Sheaf realization of Bridgeland's Hall algebra of Dynkin type

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Outline

Overview

- $\mathbf{Q}$ is a finite acyclic quiver.
- $\mathfrak{g}$ is the associated Kac-Moody Lie algebra.
- $\mathrm{U}_v(\mathfrak{g})$ is the quantum group with the positive part $\mathrm{U}_v(\mathfrak{n}^+)$.

$$\begin{array}{ccc}
\mathrm{U}_v(\mathfrak{n}^+) & \hookrightarrow & \mathrm{U}_v(\mathfrak{g}) \\
\text{Ringel-Hall algebra} & \hookrightarrow & \text{Bridgeland's Hall algebra} \\
| & & | \\
\text{Lusztig's perverse sheaf theory} & & ??? \\
| & & | \\
\text{(dual) canonical basis of } \mathrm{U}_v(\mathfrak{n}^+) & \subset & \text{a new basis of } \mathrm{U}_v(\mathfrak{g})?
\end{array}$$

Goal: Generalize Lusztig's sheaf construction to complete the ??? part and obtain a new basis in the case that $\mathbf{Q}$ is a Dynkin quiver.

Notation

- $\mathbf{Q} = (I, H, s, t)$ is a Dynkin quiver.
- $\mathcal{A} = \mathcal{A}_{\mathbb{K}} = \mathrm{rep}_{\mathbb{K}}(\mathbf{Q})$ is the category of finite-dimensional representations of $\mathbf{Q}$ over a field $\mathbb{K}$.
- $\mathcal{P} \subset \mathcal{A}$ is the full subcategory of projective objects.
- $K(\mathcal{A})$ is the Grothendieck group of $\mathcal{A}$.
- The Euler form and the symmetric Euler form $K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ are induced by

$$\langle \hat{M}, \hat{N} \rangle = \dim_{\mathbb{K}} \mathrm{Hom}_{\mathcal{A}}(M, N) - \dim_{\mathbb{K}} \mathrm{Ext}^1_{\mathcal{A}}(M, N),$$
$$(\hat{M}, \hat{N}) = \langle \hat{M}, \hat{N} \rangle + \langle \hat{N}, \hat{M} \rangle.$$

The category of two-periodic projective complexes

- $\mathcal{C}_2(\mathcal{A})$ is the category of two-periodic complexes. Its objects are of the form

$$M_\bullet = (M^1, M^0, d^1, d^0) = M^1 \underset{d^0}{\overset{d^1}{\rightleftarrows}} M^0,$$

where $M^j \in \mathcal{A}$ and $d^j \in \mathrm{Hom}_{\mathcal{A}}(M^j, M^{j+1})$ satisfies $d^{j+1}d^j = 0$ for $j \in \mathbb{Z}_2$. Its morphisms are pairs $f = (f^1, f^0) : M_\bullet \to N_\bullet$, where $f^j \in \mathrm{Hom}_{\mathcal{C}_2(\mathcal{A})}(M^j, N^j)$ satisfies $d_N^j f^j = f^{j+1} d_M^j$ for $j \in \mathbb{Z}_2$.

$$\begin{array}{ccc} M^1 & \underset{d_M^0}{\overset{d_M^1}{\rightleftarrows}} & M^0 \\ f^1 \downarrow & & \downarrow f^0 \\ N^1 & \underset{d_N^0}{\overset{d_N^1}{\rightleftarrows}} & N^0. \end{array}$$

- $\mathcal{C}_2(\mathcal{P}) \subset \mathcal{C}_2(\mathcal{A})$ is the full subcategory of two-periodic projective complexes, that is, $M_\bullet \in \mathcal{C}_2(\mathcal{P}) \Leftrightarrow M^j \in \mathcal{P}$ for $j \in \mathbb{Z}_2$.

Hall algebra for two-periodic complexes

- $\mathcal{A} = \mathcal{A}_{\mathbb{F}_q}$ over the finite field $\mathbb{F}_q$.
- $\mathcal{H}_q(\mathcal{C}_2(\mathcal{A}))$ is the Ringel-Hall algebra of $\mathcal{C}_2(\mathcal{A})$. It is a $\mathbb{C}$-algebra with a basis $\{u_{[M_\bullet]} | [M_\bullet] \in \mathrm{Iso}(\mathcal{C}_2(\mathcal{A}))\}$ and the multiplication

$$u_{[M_\bullet]} * u_{[N_\bullet]} = \sum_{[L_\bullet]} g^{L_\bullet}_{M_\bullet N_\bullet} u_{[L_\bullet]},$$

where $g^{L_\bullet}_{M_\bullet N_\bullet} = |\{\text{subobjects } L'_\bullet \subset L_\bullet | L_\bullet / L'_\bullet \cong M_\bullet, L'_\bullet \cong N_\bullet\}|$.
- Riedtmann-Peng formula

$$g^{L_\bullet}_{M_\bullet N_\bullet} = \frac{|\mathrm{Ext}^1_{\mathcal{C}_2(\mathcal{A})}(M_\bullet, N_\bullet)_{L_\bullet}|}{|\mathrm{Hom}_{\mathcal{C}_2(\mathcal{A})}(M_\bullet, N_\bullet)|} \frac{a_{L_\bullet}}{a_{M_\bullet} a_{N_\bullet}},$$

where $\mathrm{Ext}^1_{\mathcal{C}_2(\mathcal{A})}(M_\bullet, N_\bullet)_{L_\bullet} \subset \mathrm{Ext}^1_{\mathcal{C}_2(\mathcal{A})}(M_\bullet, N_\bullet)$ consists of extensions whose middle terms are isomorphic to $L_\bullet$, and $a_{M_\bullet}, a_{N_\bullet}, a_{L_\bullet}$ are the orders of the automorphism groups.

Hall algebra for two-periodic projective complexes

- The multiplication can be rewritten as

$$(a_{M_\bullet} u_{[M_\bullet]}) * (a_{N_\bullet} u_{[N_\bullet]}) = \sum_{[L_\bullet]} \frac{|\mathrm{Ext}^1_{\mathcal{C}_2(\mathcal{A})}(M_\bullet, N_\bullet)_{L_\bullet}|}{|\mathrm{Hom}_{\mathcal{C}_2(\mathcal{A})}(M_\bullet, N_\bullet)|} (a_{L_\bullet} u_{[L_\bullet]}).$$

- $\mathcal{H}_q(\mathcal{C}_2(\mathcal{P})) \subset \mathcal{H}_q(\mathcal{C}_2(\mathcal{A}))$ is the subspace spanned by $u_{[M_\bullet]}$ for $M_\bullet \in \mathcal{C}_2(\mathcal{P})$, then it is a subalgebra, since $\mathcal{C}_2(\mathcal{P})$ is closed under extensions.
- The twisted form $\mathcal{H}_q^{\mathrm{tw}}(\mathcal{C}_2(\mathcal{P}))$ is the same as $\mathcal{H}_q(\mathcal{C}_2(\mathcal{P}))$ as $\mathbb{C}$-vector spaces, with a twisted multiplication

$$u_{[M_\bullet]} * u_{[N_\bullet]} = v_q^{\langle \hat{M}_0, \hat{N}_0 \rangle + \langle \hat{M}_1, \hat{N}_1 \rangle} \sum_{[L_\bullet]} g_{M_\bullet N_\bullet}^{L_\bullet} u_{[L_\bullet]},$$

where $v_q \in \mathbb{C}$ is a fixed square root of q.

