Near-integral fusion

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Fusion categories

A fusion category is a category C such that:
(1) C is $\mathbb{C}$-linear (hom-sets are $\mathbb{C}$-vector spaces) and semisimple.
(2) C has only a finite number of simple objects.
(3) C is a monoidal category: we have a functor $\otimes : C \times C \to C$ with a unit object $1 \in C$ and natural isomorphisms $\alpha_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ satisfying the pentagon axiom and the unit axiom.
(4) C is rigid: for any $V \in C$ there exist V^* and ${}^*V$ along with some additional properties.

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Examples:
(1) the category $C = \mathrm{Rep}(G)$, where G is a finite group.
(2) the category $C = \mathrm{Rep}(H)$, where H is a semisimple finite dimensional Hopf algebra.

Motivations

Definition (Tambara-Yamagami, 1998)

Let G be a finite group. A *Tambara-Yamagami fusion category* is a fusion category C whose isomorphism classes of simple objects are represented by G and a non-invertible object X, satisfying

$$g \otimes h = gh, \forall g, h \in G, X \otimes X = \bigoplus_{g \in G} g.$$

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Definition (Tambara-Yamagami, 1998)

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$$g \otimes h = gh, \forall g, h \in G, X \otimes X = \bigoplus_{g \in G} g.$$

Definition (Siehler, 2003)

Let G be a finite group and let k be a nonnegative integer. A *near-group fusion category* of type (G, k) is a fusion category C whose isomorphism classes of simple objects are represented by G and a non-invertible object X, satisfying

$$g \otimes h = gh, \forall g, h \in G, X \otimes X = \bigoplus_{g \in G} g \oplus kX.$$

Question

The near-group fusion categories have provided an infinite family of examples which cannot be described by the classical representation theory and basic categorical constructions. Also there are several generalizations of near-group fusion categories. But these generalizations can not be applied to classification of low-rank fusion categories.

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Our question is:

How to find a suitable generalization and apply it to the classification of low-rank fusion categories?

Definition

A fusion category $\mathcal{C}$ is called a near-integral fusion category if $\mathcal{C}$ contains a fusion subcategory $\mathcal{D}$ such that $\mathrm{rank}(\mathcal{C}) = \mathrm{rank}(\mathcal{D}) + 1$.

- Let $\mathrm{Irr}(\mathcal{C}) = \{X_1, X_2, \cdots, X_n\}$, $\mathrm{Irr}(\mathcal{D}) = \{X_1, X_2, \cdots, X_{n-1}\}$. Then

$$X_n \otimes X_n^* = \bigoplus_{i=1}^{n-1} \mathrm{FPdim}(X_i)X_i \oplus kX_n.$$

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- Let $\mathcal{C} = \mathcal{C}(\mathcal{D}, k)$. Then $\mathcal{D}$ is integral.
- If $\mathcal{D}$ is pointed then $\mathcal{C} = \mathcal{C}(\mathcal{D}, k)$ is a near-group fusion category.

Premodular categories

Braided fusion categories equipped with spherical structures are called premodular fusion categories. The Müger center $\mathcal{C}'$ of a premodular category $\mathcal{C}$ is

$$\mathcal{C}' = \{X \in \mathcal{C} \mid c_{Y,X}c_{X,Y} = \mathrm{id}_{X\otimes Y}, \forall\ Y \in \mathcal{C}\}.$$

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Two extremes:
A premodular category $\mathcal{C}$ is called modular if $\mathcal{C}'$ is equivalent to the trivial category Vect.

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All premodular categories between these two extremes are called properly premodular categories.

Symmetric cases

Theorem (Deligne, 1990)

Any symmetric category is equivalent $\mathrm{Rep}(G,u)$ as braided categories, where $\mathrm{Rep}(G,u)$ is the category of finite-dimensional representation of G and u is a central element of order at most 2.

