Constructing the queer quantum supergroup using Hecke-Clifford superalgebras

Jie Du

University of New South Wales

(joint with Haixia Gu, Zhenhua Li, and Jinkui Wan)

21st International Conference on Representations of Algebras
Shanghai, August 5-9, 2024

1. Introduction: Motivation and History
2. Hecke–Clifford superalgebras and some special elements
3. The queer q-Schur superalgebras and its standardisation
4. Standard multiplication formulas and their expansions
5. The regular module for the quantum queer supergroup

1. Introduction: Motivation and History

1. Introduction: Motivation and History

1. The structure of a group with a BN-pair is hidden in its Weyl group.

1. Introduction: Motivation and History

1. The structure of a group with a BN-pair is hidden in its Weyl group.
2. The representation theory of a semismiple complex Lie algebra is hidden in its associated Hecke algebra (KL conjecture/theorem).

1. Introduction: Motivation and History

1. The structure of a group with a BN-pair is hidden in its Weyl group.
2. The representation theory of a semismiple complex Lie algebra is hidden in its associated Hecke algebra (KL conjecture/theorem).
3. The structure of a quantum linear group $\mathbf{U} = \mathbf{U}(\mathfrak{gl}_m)$ (or supergroup $\mathbf{U}(\mathfrak{gl}_{m|n})$) is hidden in the Hecke algebras of symmetric groups.

1. Introduction: Motivation and History

1. The structure of a group with a BN-pair is hidden in its Weyl group.
2. The representation theory of a semismiple complex Lie algebra is hidden in its associated Hecke algebra (KL conjecture/theorem).
3. The structure of a quantum linear group $\mathbf{U} = \mathbf{U}(\mathfrak{gl}_m)$ (or supergroup $\mathbf{U}(\mathfrak{gl}_{m|n})$) is hidden in the Hecke algebras of symmetric groups.

More precisely, the Hecke algebra $\mathcal{H} = \mathcal{H}_r = \mathcal{H}(\mathfrak{S}_r)$ is the algebra over $\mathbb{Q}(\upsilon)$ generated by $T_1, \ldots, T_{r-1}$ subject to a certain relations.

1. Introduction: Motivation and History

1. The structure of a group with a BN-pair is hidden in its Weyl group.
2. The representation theory of a semismiple complex Lie algebra is hidden in its associated Hecke algebra (KL conjecture/theorem).
3. The structure of a quantum linear group $\mathbf{U} = \mathbf{U}(\mathfrak{gl}_m)$ (or supergroup $\mathbf{U}(\mathfrak{gl}_{m|n})$) is hidden in the Hecke algebras of symmetric groups.

More precisely, the Hecke algebra $\mathcal{H} = \mathcal{H}_r = \mathcal{H}(\mathfrak{S}_r)$ is the algebra over $\mathbb{Q}(\boldsymbol{v})$ generated by $T_1, \dots, T_{r-1}$ subject to a certain relations. It has basis $\{T_w \mid w \in \mathfrak{S}_r\}$ and its regular module ${}_{\mathcal{H}}\mathcal{H}$ has the following matrix representation ($\boldsymbol{q} = \boldsymbol{v}^2$):

$$T_i T_w = \begin{cases} T_{s_i w} \quad (s_i = (i, i+1)), & \text{if } s_i w > w; \\ (\boldsymbol{q} - 1) T_w + \boldsymbol{q} T_{s_i w}, & \text{if } s_i w < w. \end{cases}$$

1. Introduction: Motivation and History

1. The structure of a group with a BN-pair is hidden in its Weyl group.
2. The representation theory of a semisimple complex Lie algebra is hidden in its associated Hecke algebra (KL conjecture/theorem).
3. The structure of a quantum linear group $\mathbf{U} = \mathbf{U}(\mathfrak{gl}_m)$ (or supergroup $\mathbf{U}(\mathfrak{gl}_{m|n})$) is hidden in the Hecke algebras of symmetric groups.

More precisely, the Hecke algebra $\mathcal{H} = \mathcal{H}_r = \mathcal{H}(\mathfrak{S}_r)$ is the algebra over $\mathbb{Q}(\boldsymbol{v})$ generated by $T_1, \ldots, T_{r-1}$ subject to a certain relations. It has basis $\{T_w \mid w \in \mathfrak{S}_r\}$ and its regular module ${}_{\mathcal{H}}\mathcal{H}$ has the following matrix representation ($\boldsymbol{q} = \boldsymbol{v}^2$):

$$T_i T_w = \begin{cases} T_{s_i w} \quad {\scriptstyle (s_i = (i, i+1))}, & \text{if } s_i w > w; \\ (\boldsymbol{q} - 1) T_w + \boldsymbol{q} T_{s_i w}, & \text{if } s_i w < w. \end{cases}$$

From this basic structure together with a sequence of constructions, it is possible to construct a basis $\{A(\mathbf{j})\}_{A,\mathbf{j}}$ for $\mathbf{U}$ such that its regular representation ${}_{\mathbf{U}}\mathbf{U}$ is given by explicit multiplication formulas for

$$E_h \cdot A(\mathbf{j}), \quad F_h \cdot A(\mathbf{j}), \quad K_i \cdot A(\mathbf{j}).$$

1. Introduction: Motivation and History

1. The structure of a group with a BN-pair is hidden in its Weyl group.
2. The representation theory of a semismiple complex Lie algebra is hidden in its associated Hecke algebra (KL conjecture/theorem).
3. The structure of a quantum linear group $\mathbf{U} = \mathbf{U}(\mathfrak{gl}_m)$ (or supergroup $\mathbf{U}(\mathfrak{gl}_{m|n})$) is hidden in the Hecke algebras of symmetric groups.

