$

is called the **cochain complex of Rota-Baxter Lie algebra** $(\mathfrak{g}, \ell, T)$*. Its cohomology, denoted by*

$$\mathrm{H}^\bullet_{\mathrm{RBLA}}(\mathfrak{g}),$$

is called the **cohomology of Rota-Baxter Lie algebra** $(\mathfrak{g}, \ell, T)$*.*

Proposition

$\mathrm{H}^\bullet_{\mathrm{RBLA}}(\mathfrak{g})$ *controls the infinitesimal deformation of* ℓ *and* T *simultaneously.*

Part II: L_∞-algebra associated to Rota-Baxter Lie algebras

Given a graded space V, define

$$\mathfrak{C}_{\mathrm{RBLA}}(V)_\lambda = \mathfrak{C}_{\mathrm{Lie}}(V) \oplus \mathfrak{C}_{\mathrm{RBO}}(V),$$

where

$$\mathfrak{C}_{\mathrm{Lie}}(V) = \mathrm{Hom}(S^c(sV), sV), \mathfrak{C}_{\mathrm{RBO}}(V) = \mathrm{Hom}(S^c(sV), V).$$

Theorem (C.-Qi-Wang-Zhou 2024)

(i) *There is an* $\mathbf{L}_\infty$**-algebra** *structure on* $\mathfrak{C}_{\mathrm{RBLA}}(V)$.

(ii) *If* V *concentrates in degree* 0*, there is a bijection*

$$\left\{ \begin{array}{c} \textit{Rota-Baxter Lie algebra} \\ \textit{structure } (\ell, T) \textit{ on } V \end{array} \right\} \leftrightarrow \mathrm{MC}(\mathfrak{C}_{\mathrm{RBLA}}(V)).$$

(continued)

J. Chen, Z. Qi, K. Wang, G. Zhou, *(De)colouring in operad theory with applications to homotopy theory of operated algebras*, preprint 2024.

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(iii) *Let $(\mathfrak{g}, \ell, T)$ be a Rota-Baxter Lie algebra. The cochain complex $C^\bullet_{\mathrm{RBLA}}(\mathfrak{g})$ of $(\mathfrak{g}, \ell, T)$ can be realized as the twisted L_∞-algebra $\mathfrak{C}_{\mathrm{RBLA}}(V)^{(\ell,T)}_\lambda$.*

(iv) *Let $(\mathfrak{g}, \ell, T)$ be a Rota-Baxter Lie algebra. The cochain complex $C^\bullet_{\mathrm{RBO}}(\mathfrak{g})$ of T can be realized as the twisted L_∞-algebra $\mathfrak{C}_{\mathrm{RBLA}}(V)^{(\ell,0)}_\lambda$.*

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Part II: Homotopy Rota-Baxter Lie algebras

$$\mathfrak{C}_{\mathrm{RBLA}}(V) = \mathrm{Hom}(\bigoplus_{n=0}^{\infty} S^n(sV), sV) \oplus \mathrm{Hom}(\bigoplus_{n=0}^{\infty} S^n(sV), V)$$

Definition (Homotopy Rota-Baxter Lie algebras)

Let $V = \bigoplus_{i\in\mathbb{Z}} V_i$ *be a graded space. A* **homotopy Rota-Baxter Lie algebra structure** *of weight* λ *on* V *is defined to be a Maurer-Cartan element in the* L_∞*-algebra* $\overline{\mathfrak{C}_{\mathrm{RBLA}}}(V)$*, where*

$$\overline{\mathfrak{C}_{\mathrm{RBLA}}}(V) = \mathrm{Hom}(\bigoplus_{n=1}^{\infty} S^n(sV), sV) \oplus \mathrm{Hom}(\bigoplus_{n=1}^{\infty} S^n(sV), V)$$

is the L_∞*-subalgebra of* $\mathfrak{C}_{\mathrm{RBLA}}(V)_\lambda$.

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Part II: Homotopy Rota-Baxter Lie algebras

By solving the Maurer-Cartan equation, we have the following equivalent definition.