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Theorem (Dong-Chen-Wang, 2022)

Suppose $\mathcal{C} = \mathrm{Rep}(G)$ is the category of finite dimensional representations of G. Then two statements below are equivalent:
(1) $\mathcal{C}$ has a fusion subcategory $\mathcal{D}$ such that $\mathrm{rank}(\mathcal{C}) = \mathrm{rank}(\mathcal{D}) + 1$;
(2) There exists $\chi \in \mathrm{Irr}(G)$ such that χ does not vanish on exactly two conjugacy classes.

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The irreducible characters of finite groups which does not vanish on exactly two conjugacy classes were initially studied by S. Gagola in the 1980's and have been called the Gagola characters.

Modular cases

Theorem (Dong-Chen-Wang, 2022)

Let $\mathcal{C} = \mathcal{C}(\mathcal{D}, k)$ be a modular near-integral fusion category. Then $\mathcal{C}$ is exactly one of the following:

(1) a pointed modular category $\mathcal{C}(\mathbb{Z}_2, \pm i)$;

(2) a Fibonacci category;

(3) an Ising category.

Cases: $\mathrm{FPdim}(C) \notin \mathbb{Z}$

Theorem

Let $C = C(\mathcal{D}, k)$ be a premodular near-integral fusion category. If $\mathrm{FPdim}(C) \notin \mathbb{Z}$, then $C \simeq C(A_1, 5, q)_{\mathrm{ad}}$ is a braided equivalence where q^2 is a primitive 5th root of unity.

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$\mathcal{C}(A_N, \ell, q)$: a premodular quantum group category defined in [Kazhdan, Wenzl, 1993] and $\mathcal{C}(A_N, \ell, q)_{ad}$ is the adjoint subcategory of $\mathcal{C}(A_N, \ell, q)$.

Proper cases: $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$

Proposition

Let $\mathcal{C} = \mathcal{C}(\mathcal{D}, k)$ be a nonsymmetrically premodular near-integral fusion category with $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$. Write $N = \mathrm{FPdim}(\mathcal{D})$. If $\mathcal{D}'$ is Tannakian, then $\mathcal{C}' = \mathcal{D}$, and either

1. $\theta_{X_n} = \pm\zeta_4$, $k = 0$, $\mathrm{FPdim}(\mathcal{C}) = 2N$, or
2. $\theta_{X_n} = \zeta_3^{\pm 1}$, $N = 2k^2$, and $\mathrm{FPdim}(\mathcal{C}) = 3N = 6k^2$, or
3. $\theta_{X_n} = -1$, $N = (3/4)k^2$, and $\mathrm{FPdim}(\mathcal{C}) = 4N = 3k^2$.

where θ_{X_n} is the twist of X_n, $\zeta_p = \exp(2\pi i/p)$. In all cases, $\mathrm{FPdim}(X_n) \in \mathbb{Z}$.

Proposition

Let $\mathcal{C}$ be a premodular near-integral fusion category with $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$. If $\mathcal{D}'$ is super-Tannakian, then $\mathcal{D} = \mathcal{D}'$, $k = 0$ and $\theta_{X_n}^{16} = 1$.

Goal

Goal: Classify all premodular fusion categories of rank ≤ 6, up to braided equivalence.

Known Results:

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3. Super-modular categories of rank ≤ 6 were classified by [Bruillard, Galindo, Ng, Plavnik, Rowell, Wang, 2018, 2019, 2021].
4. Modular categories of rank ≤ 5 were classified by [Bruillard, Galindo, Ng, Rowell, Wang, 2009, 2016].

Strategy

It suffices to classify premodular fusion categories of rank $4, 5$ and 6.

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Step 1: Determine all possible Müger center $\mathcal{C}' = \mathrm{Rep}(G, \nu)$.