More precisely, the Hecke algebra $\mathcal{H} = \mathcal{H}_r = \mathcal{H}(\mathfrak{S}_r)$ is the algebra over $\mathbb{Q}(\boldsymbol{v})$ generated by $T_1, \ldots, T_{r-1}$ subject to a certain relations. It has basis $\{T_w \mid w \in \mathfrak{S}_r\}$ and its regular module ${}_{\mathcal{H}}\mathcal{H}$ has the following matrix representation ($\boldsymbol{q} = \boldsymbol{v}^2$):

$$T_i T_w = \begin{cases} T_{s_i w} \quad {\scriptstyle (s_i = (i, i+1))}, & \text{if } s_i w > w; \\ (\boldsymbol{q} - 1) T_w + \boldsymbol{q} T_{s_i w}, & \text{if } s_i w < w. \end{cases}$$

From this basic structure together with a sequence of constructions, it is possible to construct a basis $\{A(\mathbf{j})\}_{A,\mathbf{j}}$ for $\mathbf{U}$ such that its regular representation ${}_{\mathbf{U}}\mathbf{U}$ is given by explicit multiplication formulas for

$$E_h \cdot A(\mathbf{j}), \quad F_h \cdot A(\mathbf{j}), \quad K_i \cdot A(\mathbf{j}).$$

This question was first answered by Beilinson–Lusztig–MacPherson.

BLM Theorem

Theorem (Beilinson–Lusztig–MacPherson[1] '90)

BLM Theorem

Theorem (Beilinson–Lusztig–MacPherson[1] '90)

The quantum linear group $\mathbf{U}_\upsilon(\mathfrak{gl}_n)$, *generated by* $\mathsf{K}_a, \mathsf{K}_a^{-1}, \mathsf{E}_h, \mathsf{F}_h$, *has a basis*

$$\{A(\mathbf{j}) \mid A = (a_{i,j}) \in M_n(\mathbb{N})^{0diag}, \mathbf{j} = (j_i) \in \mathbb{Z}^n\}$$

that satisfies the following multiplication rules:

(1) $\mathsf{K}_a \cdot A(\mathbf{j}) = \upsilon^{\mathrm{ro}(A)\cdot\boldsymbol{e}_a} A(\boldsymbol{j} + \boldsymbol{e}_a), \quad A(\boldsymbol{j}) \cdot \mathsf{K}_a = \upsilon^{\mathrm{co}(A)\cdot\boldsymbol{e}_a} A(\boldsymbol{j} + \boldsymbol{e}_a);$

(2)
$$\begin{aligned}\mathsf{E}_h \cdot A(\boldsymbol{j}) &= \upsilon^{f(h+1)+j_{h+1}} \overline{[\![a_{h,h+1}+1]\!]} (A + E_{h,h+1})(\boldsymbol{j}) \\ &+ \frac{\upsilon^{f(h)-j_h-1}}{1-\upsilon^{-2}} \Big((A - E_{h+1,h})(\boldsymbol{j} + \alpha_h) - (A - E_{h+1,h})(\boldsymbol{j} + \beta_h) \Big) \\ &+ \sum_{k<h, a_{h+1,k}\geqslant 1} \upsilon^{f(k)} \overline{[\![a_{h,k}+1]\!]} (A + E_{h,k} - E_{h+1,k})(\boldsymbol{j} + \alpha_h) \\ &+ \sum_{k>h+1, a_{h+1,k}\geqslant 1} \upsilon^{f(k)} \overline{[\![a_{h,k}+1]\!]} (A + E_{h,k} - E_{h+1,k})(\boldsymbol{j});\end{aligned}$$

(3) $\mathsf{F}_h \cdot A(\boldsymbol{j}) = \cdots.$

[1] A.A. Beilinson, G. Lusztig, R. MacPherson, *A geometric setting for the quantum deformation of* GL_n, Duke Math.J. **61** (1990), 655-677.

The Schur(–Weyl) duality

We may view such a construction for a quantum group via its associated Hecke algebra (${}_{\mathcal{H}}\mathcal{H} \rightsquigarrow {}_{\mathbf{U}}\mathbf{U}$) as a new development in the theory of the Schur–Weyl duality:

The Schur(–Weyl) duality

We may view such a construction for a quantum group via its associated Hecke algebra (${}_{\mathcal{H}}\mathcal{H} \rightsquigarrow {}_{\mathbf{U}}\mathbf{U}$) as a new development in the theory of the Schur–Weyl duality:

- For the natural repn V_n of $\mathfrak{gl}_n$, there are commuting actions
$$\mathcal{U}(\mathfrak{gl}_n) \curvearrowright V_n^{\otimes r} \curvearrowleft \mathfrak{S}_r.$$

The Schur(–Weyl) duality

We may view such a construction for a quantum group via its associated Hecke algebra (${}_{\mathcal{H}}\mathcal{H} \rightsquigarrow {}_{\mathbf{U}}\mathbf{U}$) as a new development in the theory of the Schur–Weyl duality:

- For the natural repn V_n of $\mathfrak{gl}_n$, there are commuting actions
$$\mathcal{U}(\mathfrak{gl}_n) \curvearrowright V_n^{\otimes r} \curvearrowleft \mathfrak{S}_r.$$
- This defines two algebra homomorphisms
$$\mathcal{U}(\mathfrak{gl}_n) \xrightarrow{\phi} \mathrm{End}(V_n^{\otimes r}) \xleftarrow{\psi} \mathbb{C}\mathfrak{S}_r.$$

The Schur(–Weyl) duality

We may view such a construction for a quantum group via its associated Hecke algebra (${}_{\mathcal{H}}\mathcal{H} \rightsquigarrow {}_{\mathbf{U}}\mathbf{U}$) as a new development in the theory of the Schur–Weyl duality:

- For the natural repn V_n of $\mathfrak{gl}_n$, there are commuting actions
$$\mathcal{U}(\mathfrak{gl}_n) \curvearrowright V_n^{\otimes r} \curvearrowleft \mathfrak{S}_r.$$
- This defines two algebra homomorphisms
$$\mathcal{U}(\mathfrak{gl}_n) \xrightarrow{\phi} \mathrm{End}(V_n^{\otimes r}) \xleftarrow{\psi} \mathbb{C}\mathfrak{S}_r.$$
- The Schur–Weyl duality (Schur, 1920s, H. Weyl, 1930s) tells