Definition (Homotopy Rota-Baxter Lie algebras)

A **homotopy Rota-Baxter Lie algebra** *of weight* λ *is a graded space* V *equipped with two family of anti-symmetric graded maps* $\{\ell_n : V^{\otimes n} \to V\}_{n\geqslant 1}$ *and* $\{T_n : V^{\otimes n} \to V\}_{n\geqslant 1}$ *with* $|\ell_n| = n-2$ *and* $|T_n| = n-1$, *subject to*

$$\sum_{i=1}^{n} \sum_{\sigma\in \mathrm{Sh}(i,n-i)} \mathrm{sgn}(\sigma)(-1)^{i(n-i)}\ell_{n-i+1}(\ell_i \otimes \mathrm{id}^{\otimes n-i})r_\sigma = 0 \tag{2}$$

and

$$\begin{aligned} &\sum \pm \ell_k(T_{r_1} \otimes \cdots \otimes T_{r_k})r_\sigma \\ = \quad &\sum \pm T_{r_1}(\ell_p(T_{r_2} \otimes \cdots \otimes T_{r_q} \otimes \mathrm{id}^{\otimes p-q+1}) \otimes \mathrm{id}^{r_1-1})r_\sigma. \end{aligned} \tag{3}$$

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Part II: Homotopy Rota-Baxter Lie algebras

$$
\begin{aligned}
&\sum \pm \ell_k(T_{r_1} \otimes \cdots \otimes T_{r_k}) r_\sigma \\
=&\sum \pm T_{r_1}(\ell_p(T_{r_2} \otimes \cdots \otimes T_{r_q} \otimes \mathrm{id}^{\otimes p-q+1}) \otimes \mathrm{id}^{r_1-1}) r_\sigma,
\end{aligned}
\tag{3}
$$

Example

- *When $n = 2$, we have*

$$
\begin{aligned}
&\ell_2 \circ (T_1 \otimes T_1) - T_1 \circ \ell_2 \circ (T_1 \otimes \mathrm{id} + \mathrm{id} \otimes T_1 + \lambda \mathrm{id} \otimes \mathrm{id}) \\
&= \partial(T_2).
\end{aligned}
$$

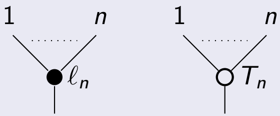
That is, T_1 is a Rota-Baxter operator of weight λ up to homotopy.

Part II: The operad of homotopy Rota-Baxter Lie algebras

Definition (Operad of homotopy Rota-Baxter Lie algebras)

The dg operad ${}_{\lambda}\mathfrak{RBL}_{\infty}$ *of homotopy Rota-Baxter Lie algebra is the free operad generated by* ℓ_n, T_n *with* $|\ell_n| = n-2, |T_n| = n-1$.

The generators $\ell_n, n \geqslant 2$, *and* $T_n, n \geqslant 1$, *are represented by the following:*



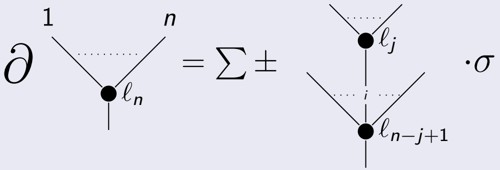
(continued)

Part II: The operad of homotopy Rota-Baxter Lie algebras

$$\sum_{i=1}^{n} \sum_{\sigma \in \mathrm{Sh}(i,n-i)} \mathrm{sgn}(\sigma)(-1)^{i(n-i)} \ell_{n-i+1}(\ell_i \otimes \mathrm{id}^{\otimes n-i}) r_\sigma = 0 \tag{2}$$

Definition (Operad of homotopy Rota-Baxter Lie algebras)

The action of differential operator ∂ on generators can be expressed by shuffle trees as follows:



(continued)

Part II: The operad of homotopy Rota-Baxter Lie algebras

$$\begin{aligned}
&\sum \pm \ell_k(T_{r_1} \otimes \cdots \otimes T_{r_k}) r_\sigma \\
=&\sum \pm T_{r_1}(\ell_p(T_{r_2} \otimes \cdots \otimes T_{r_q} \otimes \mathrm{id}^{\otimes p-q+1}) \otimes \mathrm{id}^{r_1-1}) r_\sigma,
\end{aligned} \tag{3}$$

Definition (Operad of homotopy Rota-Baxter Lie algebras)

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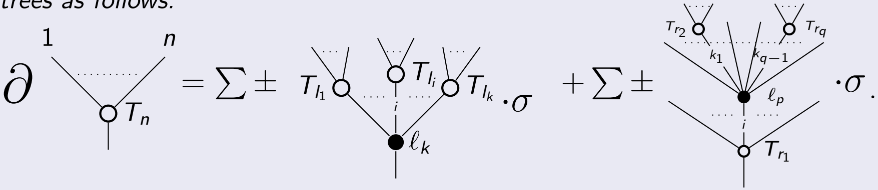


Part II: The operad of homotopy Rota-Baxter Lie algebras

Theorem (C.-Qi-Wang-Zhou 2024)

$_{\lambda}\mathfrak{RBL}_{\infty}$ *is the minimal model of the operad of Rota-Baxter Lie algebras*

J. Chen, Z. Qi, K. Wang, G. Zhou, *(De)colouring in operad theory with applications to homotopy theory of operated algebras*, preprint 2024.