G	$\lvert G\rvert$	$\#\nu$
C_1	1	1
C_2	2	2
C_3	3	1
S_3	6	1
C_4	4	2
C_2^2	4	4
D_5	10	1
A_4	12	1

G	$\lvert G\rvert$	$\#\nu$
C_5	5	1
D_4	8	2
Q_8	8	2
D_7	14	1
F_5	20	1
$C_7 \rtimes C_3$	21	1
S_4	24	1
A_5	60	1

G	$\lvert G\rvert$	$\#\nu$
C_6	6	2
D_6	12	2
Dic_3	12	2
D_9	18	1
$C_3 \rtimes S_3$	18	1
$C_3^2 \rtimes C_4$	36	1
$\mathrm{PSU}(3,2)$	72	1
$\mathrm{PSU}(2,7)$	168	1

Figure: Finite groups with 6 or less conjugacy classes

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Remark: Since $\mathrm{FPdim}(C_G) = \frac{\mathrm{FPdim}(C)}{|G|}$, C_G is simper than C

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$$\operatorname{FPdim} S_{X,\pi} = \dim\pi[G : G_X] \operatorname{FPdim} X. \tag{1}$$

Step 4: Determine all possible braidings mainly using [Nikshych, Classifying braidings on fusion categories, 2019].

Strategy

Let q be a complex number such that q^2 is a primitive root of unity of order $m \in \mathbb{Z}_{\geq 2}$ and $[n]_m := (q^n - q^{-n})/(q - q^{-1})$ for $n \in \mathbb{Z}$.

Notation	Input data	Description
$\mathrm{Rep}(G, \nu)$	Finite group G and central $\nu \in G$ with $\nu^2 = e$	$\mathrm{Rep}(G)$ with symmetric braiding
Vec	none	$\mathrm{Rep}(C_1)$
sVec	none	$\mathrm{Rep}(C_2, \nu)$, ν nontrivial
$\mathcal{I}_q$	primitive 16th root of unity q	Ising braided fusion categories
$\mathrm{Rep}(G)^\alpha$	Finite group G and root of unity α	$\mathrm{Rep}(G)$ with nonsymmetric braiding
$\mathcal{C}(G, q)$	Abelian group G and quadratic form $q : G \to \mathbb{C}^\times$	Pre-metric group categories
$\mathcal{C}(X, \ell, q)$	Dynkin label X, integer ℓ and q^2 an ℓ^{th} root of unity	Premodular quantum group categories

Figure: Fixed notation for frequently-mentioned braided categories

Main Results: $\mathrm{rank}(\mathcal{C}) = 4$

$\mathcal{C}$	$\mathrm{FPdim}(\mathcal{C})$	FPdims	$C_{\mathcal{C}}(\mathcal{C})$	#
$\mathrm{Rep}(C_4, \nu)$	4	$1, 1, 1, 1$	$\mathrm{Rep}(C_4, \nu)$	2
$\mathrm{Rep}(C_2^2, \nu)$	4	$1, 1, 1, 1$	$\mathrm{Rep}(C_2^2, \nu)$	2
$\mathrm{Rep}(D_5)$	10	$1, 1, 2, 2$	$\mathrm{Rep}(D_5)$	1
$\mathrm{Rep}(A_4)$	12	$1, 1, 1, 3$	$\mathrm{Rep}(A_4)$	1
$\mathrm{Rep}(A_4)^\epsilon$	12	$1, 1, 1, 3$	$\mathrm{Rep}(C_3)$	1
$\mathcal{C}(C_2, q) \boxtimes \mathrm{Rep}(C_2, \nu)$	4	$1, 1, 1, 1$	$\mathrm{Rep}(C_2, \nu)$	3
$\mathcal{C}(C_4, q_{\pm i})$	4	$1, 1, 1, 1$	$\mathrm{Rep}(C_2)$	2
$\mathrm{Rep}(D_5)^\mu$	10	$1, 1, 2, 2$	$\mathrm{Rep}(C_2)$	2
$\mathcal{C}(A_1, 5, q)_{\mathrm{ad}} \boxtimes \mathrm{Rep}(C_2, \nu)$	$\frac{5}{2}\csc^2(\frac{\pi}{5})$	$1, 1, [3]_5, [3]_5$	$\mathrm{Rep}(C_2, \nu)$	8
$\mathcal{C}(A_1, 8, q)_{\mathrm{ad}}$	$2\csc^2(\frac{\pi}{8})$	$1, 1, [3]_8, [3]_8$	sVec	2
$\mathcal{C}(C_4, q)$	4	$1, 1, 1, 1$	Vec	4
$\mathcal{C}(C_2^2, q)$	4	$1, 1, 1, 1$	Vec	5
$\mathcal{C}(A_1, 5, q_1)_{\mathrm{ad}} \boxtimes \mathcal{C}(C_2, q_2)$	$\frac{5}{2}\csc^2(\frac{\pi}{5})$	$1, 1, [3]_5, [3]_5$	Vec	8
$\mathcal{C}(A_1, 5, q_1)_{\mathrm{ad}} \boxtimes \mathcal{C}(A_1, 5, q_2)_{\mathrm{ad}}$	$5[3]_5^2$	$1, [3]_5, [3]_5, [3]_5^2$	Vec	10
$\mathcal{C}(A_1, 9, q)_{\mathrm{ad}}$	$\frac{9}{4}\csc^2(\frac{\pi}{9})$	$1, [3]_9, [5]_9, [7]_9$	Vec	6