Localization and Bridgeland's Hall algebra

- An object $M_\bullet \in \mathcal{C}_2(\mathcal{P})$ is said to be contractible, if it is isomorphic to $K_P \oplus K_Q^*$ for some $P, Q \in \mathcal{P}$, where

$$K_P = P \underset{0}{\overset{1}{\rightleftarrows}} P, \quad K_Q^* = Q \underset{1}{\overset{0}{\rightleftarrows}} Q.$$

We denote by elements

$$b_{K_P} = a_{K_P} u_{[K_P]}, \; b_{K_P^*} = a_{K_P^*} u_{[K_P^*]} \in \mathcal{H}_q^{\mathrm{tw}}(\mathcal{C}_2(\mathcal{P})).$$

- Bridgeland found that $\{b_{K_P}, b_{K_P^*} | P \in \mathcal{P}\}$ satisfies the Ore conditions

$$b_{K_P} * u_{[M_\bullet]} = v_q^{(\hat{P}, \hat{M}^0 - \hat{M}^1)} u_{[M_\bullet]} * b_{K_P},$$
$$b_{K_P^*} * u_{[M_\bullet]} = v_q^{-(\hat{P}, \hat{M}^0 - \hat{M}^1)} u_{[M_\bullet]} * b_{K_P^*},$$

and defined the localization with the reduced quotient

$$\mathcal{DH}_q(\mathcal{A}) = \mathcal{H}_q^{\mathrm{tw}}(\mathcal{C}_2(\mathcal{P}))[b_{K_P}^{-1}, b_{K_P^*}^{-1} | P \in \mathcal{P}],$$
$$\mathcal{DH}_q^{\mathrm{red}}(\mathcal{A}) = \mathcal{DH}_q(\mathcal{A}) / \langle b_{K_P} * b_{K_P^*} - 1 | P \in \mathcal{P} \rangle.$$

Bridgeland's Hall algebra for quiver representations

For any $\alpha \in \mathcal{K}(\mathcal{A})$, it can be written as $\alpha = \hat{P} - \hat{Q}$ for some $P, Q \in \mathcal{P}$, there is a well-defined element

$$b_\alpha = b_{K_P} * b_{K_Q^*} \in \mathcal{DH}_q^{\mathrm{red}}(\mathcal{A}).$$

For any $M \in \mathcal{A}$, there are elements

$$E_M = v_q^{\langle \hat{P}, \hat{M} \rangle} b_{-\hat{P}} * (a_{C_M} u_{[C_M]}),\ F_M = v_q^{\langle \hat{P}, \hat{M} \rangle} b_{\hat{P}} * (a_{C_M^*} u_{[C_M^*]}) \in \mathcal{DH}_q^{\mathrm{red}}(\mathcal{A}),$$

where $C_M = (P, Q, f, 0) \in \mathcal{C}_2(\mathcal{P})$ is given by the minimal projective resolution $0 \to P \xrightarrow{f} Q \xrightarrow{g} M \to 0$ of M.

Theorem (Bridgeland)

There is an algebra isomorphism

$$\mathrm{U}_{v=v_q}(\mathfrak{g}) \to \mathcal{DH}_q^{red}(\mathcal{A})$$

$$E_i \mapsto E_{S_i}/(q-1), F_i \mapsto -v_q F_{S_i}/(q-1), K_i \mapsto b_{\hat{S}_i}, K_i^{-1} \mapsto b_{-\hat{S}_i}.$$

Outline

Notation

- $k = \overline{\mathbb{F}}_q$ is the algebraic closure of $\overline{\mathbb{F}}_q$.
- For any k-variety X together with an algebraic group G-action defined over $\mathbb{F}_q$, $\mathcal{D}^b_G(X)$ is the G-equivariant bounded derived category of $\mathbb{C}$-constructible sheaves on X, and $\mathcal{D}^{b,ss}_G(X)$ is the full subcategory of semisimple complexes, that is, direct sums of simple perverse sheaves up to shifts.
- $\mathcal{D}^b_{G,m}(X) \subset \mathcal{D}^b_G(X), \mathcal{D}^{b,ss}_{G,m}(X) \subset \mathcal{D}^{b,ss}_G(X)$ are the full subcategories of mixed Weil complexes.
- $\mathcal{A} = \mathrm{rep}_k(\mathbf{Q})$.
- $\{P_i \in \mathcal{P} | i \in I\}$ is a fixed complete set of indecomposable projective objects up to isomorphisms.

Variety of $\mathcal{C}_2(\mathcal{P})$

- For any $\underline{e} \in \mathbb{N}I \times \mathbb{N}I$, we denote by $P^j = \bigoplus_{i\in I} e_i^j P_i$ for $j \in \mathbb{Z}_2$, and define an affine variety

$$\mathrm{C}_{\underline{e}} = \{(d^1, d^0) \in \mathrm{Hom}_{\mathcal{A}}(P^1, P^0) \times \mathrm{Hom}_{\mathcal{A}}(P^0, P^1) | d^{j+1}d^j = 0\}$$

  and define a connected algebraic group $\mathrm{G}_{\underline{e}}$ such that

$$\mathrm{G}_{\underline{e}} \cong \mathrm{Aut}_{\mathcal{A}}(P^1) \times \mathrm{Aut}_{\mathcal{A}}(P^0)$$

  which acts on $\mathrm{C}_{\underline{e}}$ via

$$(g^1, g^0).(d^1, d^0) = (g^0 d^1 (g^1)^{-1}, g^1 d^0 (g^0)^{-1}).$$

- There is bijection between $\{\mathrm{G}_{\underline{e}}\text{-orbits in } \mathrm{C}_{\underline{e}}\}$ and $\{$isomorphism classes of $M_\bullet \in \mathcal{C}_2(\mathcal{P})$ satisfying $M^j \cong P^j$ $\}$. Moreover

$$\mathrm{Stab}(\mathcal{O}_{(d^1,d^0)}) \cong \mathrm{Aut}_{\mathcal{C}_2(\mathcal{P})}(P^1, P^0, d^1, d^0).$$