The Schur(–Weyl) duality

We may view such a construction for a quantum group via its associated Hecke algebra (${}_{\mathcal{H}}\mathcal{H} \rightsquigarrow {}_{\mathbf{U}}\mathbf{U}$) as a new development in the theory of the Schur–Weyl duality:

- For the natural repn V_n of $\mathfrak{gl}_n$, there are commuting actions
$$\mathcal{U}(\mathfrak{gl}_n) \curvearrowright V_n^{\otimes r} \curvearrowleft \mathfrak{S}_r.$$
- This defines two algebra homomorphisms
$$\mathcal{U}(\mathfrak{gl}_n) \xrightarrow{\phi} \mathrm{End}(V_n^{\otimes r}) \xleftarrow{\psi} \mathbb{C}\mathfrak{S}_r.$$
- The Schur–Weyl duality (Schur, 1920s, H. Weyl, 1930s) tells
  - A double centraliser property: $\mathrm{im}(\phi) = \mathrm{End}_{\mathbb{C}\mathfrak{S}_r}(V_n^{\otimes r}) = S(n,r)$, the **Schur algebra**, and $\mathrm{im}(\psi) = \mathrm{End}_{\mathcal{U}(\mathfrak{gl}_n)}(V_n^{\otimes r})$;

The Schur(–Weyl) duality

We may view such a construction for a quantum group via its associated Hecke algebra (${}_{\mathcal{H}}\mathcal{H} \rightsquigarrow {}_{\mathbf{U}}\mathbf{U}$) as a new development in the theory of the Schur–Weyl duality:

- For the natural repn V_n of $\mathfrak{gl}_n$, there are commuting actions
$$\mathcal{U}(\mathfrak{gl}_n) \curvearrowright V_n^{\otimes r} \curvearrowleft \mathfrak{S}_r.$$
- This defines two algebra homomorphisms
$$\mathcal{U}(\mathfrak{gl}_n) \xrightarrow{\phi} \mathrm{End}(V_n^{\otimes r}) \xleftarrow{\psi} \mathbb{C}\mathfrak{S}_r.$$
- The Schur–Weyl duality (Schur, 1920s, H. Weyl, 1930s) tells
    - A double centraliser property: $\mathrm{im}(\phi) = \mathrm{End}_{\mathbb{C}\mathfrak{S}_r}(V_n^{\otimes r}) = S(n,r)$, the **Schur algebra**, and $\mathrm{im}(\psi) = \mathrm{End}_{\mathcal{U}(\mathfrak{gl}_n)}(V_n^{\otimes r})$;
    - A category equivalence: $S(n,r)$-**mod** $\xrightarrow{\sim} \mathbb{C}\mathfrak{S}_r$-**mod** ($n \geqslant r$) given by the Schur functors;

The Schur(–Weyl) duality

We may view such a construction for a quantum group via its associated Hecke algebra (${}_{\mathcal{H}}\mathcal{H} \rightsquigarrow {}_{\mathbf{U}}\mathbf{U}$) as a new development in the theory of the Schur–Weyl duality:

- For the natural repn V_n of $\mathfrak{gl}_n$, there are commuting actions
$$\mathcal{U}(\mathfrak{gl}_n) \curvearrowright V_n^{\otimes r} \curvearrowleft \mathfrak{S}_r.$$
- This defines two algebra homomorphisms
$$\mathcal{U}(\mathfrak{gl}_n) \xrightarrow{\phi} \mathrm{End}(V_n^{\otimes r}) \xleftarrow{\psi} \mathbb{C}\mathfrak{S}_r.$$
- The Schur–Weyl duality (Schur, 1920s, H. Weyl, 1930s) tells
    - A double centraliser property: $\mathrm{im}(\phi) = \mathrm{End}_{\mathbb{C}\mathfrak{S}_r}(V_n^{\otimes r}) = S(n,r)$, the **Schur algebra**, and $\mathrm{im}(\psi) = \mathrm{End}_{\mathcal{U}(\mathfrak{gl}_n)}(V_n^{\otimes r})$;
    - A category equivalence: $S(n,r)$-**mod** $\xrightarrow{\sim} \mathbb{C}\mathfrak{S}_r$-**mod** ($n \geqslant r$) given by the Schur functors;
    - A structural connection: ${}_{\mathcal{H}}\mathcal{H} \rightsquigarrow {}_{\mathbf{U}}\mathbf{U}$ (compare $W \rightsquigarrow$ BN-pair struct.).

The Schur(–Weyl) duality

We may view such a construction for a quantum group via its associated Hecke algebra (${}_{\mathcal{H}}\mathcal{H} \rightsquigarrow {}_{\mathbf{U}}\mathbf{U}$) as a new development in the theory of the Schur–Weyl duality:

- For the natural repn V_n of $\mathfrak{gl}_n$, there are commuting actions
$$\mathcal{U}(\mathfrak{gl}_n) \curvearrowright V_n^{\otimes r} \curvearrowleft \mathfrak{S}_r.$$
- This defines two algebra homomorphisms
$$\mathcal{U}(\mathfrak{gl}_n) \xrightarrow{\phi} \mathrm{End}(V_n^{\otimes r}) \xleftarrow{\psi} \mathbb{C}\mathfrak{S}_r.$$
- The Schur–Weyl duality (Schur, 1920s, H. Weyl, 1930s) tells
  - A double centraliser property: $\mathrm{im}(\phi) = \mathrm{End}_{\mathbb{C}\mathfrak{S}_r}(V_n^{\otimes r}) = S(n, r)$, the **Schur algebra**, and $\mathrm{im}(\psi) = \mathrm{End}_{\mathcal{U}(\mathfrak{gl}_n)}(V_n^{\otimes r})$;
  - A category equivalence: $S(n, r)$-**mod** $\xrightarrow{\sim} \mathbb{C}\mathfrak{S}_r$-**mod** ($n \geqslant r$) given by the Schur functors;
  - A structural connection: ${}_{\mathcal{H}}\mathcal{H} \rightsquigarrow {}_{\mathbf{U}}\mathbf{U}$ (compare $W \rightsquigarrow$ BN-pair struct.).