Figure: The 57 braided equivalence classes of premodular fusion categories of rank 4, separated by rank of symmetric center from greatest (top) to least (bottom)

Main Results: $\operatorname{rank}(C) = 5$

C	$\mathrm{FPdim}(C)$	FPdims	$C_C(C)$	#
$\mathrm{Rep}(C_5)$	5	1, 1, 1, 1, 1	$\mathrm{Rep}(C_5)$	1
$\mathrm{Rep}(D_4, \nu)$	8	1, 1, 1, 1, 2	$\mathrm{Rep}(D_4, \nu)$	2
$\mathrm{Rep}(Q_8, \nu)$	8	1, 1, 1, 1, 2	$\mathrm{Rep}(Q_8, \nu)$	2
$\mathrm{Rep}(D_7)$	14	1, 1, 2, 2, 2	$\mathrm{Rep}(D_7)$	1
$\mathrm{Rep}(F_5)$	20	1, 1, 1, 1, 4	$\mathrm{Rep}(F_5)$	1
$\mathrm{Rep}(C_7 \rtimes C_3)$	21	1, 1, 1, 3, 3	$\mathrm{Rep}(C_7 \rtimes C_3)$	1
$\mathrm{Rep}(S_4)$	24	1, 1, 2, 3, 3	$\mathrm{Rep}(S_4)$	1
$\mathrm{Rep}(A_5)$	60	1, 3, 3, 4, 5	$\mathrm{Rep}(A_5)$	1
$\mathrm{Rep}(D_4)^{\pm i}$	8	1, 1, 1, 1, 2	$\mathrm{Rep}(C_2^2)$	2
$\mathrm{Rep}(Q_8)^{\pm i}$	8	1, 1, 1, 1, 2	$\mathrm{Rep}(C_2^2)$	2

Figure: The 14 braided equivalence classes of rank 5 braided fusion categories with Tannakian subcategory of maximal rank 5 (above) or 4 (below)

Main Results: $\mathrm{rank}(C) = 5$

C	$\mathrm{FPdim}(C)$	FPdims	$C_C(C)$	#
$C(C_2^2, q_-)^{S_3}$	24	$1, 1, 2, 3, 3$	$\mathrm{Rep}(S_3)$	1
$(\mathcal{I}_{q_1} \boxtimes \mathcal{I}_{q_2})_{\mathbb{Q}}$	8	$1, 1, 1, 1, 2$	$\mathrm{Rep}(C_2)$	12
$\mathrm{Rep}(D_7)^{\psi}$	14	$1, 1, 2, 2, 2$	$\mathrm{Rep}(C_2)$	2
$C(A_1, 10, q)_{\mathrm{ad}}$	$10[3]_5^2$	$1, 1, 2[3]_5, [3]_5^2, [3]_5^2$	$\mathrm{Rep}(C_2)$	4
$C(C_5, q)$	5	$1, 1, 1, 1, 1$	Vec	2
$(\mathrm{Rep}(S_3)^{\omega})^{\gamma}$	12	$1, 1, 2, \sqrt{3}, \sqrt{3}$	Vec	4
$C(A_1, 11, q)_{\mathrm{ad}}$	$\frac{11}{4}\csc^2(\pi/11)$	$1, [2]_{11}, [4]_{11}, [6]_{11}, [8]_{11}$	Vec	10
$C(A_2, 7, q)_{\mathrm{ad}}$	$\frac{7^2}{2^8}\csc^6(\frac{\pi}{7})\sec^2(\frac{\pi}{7})$	$1, \frac{[4]_7[5]_7}{[2]_7}, \frac{[4]_7[5]_7}{[2]_7}, [2]_7[4]_7, \frac{[3]_7^2[6]_7}{[2]_7}$	Vec	6