Induction functor

For any $\underline{e} = \underline{e}' + \underline{e}''$, consider the diagram

$$\mathrm{C}_{\underline{e}'} \times \mathrm{C}_{\underline{e}''} \xleftarrow{p_1} \mathrm{C}' \xrightarrow{p_2} \mathrm{C}'' \xrightarrow{p_3} \mathrm{C}_{\underline{e}},$$

where

$$\begin{aligned}
\mathrm{C}'' = &\{(d^1, d^0, W^1, W^0) | (d^1, d^0) \in \mathrm{C}_{\underline{e}}, W^j \subset P^j \\
&\text{ is a direct summand such that } W^j \cong P''^j \text{ and } d^j(W^j) \subset W^{j+1}\}, \\
\mathrm{C}' = &\{(d^1, d^0, W^1, W^0, \rho_1^1, \rho_1^0, \rho_2^1, \rho_2^0) | (d^1, d^0, W^1, W^0) \in \mathrm{C}'', \\
&\rho_1^j : P^j / W^j \xrightarrow{\cong} P'^j, \rho_2^j : W^j \xrightarrow{\cong} P''^j \text{ are isomorhisms in } \mathcal{A}\}, \\
p_1&(d^1, d^0, W^1, W^0, \rho_1^1, \rho_1^0, \rho_2^1, \rho_2^0) = (\rho_{1*}(d^1, d^0), \rho_{2*}(d^1, d^0)) \\
&= (\rho_1^0 \overline{d^1} (\rho_1^1)^{-1}, \rho_1^1 \overline{d^0} (\rho_1^0)^{-1}), (\rho_2^0 d^1|_{W^1} (\rho_2^1)^{-1}, \rho_2^1 d^0|_{W^0} (\rho_2^0)^{-1}), \\
p_2&(d^1, d^0, W^1, W^0, \rho_1^1, \rho_1^0, \rho_2^1, \rho_2^0) = (d^1, d^0, W^1, W^0), \\
p_3&(d^1, d^0, W^1, W^0) = (d^1, d^0).
\end{aligned}$$

Induction functor

$$\mathrm{C}_{\underline{e}'} \times \mathrm{C}_{\underline{e}''} \xleftarrow[\text{not smooth in general}]{p_1} \mathrm{C}' \xrightarrow[\text{principal bundle}]{p_2} \mathrm{C}'' \xrightarrow[\text{proper}]{p_3} \mathrm{C}_{\underline{e}}$$

We define the induction functor to be

$$\mathcal{D}^b_{\mathrm{G}_{\underline{e}'}}(\mathrm{C}_{\underline{e}'}) \boxtimes \mathcal{D}^b_{\mathrm{G}_{\underline{e}''}}(\mathrm{C}_{\underline{e}''}) \to \mathcal{D}^b_{\mathrm{G}_{\underline{e}}}(\mathrm{C}_{\underline{e}})$$

$$\mathrm{Ind}^{\underline{e}}_{\underline{e}',\underline{e}''}(A \boxtimes B) = (p_3)_!(p_2)_\flat(p_1)^*(A \boxtimes B)[-|\underline{e}',\underline{e}''|](-\frac{|\underline{e}',\underline{e}''|}{2}),$$

where $(p_2)_\flat$ is the equivariant descent functor which is the quasi-inverse of $(p_2)^*$, and

$$\begin{aligned} |\underline{e}',\underline{e}''| =& \langle P'^1, P''^1\rangle + \langle P'^0, P''^0\rangle \\ =& \dim_k \mathrm{Hom}_{\mathcal{A}}(P'^1, P''^1) + \dim_k \mathrm{Hom}_{\mathcal{A}}(P'^0, P''^0). \end{aligned}$$

Restriction functor

For any $\underline{e} = \underline{e}' + \underline{e}''$, we have $P^j = P'^j \bigoplus P''^j$. We denote by $\rho_1^j : P^j / P''^j \xrightarrow{\cong} P'^j$ the natural isomorphism and $\rho_2^j : P''^j \xrightarrow{1} P''^j$ the identity morphism.
Consider the diagram

$$\begin{array}{ccccc} \mathrm{C}_{\underline{e}'} \times \mathrm{C}_{\underline{e}''} & \xleftarrow{\kappa} & \mathrm{F} & \xrightarrow[\text{closed embedding}]{\iota} & \mathrm{C}_{\underline{e}}, \\ (\rho_{1*}(d^1, d^0), \rho_{2*}(d^1, d^0)) & \longleftarrow\!\shortmid & (d^1, d^0) & \longmapsto & (d^1, d^0) \end{array}$$

where

$$\mathrm{F} = \{(d^1, d^0) \in \mathrm{C}_{\underline{e}} | d^j(P''^j) \subset P''^{j+1}\}.$$

We define the restriction functor to be

$$\mathcal{D}^b_{\mathrm{G}_{\underline{e}}}(\mathrm{C}_{\underline{e}}) \to \mathcal{D}^b_{\mathrm{G}_{\underline{e}'} \times \mathrm{G}_{\underline{e}''}}(\mathrm{C}_{\underline{e}'} \times \mathrm{C}_{\underline{e}''})$$

$$\mathrm{Res}^{\underline{e}}_{\underline{e}', \underline{e}''}(C) = \kappa_! \iota^*(C)[|\underline{e}', \underline{e}''|](\frac{|\underline{e}', \underline{e}''|}{2}).$$

Associativity and coassociativity

Lemma

For any $\underline{e}_1, \underline{e}_2, \underline{e}_3$, *we have*

$$\mathrm{Ind}^{\underline{e}_1+\underline{e}_2+\underline{e}_3}_{\underline{e}_1,\underline{e}_2+\underline{e}_3}(\mathrm{Id}\boxtimes \mathrm{Ind}^{\underline{e}_2+\underline{e}_3}_{\underline{e}_2,\underline{e}_3}) \cong \mathrm{Ind}^{\underline{e}_1+\underline{e}_2+\underline{e}_3}_{\underline{e}_1+\underline{e}_2,\underline{e}_3}(\mathrm{Ind}^{\underline{e}_1+\underline{e}_2}_{\underline{e}_1,\underline{e}_2}\boxtimes \mathrm{Id}),$$
$$(\mathrm{Id}\times \mathrm{Res}^{\underline{e}_2+\underline{e}_3}_{\underline{e}_2,\underline{e}_3})\mathrm{Res}^{\underline{e}_1+\underline{e}_2+\underline{e}_3}_{\underline{e}_1,\underline{e}_2+\underline{e}_3} \cong (\mathrm{Res}^{\underline{e}_1+\underline{e}_2}_{\underline{e}_1,\underline{e}_2}\times \mathrm{Id})\mathrm{Res}^{\underline{e}_1+\underline{e}_2+\underline{e}_3}_{\underline{e}_1+\underline{e}_2,\underline{e}_3}.$$

- Firstly, we use the induction and restriction functors to give two geometric constructions of Bridgeland's Hall algebra via functions $\mathcal{DH}_q^{\mathrm{red}}(\mathcal{A}) \cong \mathcal{D}\tilde{\mathcal{H}}_q^{\mathrm{red}}(\mathcal{A}) \cong \mathcal{D}\tilde{\mathcal{H}}_q^{*,\mathrm{red}}(\mathcal{A})$.
- Secondly, we use the induction functors and constructible sheaves to construct a $\mathbb{Z}[\mathbb{C}^*]$-algebra $\mathcal{DK}^{\mathrm{red}}$ such that the trace map induces an isomorphism $\mathbb{C}\otimes_{\mathbb{Z}[\mathbb{C}^*]}\mathcal{DK}^{\mathrm{red}} \xrightarrow{\cong} \mathcal{D}\tilde{\mathcal{H}}_q^{\mathrm{red}}(\mathcal{A})$.
- Thirdly, we use the restriction functors and perverse sheaves to construct a $\mathcal{Z}=\mathbb{Z}[v,v^{-1}]$-algebra $\mathcal{DK}^{ss,*,\mathrm{red}}$ such that the trace map induces an isomorphism $\mathbb{C}\otimes_{\mathcal{Z}}\mathcal{DK}^{ss,*,\mathrm{red}} \xrightarrow{\cong} \mathcal{D}\tilde{\mathcal{H}}_q^{*,\mathrm{red}}(\mathcal{A})$.