Another struct. conn.: presenting q-Schur algebras (Doty–Giaquinto).

The Schur(–Weyl) duality

We may view such a construction for a quantum group via its associated Hecke algebra (${}_{\mathcal{H}}\mathcal{H} \rightsquigarrow {}_{\mathbf{U}}\mathbf{U}$) as a new development in the theory of the Schur–Weyl duality:

- For the natural repn V_n of $\mathfrak{gl}_n$, there are commuting actions
$$\mathcal{U}(\mathfrak{gl}_n) \curvearrowright V_n^{\otimes r} \curvearrowleft \mathfrak{S}_r.$$
- This defines two algebra homomorphisms
$$\mathcal{U}(\mathfrak{gl}_n) \xrightarrow{\phi} \mathrm{End}(V_n^{\otimes r}) \xleftarrow{\psi} \mathbb{C}\mathfrak{S}_r.$$
- The Schur–Weyl duality (Schur, 1920s, H. Weyl, 1930s) tells
  - A double centraliser property: $\mathrm{im}(\phi) = \mathrm{End}_{\mathbb{C}\mathfrak{S}_r}(V_n^{\otimes r}) = S(n, r)$, the **Schur algebra**, and $\mathrm{im}(\psi) = \mathrm{End}_{\mathcal{U}(\mathfrak{gl}_n)}(V_n^{\otimes r})$;
  - A category equivalence: $S(n, r)$-**mod** $\xrightarrow{\sim} \mathbb{C}\mathfrak{S}_r$-**mod** ($n \geqslant r$) given by the Schur functors;
  - A structural connection: ${}_{\mathcal{H}}\mathcal{H} \rightsquigarrow {}_{\mathbf{U}}\mathbf{U}$ (compare $W \rightsquigarrow$ BN-pair struct.).

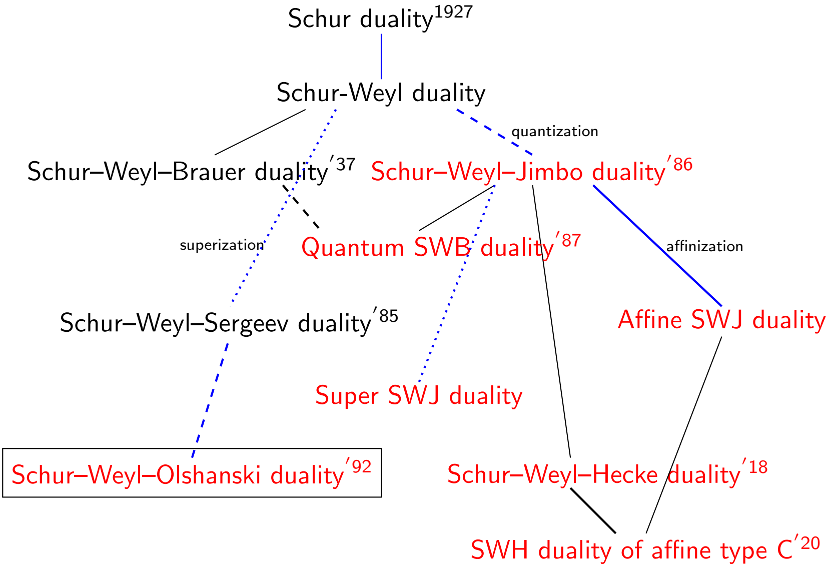
  Another struct. conn.: presenting q-Schur algebras (Doty–Giaquinto).

Since BLM's construction is geometric, one does not see directly how the constructions are originated from those in $\mathcal{H}$.

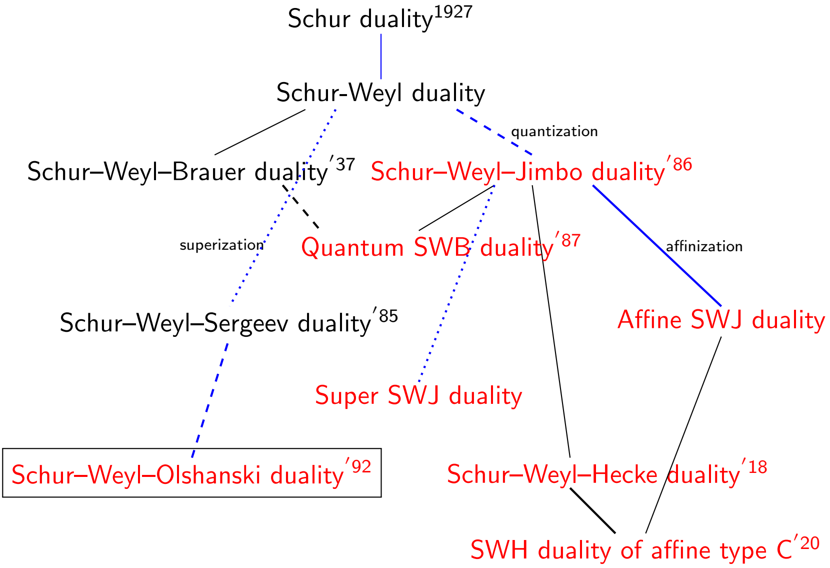
[2] J. Du, H. Gu and Z. Zhou, *Multiplication formulas and semisimplicity for q-Schur superalgebras*, Nagoya Math.J. **237** (2020), 98-126.

100 years of the Schur–Weyl duality

100 years of the Schur–Weyl duality



100 years of the Schur–Weyl duality



The monster is in Western Australia which is 6.3 times large than California!