Part III: Relative Rota-Baxter Lie algebras

Definition (Relative RB Lie algebras of weight 0)

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *be a Lie algebra and* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ *be a representation. An* **O-operator** *(or* **relative Rota-Baxter operator of weight 0***) is a linear operator* $T : V \to \mathfrak{g}$ *satisfying*

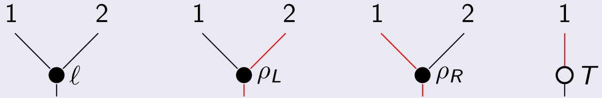
$$[T(v_1), T(v_2)]_{\mathfrak{g}} = T(\rho(T(v_1))(v_2) - \rho(T(v_2)(v_1))$$

for any $v_1, v_2 \in V$.

Part III: Operad of relative Rota-Baxter Lie algebras

Definition (Operad of relative RB Lie algebra of weight 0)

Let $C = \{b, r\}$ be a set of colors. The **colored operad** *of relative Rota-Baxter Lie algebra of weight 0 is the free colored operad with generators*



and modulo the relations of relative Rota-Baxter Lie algebras. Where

(i) ℓ *is the Lie bracket of* $\mathfrak{g}$,

(ii) ρ_L *is the left* $\mathfrak{g}$*-action on* V,

(iii) ρ_R *is the right* $\mathfrak{g}$*-action on* V,

(iv) T *is the O-operator from* V *to* $\mathfrak{g}$.

Part III: Coloring of operads

Let C be a set of colors, $\mathcal{P} = \mathcal{F}(E)/(I)$ be an operad.

(i) We introduced the concept of the **nice coloring** $\mathbb{C}$ of $\mathcal{P}$

$\Rightarrow$ Colored operad $\mathcal{N}_{\mathbb{C}}(\mathcal{P})$.

(ii) Furthermore, $\mathbb{C}$ induces a coloring of the minimal model $\mathcal{P}_\infty$

$\Rightarrow$ Minimal model $\mathcal{N}_{\mathbb{C}}(\mathcal{P}_\infty)$ for $\mathcal{N}_{\mathbb{C}}(\mathcal{P})$.

Theorem (Coloring process, C.-Qi-Wang-Zhou 2024)

Let $\mathcal{P}$ be an operad with the minimal model $\mathcal{P}_\infty$. For a **nice coloring** $\mathbb{C}$ *of $\mathcal{P}$, it induces a coloring $\mathbb{C}$ on $\mathcal{P}_\infty$ such that*

$$\mathcal{N}_{\mathbb{C}}(\mathcal{P}_\infty) = \mathcal{N}_{\mathbb{C}}(\mathcal{P})_\infty.$$

That is to say $\mathcal{N}_{\mathbb{C}}(\mathcal{P}_\infty)$ is the minimal model of $\mathcal{N}_{\mathbb{C}}(\mathcal{P})$.

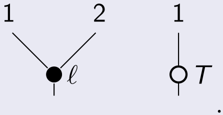
J. Chen, Z. Qi, K. Wang, G. Zhou, *(De)colouring in operad theory with applications to homotopy theory of operated algebras*, preprint 2024.

Part III: Coloring of operads

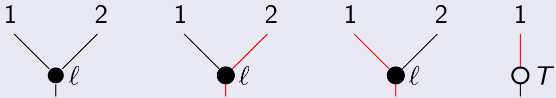
Example (Relative Rota-Baxter Lie algebra with weight 0)

Consider the operad ${}_0\mathfrak{RBL}$ *and the coloring set* $C = \{b, r\}$.

Generators of ${}_0\mathfrak{RBL}$:



There is a nice coloring $\mathbb{C}$ *of* T *and* ℓ:



The colored operad $\mathcal{N}_{\mathbb{C}}({}_0\mathfrak{RBL})$ *is exactly the colored operad of relative Rota-Baxter Lie algebra of weight zero.*

Part III: Coloring of operads

Theorem (C.-Qi-Wang-Zhou 2024)

(i) *The dg colored operad* $\mathcal{N}_{\mathbb{C}}({}_0\mathfrak{RBL}_\infty)$ *is the minimal model of the colored operad of relative Rota-Baxter Lie algebras with weight* 0.