Pullback and pushforward for functions

- $\sigma : X \to X, \sigma : G \to G$ are the Frobenius maps.
- $X^\sigma \subset X, G^\sigma \subset G$ are their σ-fixed point sets.
- $\tilde{\mathcal{H}}_{G^\sigma}(X^\sigma)$ is the $\mathbb{C}$-vector space of G^σ-invariant $\mathbb{C}$-valued functions on X^σ.
- For any G-equivariant morphism $\varphi : X \to Y$ which is compatible with $\mathbb{F}_q$-structures, there are two linear maps

$$\begin{aligned}\varphi^* : \tilde{\mathcal{H}}_{G^\sigma}(Y^\sigma) &\to \tilde{\mathcal{H}}_{G^\sigma}(X^\sigma)\\ g &\mapsto (x \mapsto g(\varphi(x))),\\ \varphi_! : \tilde{\mathcal{H}}_{G^\sigma}(X^\sigma) &\to \tilde{\mathcal{H}}_{G^\sigma}(Y^\sigma)\\ f &\mapsto (y \mapsto \sum_{x \in \varphi^{-1}(y)} f(x)).\end{aligned}$$

For any $\underline{e} = \underline{e}' + \underline{e}''$, the diagrams

$$\mathrm{C}_{\underline{e}'} \times \mathrm{C}_{\underline{e}''} \xleftarrow{p_1} \mathrm{C}' \xrightarrow{p_2} \mathrm{C}'' \xrightarrow{p_3} \mathrm{C}_{\underline{e}},$$
$$\mathrm{C}_{\underline{e}'} \times \mathrm{C}_{\underline{e}''} \xleftarrow{\kappa} \mathrm{F} \xrightarrow{\iota} \mathrm{C}_{\underline{e}},$$

are defined over $\mathbb{F}_q$. Taking the σ-fixed points set, we obtain

$$\mathrm{C}^\sigma_{\underline{e}'} \times \mathrm{C}^\sigma_{\underline{e}''} \xleftarrow{p_1} \mathrm{C}'^\sigma \xrightarrow{p_2} \mathrm{C}''^\sigma \xrightarrow{p_3} \mathrm{C}^\sigma_{\underline{e}},$$
$$\mathrm{C}^\sigma_{\underline{e}'} \times \mathrm{C}^\sigma_{\underline{e}''} \xleftarrow{\kappa} \mathrm{F}^\sigma \xrightarrow{\iota} \mathrm{C}^\sigma_{\underline{e}},$$

then we define define two linear maps

$$\tilde{\mathcal{H}}_{\mathrm{G}^\sigma_{\underline{e}'}}(\mathrm{C}^\sigma_{\underline{e}'}) \otimes \tilde{\mathcal{H}}_{\mathrm{G}^\sigma_{\underline{e}''}}(\mathrm{C}^\sigma_{\underline{e}''}) \to \tilde{\mathcal{H}}_{\mathrm{G}^\sigma_{\underline{e}}}(\mathrm{C}^\sigma_{\underline{e}})$$
$$\mathrm{ind}^{\underline{e}}_{\underline{e}',\underline{e}''}(f \otimes g) = \frac{v_q^{|\underline{e}',\underline{e}''|}}{|\mathrm{G}^\sigma_{\underline{e}'} \times \mathrm{G}^\sigma_{\underline{e}''}|}(p_3)_!(p_2)_!(p_1)^*(f \otimes g),$$
$$\tilde{\mathcal{H}}_{\mathrm{G}^\sigma_{\underline{e}}}(\mathrm{C}^\sigma_{\underline{e}}) \to \tilde{\mathcal{H}}_{\mathrm{G}^\sigma_{\underline{e}'}}(\mathrm{C}^\sigma_{\underline{e}'}) \otimes \tilde{\mathcal{H}}_{\mathrm{G}^\sigma_{\underline{e}''}}(\mathrm{C}^\sigma_{\underline{e}''})$$
$$\mathrm{res}^{\underline{e}}_{\underline{e}',\underline{e}''}(h) = v_q^{-|\underline{e}',\underline{e}''|}\kappa_!\iota^*(h).$$

Induction and restriction linear maps for functions

- Then all $\mathrm{ind}^{\underline{e}}_{\underline{e}',\underline{e}''}$ define a multiplication and all $\mathrm{res}^{\underline{e}}_{\underline{e}',\underline{e}''}$ define a comultiplication on the direct sum

$$\tilde{\mathcal{H}}_q(\mathcal{C}_2(\mathcal{P})) = \bigoplus_{\underline{e}} \tilde{\mathcal{H}}_{\mathrm{G}^{\sigma}_{\underline{e}}}(\mathrm{C}^{\sigma}_{\underline{e}}).$$

- The comultiplication induces a multiplication on the graded dual

$$\tilde{\mathcal{H}}^*_q(\mathcal{C}_2(\mathcal{P})) = \bigoplus_{\underline{e}} \tilde{\mathcal{H}}^*_{\mathrm{G}^{\sigma}_{\underline{e}}}(\mathrm{C}^{\sigma}_{\underline{e}}) = \bigoplus_{\underline{e}} \mathrm{Hom}_{\mathbb{C}}(\tilde{\mathcal{H}}_{\mathrm{G}^{\sigma}_{\underline{e}}}(\mathrm{C}^{\sigma}_{\underline{e}}), \mathbb{C}).$$

Any point $(d^1, d^0) \in \mathrm{C}^\sigma_{\underline{e}}$ determines an object $M_\bullet \in \mathcal{C}_2(\mathcal{P}_{\mathbb{F}_q})$ of projective dimension vector pair $\underline{e}$. In this case, we also write $M_\bullet \in \mathrm{C}^\sigma_{\underline{e}}$, and denote by $\mathcal{O}_{M_\bullet} \subset \mathrm{C}^\sigma_{\underline{e}}$ the corresponding $\mathrm{G}^\sigma_{\underline{e}}$-orbit and $1_{\mathcal{O}_{M_\bullet}} \in \tilde{\mathcal{H}}_{\mathrm{G}^\sigma_{\underline{e}}}(\mathrm{C}^\sigma_{\underline{e}})$ the corresponding characteristic function.