Figure: The 41 braided equivalence classes of premodular fusion categories of rank 5 with rank ≤ 3 maximal Tannakian subcategory

Main Results: $\mathrm{rank}(\mathcal{C}) = 6$

$\mathcal{C}$	$\mathrm{FPdim}(\mathcal{C})$	FPdims	$C_{\mathcal{C}}(\mathcal{C})$	#
$\mathrm{Rep}(C_6)$	6	$1,1,1,1,1,1$	$\mathrm{Rep}(C_6)$	1
$\mathrm{Rep}(D_6)$	12	$1,1,1,1,2,2$	$\mathrm{Rep}(D_6)$	1
$\mathrm{Rep}(\mathrm{Dic}_3)$	12	$1,1,1,1,2,2$	$\mathrm{Rep}(\mathrm{Dic}_3)$	1
$\mathrm{Rep}(D_9)$	18	$1,1,2,2,2,2$	$\mathrm{Rep}(D_9)$	1
$\mathrm{Rep}(C_3 \rtimes S_3)$	18	$1,1,2,2,2,2$	$\mathrm{Rep}(C_3 \rtimes S_3)$	1
$\mathrm{Rep}(C_3^2 \rtimes C_4)$	36	$1,1,1,1,4,4$	$\mathrm{Rep}(C_3^2 \rtimes C_4)$	1
$\mathrm{Rep}(\mathrm{PSU}(3,2))$	72	$1,1,1,1,2,8$	$\mathrm{Rep}(\mathrm{PSU}(3,2))$	1
$\mathrm{Rep}(\mathrm{GL}(3,2))$	168	$1,3,3,6,7,8$	$\mathrm{Rep}(\mathrm{GL}(3,2))$	1

Figure: The 8 braided equivalence classes of premodular fusion categories of rank 6 with Tannakian subcategory $\mathcal{D}$ of maximal rank 6; there are none with $\mathrm{rank}(\mathcal{D}) = 5$.

Main Results: $\mathrm{rank}(\mathcal{C}) = 6$

$\mathcal{C}$	$\mathrm{FPdim}(\mathcal{C})$	FPdims	$\mathcal{C}_{\mathcal{C}}(\mathcal{C})$	#
$\mathrm{Rep}(C_2) \boxtimes \mathrm{Rep}(S_3)^{\omega}$	12	1, 1, 1, 1, 2, 2	$\mathrm{Rep}(C_2^2)$	2
$\mathrm{Rep}(\mathrm{Dic}_3)^{\omega}$	12	1, 1, 1, 1, 2, 2	$\mathrm{Rep}(C_4)$	2
$\mathrm{Rep}(C_3^2 \rtimes C_4)^{\omega}$	36	1, 1, 1, 1, 4, 4	$\mathrm{Rep}(C_4)$	1

Figure: The 5 braided equivalence classes of premodular fusion categories of rank 6 with Tannakian subcategory of maximal rank 4