(ii) *Moreover, the* L_∞*-algebra induced by the minimal model* $\mathcal{N}_{\mathbb{C}}({}_0\mathfrak{RBL}_\infty)$ *is isomorphic to the* L_∞*-algebra found by Lazarev-Sheng-Tang 2021.*

- A. Lazarev, Y. Sheng, R. Tang, *Deformations and Homotopy Theory of Relative Rota-Baxter Lie Algebras*. Comm. Math. Phys. **383** (2021), no. 1, 595-631.
- J. Chen, Z. Qi, K. Wang, G. Zhou, *(De)colouring of (coloured) operads and homotopy theory of (relative) Rota-Baxter Lie algebras with arbitrary weight*, preprint 2024.

Definition (Relative RB Lie algebras with weight λ)

(i) *A LieAct triple* $(\mathfrak{g}, \mathfrak{h}, \rho)$ *consists of two Lie algebras* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$, $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ *and a Lie algebra homomorphism* $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$.

(ii) *A* **relative Rota-Baxter Lie algebra** *of weight* λ *is a LieAct triple* $(\mathfrak{g}, \mathfrak{h}, \rho)$ *equipped with an operator* $T : \mathfrak{h} \to \mathfrak{g}$ *satisfying*

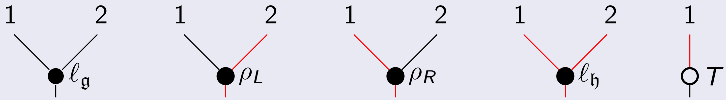
$$[T(h_1), T(h_2)]_{\mathfrak{g}} = T(\rho(T(h_1))(h_2) - \rho(T(h_2))(h_1) + \lambda[h_1, h_2]_{\mathfrak{h}})$$

for any $h_1, h_2 \in \mathfrak{h}$.

Part III: Operad of relative Rota-Baxter Lie algebras

Definition (Operad of relative RB Lie algebra with weight λ)

Let $C = \{b, r\}$ be a set of colors. The colored operad of relative Rota-Baxter Lie algebra is the free colored operad with generators



,

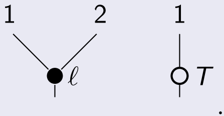
and modulo the relations of relative Rota-Baxter Lie algebras with weight.

Part III: Coloring of operads

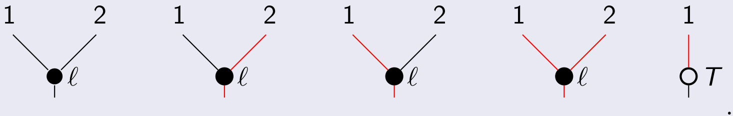
Example (Relative RB Lie algebra with weight λ)

Consider the operad ${}_{\lambda}\mathfrak{RBL}$ *and the coloring set* $C = \{b, r\}$.

Generators of ${}_{\lambda}\mathfrak{RBL}$:



There is a nice coloring $\mathbb{C}'$ *of generators* T *and* ℓ:



The colored operad $\mathcal{N}_{\mathbb{C}'}({}_{\lambda}\mathfrak{RBL})$ *is exactly the colored operad of relative Rota-Baxter Lie algebra with weight* λ.

Part III: Coloring of operads

Theorem (C.-Qi-Wang-Zhou 2024)

$\mathcal{N}_{\mathbb{C}'}({}_{\lambda}\mathfrak{RBL}_{\infty})$ *is the minimal model of the colored operad of relative Rota-Baxter Lie algebra with weight* λ*.*

J. Chen, Z. Qi, K. Wang, G. Zhou, *(De)colouring of (coloured) operads and homotopy theory of (relative) Rota-Baxter Lie algebras with arbitrary weight*, preprint 2024.

Part III: Decoloring of colored operads

Let C be a set of colors, $\mathcal{P}_C$ be a C-colored operad and $(\mathcal{P}_C)_\infty$ be the minimal model of $\mathcal{P}_C$. Assume that $\mathcal{P}_C = \mathcal{N}_{\mathbb{C}}(\mathcal{P})$ for a nice coloring $\mathbb{C}$ and an operad $\mathcal{P}$.

Theorem (Decoloring process, C.-Qi-Wang-Zhou 2024)

By the process of decolorization of $(\mathcal{P}_C)_\infty$, we obtain a minimal model $\mathcal{P}_\infty$ of $\mathcal{P}$.

J. Chen, Z. Qi, K. Wang, G. Zhou, *(De)colouring of (coloured) operads and homotopy theory of (relative) Rota-Baxter Lie algebras with arbitrary weight*, preprint 2024.

Thank you very much