Lemma

For any $\underline{e} = \underline{e}' + \underline{e}'' \in \mathbb{N}I \times \mathbb{N}I, M_\bullet \in \mathrm{C}^\sigma_{\underline{e}'}, N_\bullet \in \mathrm{C}^\sigma_{\underline{e}''}, L_\bullet \in \mathrm{C}^\sigma_{\underline{e}}$, *we have*

$$\mathrm{ind}^{\underline{e}}_{\underline{e}',\underline{e}''}(1_{\mathcal{O}_{M_\bullet}} \otimes 1_{\mathcal{O}_{N_\bullet}})(L_\bullet) = v_q^{|\underline{e}',\underline{e}''|} g^{L_\bullet}_{M_\bullet,N_\bullet},$$

$$\mathrm{res}^{\underline{e}}_{\underline{e}',\underline{e}''}(1_{\mathcal{O}_{L_\bullet}})(M_\bullet, N_\bullet) = v_q^{|\underline{e}',\underline{e}''|} \frac{|\mathrm{Ext}^1_{\mathcal{C}_2(\mathcal{P}_{\mathbb{F}_q})}(M_\bullet, N_\bullet)_{L_\bullet}|}{|\mathrm{Hom}_{\mathcal{C}_2(\mathcal{P}_{\mathbb{F}_q})}(M_\bullet, N_\bullet)|}.$$

Bridgeland's Hall algebra via functions

Corollary

There are algebra isomorphisms

$$\tilde{\mathcal{H}}_q(\mathcal{C}_2(\mathcal{P})) \xrightarrow{\cong} \mathcal{H}_q^{tw}(\mathcal{C}_2(\mathcal{P})) \qquad \tilde{\mathcal{H}}_q^*(\mathcal{C}_2(\mathcal{P})) \xrightarrow{\cong} \mathcal{H}_q^{tw}(\mathcal{C}_2(\mathcal{P}))$$

$$1_{\mathcal{O}_{M_\bullet}} \mapsto u_{[M_\bullet]}, \qquad 1^*_{\mathcal{O}_{M_\bullet}} \mapsto (a_{M_\bullet} u_{[M_\bullet]}),$$

There are algebra isomorphisms between localizations, and between reduced quotients

$$\mathcal{D}\tilde{\mathcal{H}}_q(\mathcal{A}) = \tilde{\mathcal{H}}_q(\mathcal{C}_2(\mathcal{P}))[(a_{K_P} 1_{\mathcal{O}_{K_P}})^{-1}, (a_{K_P^*} 1_{\mathcal{O}_{K_P^*}})^{-1} | P \in \mathcal{P}_{\mathbb{F}_q}] \cong \mathcal{D}\mathcal{H}_q(\mathcal{A}),$$

$$\mathcal{D}\tilde{\mathcal{H}}^*_q(\mathcal{A}) = \tilde{\mathcal{H}}^*_q(\mathcal{C}_2(\mathcal{P}))[(1^*_{\mathcal{O}_{K_P}})^{-1}, (1^*_{\mathcal{O}_{K_P^*}})^{-1} | P \in \mathcal{P}_{\mathbb{F}_q}] \cong \mathcal{D}\mathcal{H}_q(\mathcal{A}),$$

$$\mathcal{D}\tilde{\mathcal{H}}^{\mathrm{red}}_q(\mathcal{A}) = \mathcal{D}\tilde{\mathcal{H}}_q(\mathcal{A}) / \langle a_{K_P} 1_{\mathcal{O}_{K_P}} * a_{K_P^*} 1_{\mathcal{O}_{K_P^*}} - 1 | P \in \mathcal{P}_{\mathbb{F}_q} \rangle \cong \mathcal{D}\mathcal{H}^{\mathrm{red}}_q(\mathcal{A}),$$

$$\mathcal{D}\tilde{\mathcal{H}}^{*,\mathrm{red}}_q(\mathcal{A}) = \mathcal{D}\tilde{\mathcal{H}}^*_q(\mathcal{A}) / \langle 1^*_{\mathcal{O}_{K_P}} *_r 1^*_{\mathcal{O}_{K_P^*}} - 1 | P \in \mathcal{P}_{\mathbb{F}_q} \rangle \cong \mathcal{D}\mathcal{H}^{\mathrm{red}}_q(\mathcal{A}).$$

Sheaf-function correspondence

Any object in $\mathcal{D}^b_{G,m}(X)$ is of the form (L, φ), where $L \in \mathcal{D}^b_G(X)$ and $\varphi : \sigma^*(L) \xrightarrow{\cong} L$ is an isomorphism. For any $x \in X^\sigma$, there is an isomorphism $\varphi_x : L_x \xrightarrow{\cong} L_x$, then for any $s \in \mathbb{Z}$, there is an isomorphism $H^s(\varphi_x) : H^s(L_x) \xrightarrow{\cong} H^s(L_x)$ between $\mathbb{C}$-vector spaces. Taking the alternative sum of their traces

$$\chi_L(x) = \sum_{s \in \mathbb{Z}} (-1)^s \mathrm{tr}(H^s(\varphi_x)) \in \mathbb{C}$$

defines a G^σ-invariant function on X^σ.

Lemma

For any $A \in \mathcal{D}^b_{\mathrm{G}_{\underline{e}',m}}(\mathrm{C}_{\underline{e}'}), B \in \mathcal{D}^b_{\mathrm{G}_{\underline{e}'',m}}(\mathrm{C}_{\underline{e}''}), C \in \mathcal{D}^b_{\mathrm{G}_{\underline{e},m}}(\mathrm{C}_{\underline{e}})$, *we have*

$$\chi_{\mathrm{Ind}^{\underline{e}}_{\underline{e}',\underline{e}''}(A \boxtimes B)} = \mathrm{ind}^{\underline{e}}_{\underline{e}',\underline{e}''}(\chi_A \otimes \chi_B),$$

$$\chi_{\mathrm{Res}^{\underline{e}}_{\underline{e}',\underline{e}''}(C)} = \mathrm{res}^{\underline{e}}_{\underline{e}',\underline{e}''}(\chi_C).$$

Grothendieck group

- For any $\underline{e} \in \mathbb{N}I \times \mathbb{N}I$, we define $\mathcal{K}_{\underline{e}}$ to be the Grothendieck group of $\mathcal{D}^b_{\mathrm{G}_{\underline{e}},m}(\mathrm{C}_{\underline{e}})$, and define the direct sum $\mathcal{K} = \bigoplus_{\underline{e}} \mathcal{K}_{\underline{e}}$.
- For any $\mathrm{G}_{\underline{e}}$-orbit $\mathcal{O}_{M_\bullet} \subset \mathrm{C}_{\underline{e}}$, we define $S_{M_\bullet} \in \mathcal{K}_{\underline{e}}$ to be the image of $(j_{M_\bullet})_!(\overline{\mathbb{Q}}_l|_{\mathcal{O}_{M_\bullet}})[\mathrm{dim}\mathcal{O}_{M_\bullet}](\frac{\mathrm{dim}\mathcal{O}_{M_\bullet}}{2})$, where $j_{M_\bullet} : \mathcal{O}_{M_\bullet} \to \mathrm{C}_{\underline{e}}$ is the inclusion. Then $\mathcal{K}_{\underline{e}}$ is a $\mathbb{Z}[\mathbb{C}^*]$-module with a basis $\{S_{M_\bullet} | \mathcal{O}_{M_\bullet} \subset \mathrm{C}_{\underline{e}}\}$.
- For any $P \in \mathcal{P}$, suppose e_{K_P} is the projective dimension vector pair of K_P, we define

$$B_{K_P} = (j_{K_P})_!(f_{K_P})_\flat(\overline{\mathbb{Q}}_l|_{\mathrm{G}_{\underline{e}_{K_P}}}) \in \mathcal{D}^b_{\mathrm{G}_{\underline{e}_{K_P}},m}(\mathrm{C}_{\underline{e}_{K_P}}),$$

associated to $K_P = (1,0) \in \mathrm{C}_{\underline{e}_{K_P}}$, where

$$f_{K_P} : \mathrm{G}_{\underline{e}_{K_P}} \to \mathcal{O}_{K_P}$$
$$(g^1, g^0) \mapsto (g^1, g^0).(1,0)$$

is the principal $\mathrm{Aut}_{\mathcal{C}_2(\mathcal{P})}(K_P)$-bundle. Similarly, we define $B_{K_P^*}$ associated to $K_P^* = (0,1)$.