Aim of the project

Construct the quantum queer supergroup $\mathbf{U}_v(\mathfrak{q}_n)$ using Hecke–Clifford superalgebras $\mathcal{H}_r^c$. Thus, $\mathcal{H}_r^c$ plays the role as the Weyl group for a group with a BN-pair.

Aim of the project

Construct the quantum queer supergroup $\mathbf{U}_v(\mathfrak{q}_n)$ using Hecke–Clifford superalgebras $\mathcal{H}_r^c$. Thus, $\mathcal{H}_r^c$ plays the role as the Weyl group for a group with a BN-pair.

Justification of the project

Aim of the project

Construct the quantum queer supergroup $\mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n)$ using Hecke–Clifford superalgebras $\mathcal{H}_r^c$. Thus, $\mathcal{H}_r^c$ plays the role as the Weyl group for a group with a BN-pair.

Justification of the project

- Olshanski (1992) proved that there exist epi-morphisms

$$\mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n) \longrightarrow \mathrm{End}_{\mathcal{H}^c_{\boldsymbol{\upsilon},\mathbb{C}(\boldsymbol{\upsilon})}}(V(n|n)^{\otimes r}) \text{ for every } r > 0.$$

Aim of the project

Construct the quantum queer supergroup $\mathbf{U}_{\boldsymbol{v}}(\mathfrak{q}_n)$ using Hecke–Clifford superalgebras $\mathcal{H}_r^c$. Thus, $\mathcal{H}_r^c$ plays the role as the Weyl group for a group with a BN-pair.

Justification of the project

- Olshanski (1992) proved that there exist epi-morphisms
$$\mathbf{U}_{\boldsymbol{v}}(\mathfrak{q}_n) \longrightarrow \mathrm{End}_{\mathcal{H}^c_{\boldsymbol{v},\mathbb{C}(\boldsymbol{v})}}(V(n|n)^{\otimes r}) \text{ for every } r > 0.$$
- D.–Wan[4] proved $\mathrm{End}_{\mathcal{H}^c_{\boldsymbol{v},\mathbb{C}(\boldsymbol{v})}}(V(n|n)^{\otimes r}) \cong \mathfrak{Q}_{\boldsymbol{v}}(n, r; \mathbb{C}(\boldsymbol{v}))$ defined by queer permutation modules.

Aim of the project

Construct the quantum queer supergroup $\mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n)$ using Hecke–Clifford superalgebras $\mathcal{H}_r^c$. Thus, $\mathcal{H}_r^c$ plays the role as the Weyl group for a group with a BN-pair.

Justification of the project

- Olshanski (1992) proved that there exist epi-morphisms
$$\mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n) \longrightarrow \mathrm{End}_{\mathcal{H}^c_{\boldsymbol{\upsilon},\mathbb{C}(\boldsymbol{\upsilon})}}(V(n|n)^{\otimes r}) \text{ for every } r > 0.$$
- D.–Wan[4] proved $\mathrm{End}_{\mathcal{H}^c_{\boldsymbol{\upsilon},\mathbb{C}(\boldsymbol{\upsilon})}}(V(n|n)^{\otimes r}) \cong \mathfrak{Q}_{\boldsymbol{\upsilon}}(n, r; \mathbb{C}(\boldsymbol{\upsilon}))$ defined by queer permutation modules.

Almost 10 year efforts

Aim of the project

Construct the quantum queer supergroup $\mathbf{U}_{\upsilon}(\mathfrak{q}_n)$ using Hecke–Clifford superalgebras $\mathcal{H}_r^c$. Thus, $\mathcal{H}_r^c$ plays the role as the Weyl group for a group with a BN-pair.

Justification of the project

- Olshanski (1992) proved that there exist epi-morphisms
$$\mathbf{U}_{\upsilon}(\mathfrak{q}_n) \longrightarrow \mathrm{End}_{\mathcal{H}^c_{\upsilon,\mathbb{C}(\upsilon)}}(V(n|n)^{\otimes r}) \text{ for every } r > 0.$$
- D.–Wan[4] proved $\mathrm{End}_{\mathcal{H}^c_{\upsilon,\mathbb{C}(\upsilon)}}(V(n|n)^{\otimes r}) \cong \mathfrak{Q}_{\upsilon}(n, r; \mathbb{C}(\upsilon))$ defined by queer permutation modules.

Almost 10 year efforts

- The project started in 2014 during Gu's visit.

Aim of the project

Construct the quantum queer supergroup $\mathbf{U}_{\upsilon}(\mathfrak{q}_n)$ using Hecke–Clifford superalgebras $\mathcal{H}_r^c$. Thus, $\mathcal{H}_r^c$ plays the role as the Weyl group for a group with a BN-pair.

Justification of the project

- Olshanski (1992) proved that there exist epi-morphisms
$$\mathbf{U}_{\upsilon}(\mathfrak{q}_n) \longrightarrow \mathrm{End}_{\mathcal{H}^c_{\upsilon,\mathbb{C}(\upsilon)}}(V(n|n)^{\otimes r}) \text{ for every } r > 0.$$
- D.–Wan[4] proved $\mathrm{End}_{\mathcal{H}^c_{\upsilon,\mathbb{C}(\upsilon)}}(V(n|n)^{\otimes r}) \cong \mathfrak{Q}_{\upsilon}(n, r; \mathbb{C}(\upsilon))$ defined by queer permutation modules.

Almost 10 year efforts

- The project started in 2014 during Gu's visit. Since the odd case is too complicated, we doubted the existence of such a theory.

Aim of the project

Construct the quantum queer supergroup $\mathbf{U}_{\boldsymbol{v}}(\mathfrak{q}_n)$ using Hecke–Clifford superalgebras $\mathcal{H}_r^c$. Thus, $\mathcal{H}_r^c$ plays the role as the Weyl group for a group with a BN-pair.