Main Results: $\mathrm{rank}(C) = 6$

C	$\mathrm{FPdim}(C)$	FPdims	$C_C(C)$	#
$\mathrm{sVec} \boxtimes \mathrm{Rep}(C_3)$	6	$1, 1, 1, 1, 1, 1$	$\mathrm{Rep}(C_6, \nu)$	1
$C(C_2, q) \boxtimes \mathrm{Rep}(C_3)$	6	$1, 1, 1, 1, 1, 1$	$\mathrm{Rep}(C_3)$	2
$C(A_1, 5, q)_{\mathrm{ad}} \boxtimes \mathrm{Rep}(C_3)$	$\frac{15}{4}\csc^2(\frac{\pi}{5})$	$1, 1, 1, [3]_5, [3]_5, [3]_5$	$\mathrm{Rep}(C_3)$	4
$C(C_2, q) \boxtimes \mathrm{Rep}(S_3)$	12	$1, 1, 1, 1, 2, 2$	$\mathrm{Rep}(S_3)$	2
$\mathrm{sVec} \boxtimes \mathrm{Rep}(S_3)$	12	$1, 1, 1, 1, 2, 2$	$\mathrm{Rep}(S_3)$	1
$\mathrm{Rep}(C_3 \rtimes S_3)^{\omega}$	18	$1, 1, 2, 2, 2, 2$	$\mathrm{Rep}(C_2)$	3
$\mathrm{Rep}(D_9)^{\beta}$	18	$1, 1, 2, 2, 2, 2$	$\mathrm{Rep}(C_2)$	4
$C(A_1, 5, q)_{\mathrm{ad}} \boxtimes \mathrm{Rep}(S_3)$	$\frac{15}{2}\csc^2(\frac{\pi}{5})$	$1, 1, [3]_5, [3]_5, 2, 2[3]_5$	$\mathrm{Rep}(S_3)$	4

Figure: The 21 braided equivalence classes of premodular fusion categories of rank 6 with Tannakian subcategory $\mathcal{D}$ of maximal rank 3

Main Results: $\mathrm{rank}(\mathcal{C}) = 6$

$\mathcal{C}$	$\mathrm{FPdim}(\mathcal{C})$	FPdims	$\mathcal{C}_{\mathcal{C}}(\mathcal{C})$	#
$\mathrm{Rep}(C_2) \boxtimes \mathcal{C}(C_3, q)$	6	$1, 1, 1, 1, 1, 1$	$\mathrm{Rep}(C_2)$	2
$\mathrm{Rep}(C_2, \nu) \boxtimes \mathcal{I}_q$	8	$1, 1, 1, 1, \sqrt{2}, \sqrt{2}$	$\mathrm{Rep}(C_2, \nu)$	16
$\mathcal{I}_2$	8	$1, 1, 1, 1, \sqrt{2}, \sqrt{2}$	$\mathrm{Rep}(C_2)$	8
$\mathcal{C}(C_2, q) \boxtimes \mathrm{Rep}(S_3)^\omega$	12	$1, 1, 1, 1, 2, 2$	$\mathrm{Rep}(C_2)$	4
$\mathrm{sVec} \boxtimes \mathrm{Rep}(S_3)^\omega$	12	$1, 1, 1, 1, 2, 2$	$\mathrm{Rep}(C_2^2, \nu)$	2
$\mathrm{Rep}(\mathrm{Dic}_3, \nu)^\omega$	12	$1, 1, 1, 1, 2, 2$	$\mathrm{Rep}(C_2)$	3
$\mathcal{C}(C_6, q_{x,\omega})^{C_2}$	12	$1, 1, 1, 1, 2, 2$	$\mathrm{Rep}(C_2)$	6
$\mathrm{Rep}(C_3 \rtimes S_3)^\omega$	18	$1, 1, 2, 2, 2, 2$	$\mathrm{Rep}(C_2)$	1
$\mathcal{C}(A_1, 5, q)_{\mathrm{ad}} \boxtimes \mathrm{Rep}(S_3)^\omega$	$\frac{5}{2}\csc^2(\frac{\pi}{5})$	$1, 1, [3]_5, [3]_5, 2, 2[3]_5$	$\mathrm{Rep}(C_2)$	8
$\mathrm{Rep}(C_2) \boxtimes \mathcal{C}(A_1, 7, q)_{\mathrm{ad}}$	$\frac{7}{2}\csc^2(\frac{\pi}{7})$	$1, 1, [3]_7, [3]_7, [5]_7, [5]_7$	$\mathrm{Rep}(C_2)$	3