Bridgeland's Hall algebra via constructible sheaves

Theorem

All induction functors $\mathrm{Ind}^{\underline{e}}_{\underline{e}',\underline{e}''}$ *for* $\underline{e} = \underline{e}' + \underline{e}''$ *induce a multiplication on* $\mathcal{K}$ *such that the trace map induces an algebra isomorphism*

$$\mathbb{C} \otimes_{\mathbb{Z}[\mathbb{C}^*]} \mathcal{K} \xrightarrow{\cong} \tilde{\mathcal{H}}_q(\mathcal{C}_2(\mathcal{P})).$$

Moreover, the subset $\{[B_{K_P}], [B_{K_P^*}] | P \in \mathcal{P}\}$ *satisfies the Ore conditions, and so there is a well-defined localization with a reduced quotient*

$$\mathcal{DK} = \mathcal{K}[[B_{K_P}]^{-1}, [B_{K_P^*}]^{-1} | P \in \mathcal{P}],$$
$$\mathcal{DK}^{\mathrm{red}} = \mathcal{DK}/\langle [B_{K_P}] * [B_{K_P^*}] - 1 | P \in \mathcal{P}\rangle$$

such that the trace map induces algebra isomorphisms

$$\mathbb{C} \otimes_{\mathbb{Z}[\mathbb{C}^*]} \mathcal{DK} \cong \mathcal{D}\tilde{\mathcal{H}}_q(\mathcal{A}),\ \mathbb{C} \otimes_{\mathbb{Z}[\mathbb{C}^*]} \mathcal{DK}^{\mathrm{red}} \cong \mathcal{D}\tilde{\mathcal{H}}_q^{\mathrm{red}}(\mathcal{A}).$$

Semisimple Grothendieck group

- For any $\underline{e} \in \mathbb{N}I \times \mathbb{N}I$, we define $\mathcal{K}^{ss}_{\underline{e}}$ to be the Grothendieck group of $\mathcal{D}^{b,ss}_{\mathrm{G}_{\underline{e}},m}(\mathrm{C}_{\underline{e}})$, and define a $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$-module structure on it via $v.[L] = [L[-1](-\frac{1}{2})]$. We define the direct sum and its graded dual

$$\mathcal{K}^{ss} = \bigoplus_{\underline{e}} \mathcal{K}^{ss}_{\underline{e}}, \quad \mathcal{K}^{ss,*} = \bigoplus_{\underline{e}} \mathcal{K}^{ss,*}_{\underline{e}} = \bigoplus_{\underline{e}} \mathrm{Hom}_{\mathcal{Z}}(\mathcal{K}^{ss}_{\underline{e}}, \mathcal{Z}).$$

- For any $\mathrm{G}_{\underline{e}}$-orbit $\mathcal{O}_{M_\bullet} \subset \mathrm{C}_{\underline{e}}$, we define $I_{M_\bullet} \in \mathcal{K}^{ss}_{\underline{e}}$ to be the image of

$$\mathrm{IC}(\mathcal{O}_{M_\bullet}, \overline{\mathbb{Q}}_l)(\frac{\mathrm{dim}\mathcal{O}_{M_\bullet}}{2}).$$

- Then $\mathcal{K}^{ss}_{\underline{e}}$ has a $\mathcal{Z}$-basis $\mathcal{I}_{\underline{e}} = \{I_{M_\bullet} | \mathcal{O}_{M_\bullet} \subset \mathrm{C}_{\underline{e}}\}$ and $\mathcal{K}^{ss}$ has a $\mathcal{Z}$-basis $\mathcal{I} = \bigsqcup_{\underline{e}} \mathcal{I}_{\underline{e}}$. We define

$$\mathcal{I}^*_{\underline{e}} = \{I^*_{M_\bullet} | \mathcal{O}_{M_\bullet} \subset \mathrm{C}_{\underline{e}}\} \subset \mathcal{K}^{ss,*}_{\underline{e}}$$

to be the dual basis of $\mathcal{I}_{\underline{e}}$, and $\mathcal{I}^* = \bigsqcup_{\underline{e}} \mathcal{I}^*_{\underline{e}}$.

Hall algebra for $\mathcal{C}_2(\mathcal{P})$ via perverse sheaves

Lemma

For any $\underline{e} = \underline{e}' + \underline{e}'' \in \mathbb{N}I \times \mathbb{N}I$, the restriction functor can be restricted to

$$\mathrm{Res}^{\underline{e}}_{\underline{e}',\underline{e}''} : \mathcal{D}^{b,ss}_{\mathrm{G}_{\underline{e}},m}(\mathrm{C}_{\underline{e}}) \to \mathcal{D}^{b,ss}_{\mathrm{G}_{\underline{e}'},m}(\mathrm{C}_{\underline{e}'}) \boxtimes \mathcal{D}^{b,ss}_{\mathrm{G}_{\underline{e}''},m}(\mathrm{C}_{\underline{e}''}).$$

Theorem

All restriction functors $\mathrm{Res}^{\underline{e}}_{\underline{e}',\underline{e}''}$ for $\underline{e} = \underline{e}' + \underline{e}''$ induce a comultiplication r on $\mathcal{K}^{ss}$ such that the trace map induces an isomorphism

$$\mathbb{C} \otimes_{\mathcal{Z}} \mathcal{K}^{ss} \xrightarrow{\cong} \tilde{\mathcal{H}}_q(\mathcal{C}_2(\mathcal{P})),$$

where $\mathbb{C}$ is viewed as a $\mathcal{Z}$-module via $v.z = v_q z$. Dually, the comultiplication r induces a multiplication $*_r$ on $\mathcal{K}^{ss,*}$ such that the dual of the trace map induces an algebra isomorphism

$$\tilde{\mathcal{H}}^*_q(\mathcal{C}_2(\mathcal{P})) \xrightarrow{\cong} \mathbb{C} \otimes_{\mathcal{Z}} \mathcal{K}^{ss,*}.$$

Images of $b_{K_P}, b_{K_P^*}$

- There are algebra isomorphisms

$$\mathcal{H}_q^{tw}(\mathcal{C}_2(\mathcal{P})) \xrightarrow{\cong} \tilde{\mathcal{H}}_q^*(\mathcal{C}_2(\mathcal{P})) \xrightarrow{\cong} \mathbb{C} \otimes_{\mathcal{Z}} \mathcal{K}^{ss,*}.$$

- We observe that $b_{K_P} \mapsto 1 \otimes v^{-\langle \hat{P},\hat{P} \rangle} I_{K_P}^*,\ b_{K_P^*} \mapsto 1 \otimes v^{-\langle \hat{P},\hat{P} \rangle} I_{K_P^*}^*$. This inspires us to localize $\mathcal{K}^{ss,*}$ with respect to

$$\tilde{I}_{K_P}^* = v^{-\langle \hat{P},\hat{P} \rangle} I_{K_P}^*,\ \tilde{I}_{K_P^*}^* = v^{-\langle \hat{P},\hat{P} \rangle} I_{K_P^*}^*,$$

as Bridgeland localized $\mathcal{H}_q^{tw}(\mathcal{C}_2(\mathcal{P}))$ with respect to $b_{K_P}, b_{K_P^*}$.