Justification of the project

- Olshanski (1992) proved that there exist epi-morphisms

$$\mathbf{U}_{\boldsymbol{v}}(\mathfrak{q}_n) \longrightarrow \mathrm{End}_{\mathcal{H}^c_{\boldsymbol{v},\mathbb{C}(\boldsymbol{v})}}(V(n|n)^{\otimes r}) \text{ for every } r > 0.$$

- D.–Wan[4] proved $\mathrm{End}_{\mathcal{H}^c_{\boldsymbol{v},\mathbb{C}(\boldsymbol{v})}}(V(n|n)^{\otimes r}) \cong \mathfrak{Q}_{\boldsymbol{v}}(n, r; \mathbb{C}(\boldsymbol{v}))$ defined by queer permutation modules.

Almost 10 year efforts

- The project started in 2014 during Gu's visit. Since the odd case is too complicated, we doubted the existence of such a theory.
- Testing the $\boldsymbol{v} = 1$ case first.

Aim of the project

Construct the quantum queer supergroup $\mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n)$ using Hecke–Clifford superalgebras $\mathcal{H}_r^c$. Thus, $\mathcal{H}_r^c$ plays the role as the Weyl group for a group with a BN-pair.

Justification of the project

- Olshanski (1992) proved that there exist epi-morphisms
$$\mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n) \longrightarrow \mathrm{End}_{\mathcal{H}^c_{\boldsymbol{\upsilon},\mathbb{C}(\boldsymbol{\upsilon})}}(V(n|n)^{\otimes r}) \text{ for every } r > 0.$$
- D.–Wan[4] proved $\mathrm{End}_{\mathcal{H}^c_{\boldsymbol{\upsilon},\mathbb{C}(\boldsymbol{\upsilon})}}(V(n|n)^{\otimes r}) \cong \mathfrak{Q}_{\boldsymbol{\upsilon}}(n, r; \mathbb{C}(\boldsymbol{\upsilon}))$ defined by queer permutation modules.

Almost 10 year efforts

- The project started in 2014 during Gu's visit. Since the odd case is too complicated, we doubted the existence of such a theory.
- Testing the $\boldsymbol{\upsilon} = 1$ case first.
- Seeking a new approach to the regular module—the differential operator approach.

This convinced us that there must be a way to solve the problem.

References

[3] J. Du and J. Wan, *Presenting queer Schur superalgebras*, Int. Math. Res. Notices, **no. 8** (2015) 2210–2272.

[4] J. Du and J. Wan, *The queer q-Schur superalgebras*, J. Aust. Math. Soc., **105** (2018) 316–346.

[5] H. Gu, Z. Li, Y. Lin. *The integral Schur-Weyl-Sergeev duality*. J. Pure Appl. Algebra **226** (2022), 107044.

[6] J. Du and Z. Zhou, *The regular representation of* $U_{\upsilon}(\mathfrak{gl}_{m|n})$, Proc. Amer. Math. Soc., **148** (2020) 111-124.

[7] J. Du, Y. Lin and Z. Zhou, *Quantum queer supergroups via differential operators*, J. Algebra **599** (2022), 48–103.

References

[3] J. Du and J. Wan, *Presenting queer Schur superalgebras*, Int. Math. Res. Notices, **no. 8** (2015) 2210–2272.

[4] J. Du and J. Wan, *The queer q-Schur superalgebras*, J. Aust. Math. Soc., **105** (2018) 316–346.

[5] H. Gu, Z. Li, Y. Lin. *The integral Schur-Weyl-Sergeev duality*. J. Pure Appl. Algebra **226** (2022), 107044.

[6] J. Du and Z. Zhou, *The regular representation of* $U_{\upsilon}(\mathfrak{gl}_{m|n})$, Proc. Amer. Math. Soc., **148** (2020) 111-124.

[7] J. Du, Y. Lin and Z. Zhou, *Quantum queer supergroups via differential operators*, J. Algebra **599** (2022), 48–103.

A roadmap of the construction

References

[3] J. Du and J. Wan, *Presenting queer Schur superalgebras*, Int. Math. Res. Notices, **no. 8** (2015) 2210–2272.

[4] J. Du and J. Wan, *The queer q-Schur superalgebras*, J. Aust. Math. Soc., **105** (2018) 316–346.

[5] H. Gu, Z. Li, Y. Lin. *The integral Schur-Weyl-Sergeev duality*. J. Pure Appl. Algebra **226** (2022), 107044.

[6] J. Du and Z. Zhou, *The regular representation of $U_{\upsilon}(\mathfrak{gl}_{m|n})$*, Proc. Amer. Math. Soc., **148** (2020) 111-124.

[7] J. Du, Y. Lin and Z. Zhou, *Quantum queer supergroups via differential operators*, J. Algebra **599** (2022), 48–103.

A roadmap of the construction

1. Some special elements in the Hecke–Clifford superalgebra $\mathcal{H}^c_{r,R}$.

References

[3] J. Du and J. Wan, *Presenting queer Schur superalgebras*, Int. Math. Res. Notices, **no. 8** (2015) 2210–2272.

[4] J. Du and J. Wan, *The queer q-Schur superalgebras*, J. Aust. Math. Soc., **105** (2018) 316–346.

[5] H. Gu, Z. Li, Y. Lin. *The integral Schur-Weyl-Sergeev duality*. J. Pure Appl. Algebra **226** (2022), 107044.

[6] J. Du and Z. Zhou, *The regular representation of $U_{\upsilon}(\mathfrak{gl}_{m|n})$*, Proc. Amer. Math. Soc., **148** (2020) 111-124.

[7] J. Du, Y. Lin and Z. Zhou, *Quantum queer supergroups via differential operators*, J. Algebra **599** (2022), 48–103.

A roadmap of the construction

1. Some special elements in the Hecke–Clifford superalgebra $\mathcal{H}^{c}_{r,R}$.
2. Some commutation formulas in $\mathcal{H}^{c}_{r,R}$.