Figure: The 53 braided equivalence classes of premodular fusion categories of rank 6 with Tannakian subcategory $\mathcal{D}$ of maximal rank 2

$\mathcal{C}$	FPdim($\mathcal{C}$)	FPdims	#
$\mathcal{C}(C_2, q_1) \boxtimes \mathcal{C}(C_3, q_2)$	6	$1, 1, 1, 1, 1, 1$	4
$\mathcal{C}(C_2, q_1) \boxtimes \mathcal{I}_{q_2}$	8	$1, 1, 1, 1, \sqrt{2}, \sqrt{2}$	16
$\mathcal{C}(C_3, q_2) \boxtimes \mathcal{C}(A_1, 5, q)_{\text{ad}}$	$\frac{15}{4}\csc^2(\frac{\pi}{5})$	$1, 1, 1, [3]_5, [3]_5, [3]_5$	8
$\mathcal{C}(A_1, 5, q_1)_{\text{ad}} \boxtimes \mathcal{I}_{q_2}$	$5\csc^2(\frac{\pi}{5})$	$1, 1, [3]_5, [3]_5,$ $[3]_5\sqrt{2}, [3]_5\sqrt{2}$	32
$\mathcal{C}(C_2, q_1) \boxtimes \mathcal{C}(A_1, 7, q_2)_{\text{ad}}$	$\frac{7}{2}\csc^2(\frac{\pi}{7})$	$1, 1, [3]_7, [3]_7, [5]_7, [5]_7$	12
$\mathcal{C}(A_1, 5, q_1)_{\text{ad}} \boxtimes \mathcal{C}(A_1, 7, q_2)_{\text{ad}}$	$\frac{35\csc^2(\frac{\pi}{5})}{2(5-\sqrt{5})}$	$1, [3]_7, [5]_7,$ $[3]_5, [3]_5[3]_7, [3]_5[5]_7$	24
$\mathcal{C}(G_2, 21, q)$	$\frac{21}{2}(5+\sqrt{21})$	$1, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2},$ $\frac{3+\sqrt{21}}{2}, \frac{5+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}$	$12^{\dagger}$
$(\text{Rep}(D_5)^{\mu})^{\gamma}$	20	$1, 1, 2, 2, \sqrt{5}, \sqrt{5}$	4
$\mathcal{C}(A_1, 13, q)_{\text{ad}}$	$\frac{13}{4}\csc^2(\frac{\pi}{13})$	$1, [3]_{13}, [5]_{13},$ $[7]_{13}, [9]_{13}, [11]_{13}$	12
$\mathcal{C}(B_2, 9, q)_{\text{ad}}$	$9u_1^2$	$1, u_1, u_1, u_1, u_1u_2, u_1^2u_2^{-1}$	3
$\text{sVec} \boxtimes \mathcal{C}(C_3, q)$	6	$1, 1, 1, 1, 1, 1$	2
$\text{sVec} \boxtimes \mathcal{C}(A_1, 7, q)_{\text{ad}}$	$\frac{7}{2}\csc^2(\frac{\pi}{7})$	$1, 1, [3]_7, [3]_7, [5]_7, [5]_7$	6
$\mathcal{C}(A_1, 12, q)_{\text{ad}}$	$3\csc^2(\frac{\pi}{12})$	$1, 1, [3]_{12}, [5]_{12},$ $[7]_{12}, [9]_{12}$	2

Figure: The 137 braided equivalence classes of premodular fusion categories of rank 6 whose Tannakian subcategory of maximal rank is Vec, separated by modular (above) and supermodular (below). We abbreviate $u_1 = 1 - \zeta_9^4 - \zeta_9^5$ and $u_2 = \zeta_9 - \zeta_9^2 - \zeta_9^5$.

Thank you for your attention!