Bridgeland's Hall algebra via perverse sheaves

Theorem

The subset $\{\tilde{I}^*_{K_P}, \tilde{I}^*_{K^*_P} | P \in \mathcal{P}\} \subset \mathcal{K}^{ss,*}$ *satisfies the Ore conditions, and so there is a well-defined localization*

$$\mathcal{DK}^{ss,*} = \mathcal{K}^{ss,*}[(\tilde{I}^*_{K_P})^{-1}, (\tilde{I}^*_{K^*_P})^{-1} | P \in \mathcal{P}]$$

with a reduced quotient

$$\mathcal{DK}^{ss,*,\mathrm{red}} = \mathcal{DK}^{ss,*} / \langle \tilde{I}^*_{K_P} *_r \tilde{I}^*_{K^*_P} - 1 | P \in \mathcal{P} \rangle$$

such that

$$\mathcal{DH}_q(\mathcal{A}) \cong \mathcal{D}\tilde{\mathcal{H}}^*_q(\mathcal{A}) \cong \mathbb{C} \otimes_{\mathcal{Z}} \mathcal{DK}^{ss,*},$$

$$\mathcal{DH}^{\mathrm{red}}_q(\mathcal{A}) \cong \mathcal{D}\tilde{\mathcal{H}}^{*,\mathrm{red}}_q(\mathcal{A}) \cong \mathbb{C} \otimes_{\mathcal{Z}} \mathcal{DK}^{ss,*,\mathrm{red}}.$$

Outline

Bar-involution

- The bar-involution on the Laurent polynomial ring $\mathcal{Z}$ is defined to be the $\mathbb{Z}$-linear isomorphism interchanging v and v^{-1}, denoted by $\overline{\zeta(v)} = \zeta(v^{-1})$.
- The bar-involution on $\mathcal{K}^{ss}$ is induced by the Verdier dual $\mathbb{D}$, denoted by $[\bar{L}] = [\mathbb{D}L]$, which is compatible with the $\mathcal{Z}$-module structure and the bar-involution on $\mathcal{Z}$.
- The bar-involution on $\mathcal{K}^{ss,*}$ is induced by the bar-involutions on $\mathcal{Z}$ and $\mathcal{K}^{ss}$, that is, for any $f \in \mathcal{K}^{ss,*}, x \in \mathcal{K}^{ss}$, we have
$$\bar{f}(x) = \overline{f(\bar{x})}.$$
- We set $\|\underline{e}', \underline{e}''\| = |\underline{e}', \underline{e}''| + |\underline{e}'', \underline{e}'|$ for any $\underline{e}', \underline{e}'' \in \mathbb{N}I \times \mathbb{N}I$.

Lemma

(a) For any $x \in \mathcal{K}^{ss}$, *if* $r(x) = \sum x_1 \otimes x_2$, *where* x_1, x_2 *are homogeneous of degree* $|x_1|, |x_2|$, *we have* $r(\bar{x}) = \sum v^{-\||x_1|,|x_2|\|} \overline{x_2} \otimes \overline{x_1}$.
(b) Dually, for any $y_1, y_2 \in \mathcal{K}^{ss,*}$ *which are homogeneous of degree* $|y_1|, |y_2|$, *we have* $\overline{y_1 *_r y_2} = v^{\||y_1|,|y_2|\|} \overline{y_2} *_r \overline{y_1}$.

Lemma

(a) The $\mathcal{Z}$-basis $\mathcal{I}$ of $\mathcal{K}^{ss}$ is bar-invariant and has positivity.
(b) Dually, the $\mathcal{Z}$-basis $\mathcal{I}^*$ of $\mathcal{K}^{ss,*}$ is bar-invariant and has positivity.
More precisely, we have $\overline{I_{M_\bullet}} = I_{M_\bullet}, \overline{I^*_{M_\bullet}} = I^*_{M_\bullet}$, and if

$$r(I_{L_\bullet}) = \sum_{M_\bullet, N_\bullet} \zeta^{L_\bullet}_{M_\bullet, N_\bullet}(v) I_{M_\bullet} \otimes I_{N_\bullet}, \; I^*_{M_\bullet} *_r I^*_{N_\bullet} = \sum_{L_\bullet} \xi^{L_\bullet}_{M_\bullet, N_\bullet}(v) I^*_{L_\bullet}$$

then $\zeta^{L_\bullet}_{M_\bullet, N_\bullet}(v), \xi^{L_\bullet}_{M_\bullet N_\bullet}(v) \in \mathbb{N}[v, v^{-1}]$. Moreover, we have

$$\xi^{L_\bullet}_{M_\bullet N_\bullet}(v) = \zeta^{L_\bullet}_{M_\bullet N_\bullet}(v) = v^{-\|\underline{e}_{M_\bullet}, \underline{e}_{N_\bullet}\|} \zeta^{L_\bullet}_{N_\bullet M_\bullet}(v^{-1}).$$

Free module structure

- We define $\mathcal{T}^{\mathrm{red}}$ to be the $\mathcal{Z}$-subalgebra of $\mathcal{DK}^{ss,*,\mathrm{red}}$ generated by $\tilde{I}^*_{K_{P_i}}, \tilde{I}^*_{K^*_{P_i}}$ for $i \in I$, which is a commutative algebra. Then the multiplication $*_r : \mathcal{T}^{\mathrm{red}} \times \mathcal{DK}^{ss,*,\mathrm{red}} \to \mathcal{DK}^{ss,*,\mathrm{red}}$ defines a $\mathcal{T}^{\mathrm{red}}$-module structure on $\mathcal{DK}^{ss,*,\mathrm{red}}$.
- An object $M_\bullet = (M^1, M^0, d^1, d^0) \in \mathcal{C}_2(\mathcal{P})$ is said to be radical, if $\mathrm{Im}\, d^j \subset \mathrm{rad} M^{j+1}$. We define the subset

$$\mathcal{I}^{*,\mathrm{rad}} = \{I^*_{M_\bullet} \in \mathcal{I}^* | M_\bullet \text{ is radical}\} \subset \mathcal{I}^*.$$

Theorem

The $\mathcal{T}^{\mathrm{red}}$-module $\mathcal{DK}^{ss,*,\mathrm{red}}$ is free with a basis $\mathcal{I}^{*,\mathrm{rad}}$.