References

[3] J. Du and J. Wan, *Presenting queer Schur superalgebras*, Int. Math. Res. Notices, **no. 8** (2015) 2210–2272.

[4] J. Du and J. Wan, *The queer q-Schur superalgebras*, J. Aust. Math. Soc., **105** (2018) 316–346.

[5] H. Gu, Z. Li, Y. Lin. *The integral Schur-Weyl-Sergeev duality*. J. Pure Appl. Algebra **226** (2022), 107044.

[6] J. Du and Z. Zhou, *The regular representation of* $U_{\upsilon}(\mathfrak{gl}_{m|n})$, Proc. Amer. Math. Soc., **148** (2020) 111-124.

[7] J. Du, Y. Lin and Z. Zhou, *Quantum queer supergroups via differential operators*, J. Algebra **599** (2022), 48–103.

A roadmap of the construction

1. Some special elements in the Hecke–Clifford superalgebra $\mathcal{H}^{c}_{r,R}$.
2. Some commutation formulas in $\mathcal{H}^{c}_{r,R}$.
3. Queer q-Schur superalgebras $\mathcal{Q}_q(n,r)$ and its natural basis[3,4].

References

[3] J. Du and J. Wan, *Presenting queer Schur superalgebras*, Int. Math. Res. Notices, **no. 8** (2015) 2210–2272.

[4] J. Du and J. Wan, *The queer q-Schur superalgebras*, J. Aust. Math. Soc., **105** (2018) 316–346.

[5] H. Gu, Z. Li, Y. Lin. *The integral Schur-Weyl-Sergeev duality*. J. Pure Appl. Algebra **226** (2022), 107044.

[6] J. Du and Z. Zhou, *The regular representation of* $U_{\upsilon}(\mathfrak{gl}_{m|n})$, Proc. Amer. Math. Soc., **148** (2020) 111-124.

[7] J. Du, Y. Lin and Z. Zhou, *Quantum queer supergroups via differential operators*, J. Algebra **599** (2022), 48–103.

A roadmap of the construction

1. Some special elements in the Hecke–Clifford superalgebra $\mathcal{H}^{c}_{r,R}$.
2. Some commutation formulas in $\mathcal{H}^{c}_{r,R}$.
3. Queer q-Schur superalgebras $\mathcal{Q}_q(n,r)$ and its natural basis[3,4].
4. Fundamental multiplication formulas[8] in $\mathcal{Q}_q(n,r)$.

References

[3] J. Du and J. Wan, *Presenting queer Schur superalgebras*, Int. Math. Res. Notices, **no. 8** (2015) 2210–2272.

[4] J. Du and J. Wan, *The queer q-Schur superalgebras*, J. Aust. Math. Soc., **105** (2018) 316–346.

[5] H. Gu, Z. Li, Y. Lin. *The integral Schur-Weyl-Sergeev duality*. J. Pure Appl. Algebra **226** (2022), 107044.

[6] J. Du and Z. Zhou, *The regular representation of $U_{\upsilon}(\mathfrak{gl}_{m|n})$*, Proc. Amer. Math. Soc., **148** (2020) 111-124.

[7] J. Du, Y. Lin and Z. Zhou, *Quantum queer supergroups via differential operators*, J. Algebra **599** (2022), 48–103.

A roadmap of the construction

1. Some special elements in the Hecke–Clifford superalgebra $\mathcal{H}^{c}_{r,R}$.
2. Some commutation formulas in $\mathcal{H}^{c}_{r,R}$.
3. Queer q-Schur superalgebras $\mathcal{Q}_q(n,r)$ and its natural basis[3,4].
4. Fundamental multiplication formulas[8] in $\mathcal{Q}_q(n,r)$.
5. Standardisation of everything: $\mathcal{Q}^{s}_{\upsilon}(n,r)$, basis $[A^{\star}]$, and new SMFs.

References

[3] J. Du and J. Wan, *Presenting queer Schur superalgebras*, Int. Math. Res. Notices, **no. 8** (2015) 2210–2272.

[4] J. Du and J. Wan, *The queer q-Schur superalgebras*, J. Aust. Math. Soc., **105** (2018) 316–346.

[5] H. Gu, Z. Li, Y. Lin. *The integral Schur-Weyl-Sergeev duality*. J. Pure Appl. Algebra **226** (2022), 107044.

[6] J. Du and Z. Zhou, *The regular representation of* $U_{\boldsymbol{\upsilon}}(\mathfrak{gl}_{m|n})$, Proc. Amer. Math. Soc., **148** (2020) 111-124.

[7] J. Du, Y. Lin and Z. Zhou, *Quantum queer supergroups via differential operators*, J. Algebra **599** (2022), 48–103.

A roadmap of the construction

1. Some special elements in the Hecke–Clifford superalgebra $\mathcal{H}^c_{r,R}$.
2. Some commutation formulas in $\mathcal{H}^c_{r,R}$.
3. Queer q-Schur superalgebras $\mathcal{Q}_q(n,r)$ and its natural basis[3,4].
4. Fundamental multiplication formulas[8] in $\mathcal{Q}_q(n,r)$.
5. Standardisation of everything: $\mathcal{Q}^s_{\boldsymbol{\upsilon}}(n,r)$, basis $[A^\star]$, and new SMFs.
6. Long multiplication formulas in $\mathcal{Q}^s_{\boldsymbol{\upsilon}}(n,r)$.

References

[3] J. Du and J. Wan, *Presenting queer Schur superalgebras*, Int. Math. Res. Notices, **no. 8** (2015) 2210–2272.

[4] J. Du and J. Wan, *The queer q-Schur superalgebras*, J. Aust. Math. Soc., **105** (2018) 316–346.

[5] H. Gu, Z. Li, Y. Lin. *The integral Schur-Weyl-Sergeev duality*. J. Pure Appl. Algebra **226** (2022), 107044.