Basis of $\mathcal{DK}^{ss,*,\mathrm{red}}$

Lemma

For any $\alpha \in K(\mathcal{A})$, it can be written as $\alpha = \hat{P} - \hat{Q}$ for some $P, Q \in \mathcal{P}$, and there is a well-defined element

$$\tilde{I}_\alpha^* = \tilde{I}^*_{K_P \oplus K_Q^*} = \tilde{I}^*_{K_P} *_r \tilde{I}^*_{K_Q^*} \in \mathcal{T}^{\mathrm{red}}.$$

Moreover, $\mathcal{T}$ is isomorphic to the torus $\mathcal{Z}[K(\mathcal{A})]$, and it has a $\mathcal{Z}$-basis

$$\tilde{\mathcal{I}}^{*,\mathrm{red}} = \{\tilde{I}_\alpha^* | \alpha \in K(\mathcal{A})\}.$$

Theorem

The algebra $\mathcal{DK}^{ss,*,\mathrm{red}}$ has a $\mathcal{Z}$-basis

$$\tilde{\mathcal{I}}^{*,\mathrm{red}} *_r \mathcal{I}^{*,\mathrm{rad}} = \{\tilde{I}_\alpha^* *_r I^*_{M_\bullet} | \tilde{I}_\alpha^* \in \tilde{\mathcal{I}}^{*,\mathrm{red}}, I^*_{M_\bullet} \in \mathcal{I}^{*,\mathrm{rad}}\}.$$

Comparison with Lusztig's categorification

- There are algebra isomorphisms

$$\begin{aligned} \mathcal{DH}_q^{\mathrm{red}}(\mathcal{A}) &\xrightarrow{\cong} \mathcal{D}\tilde{\mathcal{H}}_q^{*,\mathrm{red}}(\mathcal{A}) \xrightarrow{\cong} \mathbb{C}\otimes_{\mathcal{Z}} \mathcal{DK}^{ss,*,\mathrm{red}} \\ a_{M_\bullet}u_{[M_\bullet]} &\mapsto \quad 1^*_{\mathcal{O}_{M_\bullet}} \quad \mapsto \quad \chi^*(1^*_{\mathcal{O}_{M_\bullet}}). \end{aligned}$$

- Similarly, by using of Lusztig's restriction functor $\mathrm{Res}^{\nu}_{\nu',\nu''}[2\sum_{i\in I}\nu'_i\nu''_i](\sum_{i\in I}\nu'_i\nu''_i)$, there are algebra isomorphisms

$$\begin{aligned} \mathcal{H}_q^{\mathrm{tw}}(\mathcal{A}) &\xrightarrow{\cong} \tilde{\mathcal{H}}_q^{*,\mathrm{red}}(\mathcal{A}) \xrightarrow{\cong} \mathbb{C}\otimes_{\mathcal{Z}} \mathcal{K}^{+,*} \\ a_M u_{[M]} &\mapsto \quad 1^*_{\mathcal{O}_M} \quad \mapsto \chi^*(1^*_{\mathcal{O}_M}), \end{aligned}$$

  where $\mathcal{K}^{+,*} = \bigoplus_\nu \mathrm{Hom}_{\mathcal{Z}}(\mathcal{K}^+_\nu, \mathcal{Z})$ is the graded dual of $\mathcal{K}^+$.

- Bridgeland prove that there is an injective algebra homomorphism

$$\begin{aligned} \mathcal{H}_q^{\mathrm{tw}}(\mathcal{A}) &\to \mathcal{DH}_q^{\mathrm{red}}(\mathcal{A}) \\ a_M u_{[M]} &\mapsto E_M = v_q^{\langle \hat{P},\hat{M}\rangle} b_{-\hat{P}} * (a_{C_M} u_{[C_M]}). \end{aligned}$$

Comparison with (dual) canonical basis

- Hence there is an injective algebra homomorphism
$$\Phi_q^+ : \mathbb{C} \otimes_{\mathcal{Z}} \mathcal{K}^{+,*} \to \mathbb{C} \otimes_{\mathcal{Z}} \mathcal{DK}^{ss,*,\mathrm{red}}.$$
- The canonical basis of $\mathcal{K}_\nu^+$ is the $\mathcal{Z}$-basis $\mathcal{B}_\nu = \{I_M | \mathcal{O}_M \subset \mathrm{E}_\nu\}$, where I_M is the image of $\mathrm{IC}(\mathcal{O}_M, \overline{\mathbb{Q}}_l)(\frac{\mathrm{dim}\mathcal{O}_M}{2})$. Let $\mathcal{B}_\nu^* = \{I_M^* | \mathcal{O}_M \subset \mathrm{E}_\nu\} \subset \mathcal{K}^{+,*}$ be the dual basis of $\mathcal{B}_\nu$ and
$$\mathcal{B}^* = \bigsqcup_\nu \mathcal{B}_\nu^*.$$
- For any $M \in \mathcal{A}$, it can be decomposed as $M \cong M_1 \oplus ... M_n$ such that $\mathrm{Ext}_{\mathcal{A}}^1(M_s, M_t) = 0$ for any $s \leqslant t$. Let $0 \to P_s \to Q_s \to M_s \to 0$ be the minimal projective resolution of M_s.

Lemma

With the same notations as above, we have

$$\Phi_q^+(1 \otimes v^{-\mathrm{dim}\mathcal{O}_M} I_M^*) = 1 \otimes v^{-\langle \hat{P}, \hat{P} \rangle + \sum_{s=1}^n \sum_{t=s+1}^n \langle \hat{P}_t, \hat{Q}_s \rangle} \tilde{I}^*_{-\hat{P}} *_r I^*_{C_M}.$$

Comparison with (dual) canonical basis

- By the Hall polynomials for $\mathcal{A}$ and $\mathcal{C}_2(\mathcal{P})$, there are the generic algebras $\mathcal{H}^{\mathrm{tw}}_{v^2}(\mathcal{A})$ and $\mathcal{DH}^{\mathrm{red}}_{v^2}(\mathcal{A})$, which are $\mathcal{Z}$-algebras.
- There are $\mathcal{Z}$-algebra isomorphisms

$$\mathcal{H}^{\mathrm{tw}}_{v^2}(\mathcal{A}) \cong \mathcal{K}^{+,*},\ \mathcal{DH}^{\mathrm{red}}_{v^2}(\mathcal{A}) \cong \mathcal{DK}^{ss,*,\mathrm{red}},$$

and there is an injective $\mathcal{Z}$-algebra homomorphism

$$\Phi_{v^2} : \mathcal{K}^{+,*} \to \mathcal{DK}^{ss,*,\mathrm{red}}$$

such that

$$\Phi_{v^2}(v^{-\dim \mathcal{O}_M} I^*_M) = v^{-\langle \hat{P},\hat{P}\rangle + \sum_{s=1}^n \sum_{t=s+1}^n \langle \hat{P}_t, \hat{Q}_s\rangle} \tilde{I}^*_{-\hat{P}} *_r I^*_{C_M},$$

and so $\Phi_{v^2}(\mathcal{B}^*) \subset \tilde{\mathcal{I}}^{*,\mathrm{red}} *_r \mathcal{I}^{*,\mathrm{rad}}$ up to powers of v.

Thank you

Thank you!