[6] J. Du and Z. Zhou, *The regular representation of* $U_{\boldsymbol{\upsilon}}(\mathfrak{gl}_{m|n})$, Proc. Amer. Math. Soc., **148** (2020) 111-124.

[7] J. Du, Y. Lin and Z. Zhou, *Quantum queer supergroups via differential operators*, J. Algebra **599** (2022), 48–103.

A roadmap of the construction

1. Some special elements in the Hecke–Clifford superalgebra $\mathcal{H}^{\mathsf{c}}_{r,R}$.
2. Some commutation formulas in $\mathcal{H}^{\mathsf{c}}_{r,R}$.
3. Queer q-Schur superalgebras $\mathcal{Q}_q(n,r)$ and its natural basis[3,4].
4. Fundamental multiplication formulas[8] in $\mathcal{Q}_q(n,r)$.
5. Standardisation of everything: $\mathcal{Q}^s_{\boldsymbol{\upsilon}}(n,r)$, basis $[A^\star]$, and new SMFs.
6. Long multiplication formulas in $\mathcal{Q}^s_{\boldsymbol{\upsilon}}(n,r)$.
7. Embedding $\mathbf{U}_{\upsilon}(\mathfrak{q}_n)$ into $\prod_{r\geqslant 0}\mathcal{Q}^s_{\boldsymbol{\upsilon}}(n,r)$ via an explicit basis and MFs.

References

[3] J. Du and J. Wan, *Presenting queer Schur superalgebras*, Int. Math. Res. Notices, **no. 8** (2015) 2210–2272.

[4] J. Du and J. Wan, *The queer q-Schur superalgebras*, J. Aust. Math. Soc., **105** (2018) 316–346.

[5] H. Gu, Z. Li, Y. Lin. *The integral Schur-Weyl-Sergeev duality*. J. Pure Appl. Algebra **226** (2022), 107044.

[6] J. Du and Z. Zhou, *The regular representation of $U_{\boldsymbol{\upsilon}}(\mathfrak{gl}_{m|n})$*, Proc. Amer. Math. Soc., **148** (2020) 111-124.

[7] J. Du, Y. Lin and Z. Zhou, *Quantum queer supergroups via differential operators*, J. Algebra **599** (2022), 48–103.

A roadmap of the construction

1. Some special elements in the Hecke–Clifford superalgebra $\mathcal{H}^{\mathrm{c}}_{r,R}$.
2. Some commutation formulas in $\mathcal{H}^{\mathrm{c}}_{r,R}$.
3. Queer q-Schur superalgebras $\mathcal{Q}_q(n,r)$ and its natural basis[3,4].
4. Fundamental multiplication formulas[8] in $\mathcal{Q}_q(n,r)$.
5. Standardisation of everything: $\mathcal{Q}^s_{\boldsymbol{\upsilon}}(n,r)$, basis $[A^\star]$, and new SMFs.
6. Long multiplication formulas in $\mathcal{Q}^s_{\boldsymbol{\upsilon}}(n,r)$.
7. Embedding $\mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n)$ into $\prod_{r\geqslant 0}\mathcal{Q}^s_{\boldsymbol{\upsilon}}(n,r)$ via an explicit basis and MFs.

Applications: Integral Schur duality, root of 1 theory, bar involution, ...

References

[3] J. Du and J. Wan, *Presenting queer Schur superalgebras*, Int. Math. Res. Notices, **no. 8** (2015) 2210–2272.

[4] J. Du and J. Wan, *The queer q-Schur superalgebras*, J. Aust. Math. Soc., **105** (2018) 316–346.

[5] H. Gu, Z. Li, Y. Lin. *The integral Schur-Weyl-Sergeev duality*. J. Pure Appl. Algebra **226** (2022), 107044.

[6] J. Du and Z. Zhou, *The regular representation of* $U_{\boldsymbol{\upsilon}}(\mathfrak{gl}_{m|n})$, Proc. Amer. Math. Soc., **148** (2020) 111-124.

[7] J. Du, Y. Lin and Z. Zhou, *Quantum queer supergroups via differential operators*, J. Algebra **599** (2022), 48–103.

A roadmap of the construction

1. Some special elements in the Hecke–Clifford superalgebra $\mathcal{H}^{c}_{r,R}$.
2. Some commutation formulas in $\mathcal{H}^{c}_{r,R}$.
3. Queer q-Schur superalgebras $\mathcal{Q}_q(n,r)$ and its natural basis[3,4].
4. Fundamental multiplication formulas[8] in $\mathcal{Q}_q(n,r)$.
5. Standardisation of everything: $\mathcal{Q}^s_{\boldsymbol{\upsilon}}(n,r)$, basis $[A^\star]$, and new SMFs.
6. Long multiplication formulas in $\mathcal{Q}^s_{\boldsymbol{\upsilon}}(n,r)$.
7. Embedding $\mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n)$ into $\prod_{r\geqslant 0}\mathcal{Q}^s_{\boldsymbol{\upsilon}}(n,r)$ via an explicit basis and MFs.

Applications: Integral Schur duality, root of 1 theory, bar involution, ...

[8] J. Du, H. Gu, Z. Li, and J. Wan, *Some multiplication formulas in queer q-Schur superalgebras*, Transf. Groups (to appear).

[9] J. Du, H. Gu, Z. Li, and J. Wan, *Constructing the quantum queer supergroup using Hecke–Clifford superalgebras*, in preparation.

2. Hecke–Clifford superalgebras and some special elements

2. Hecke–Clifford superalgebras and some special elements

- Let R be a commutative ring of characteristic not equal to 2.

2. Hecke–Clifford superalgebras and some special elements

- Let R be a commutative ring of characteristic not equal to 2.
- Let $\mathcal{C}_r$ denote the **Clifford superalgebra** over R generated by odd elements $c_1, \ldots, c_r$ subject to the relations

$$c_i^2 = -1, \quad c_i c_j = -c_j c_i, \quad 1 \leqslant i \neq j \leqslant r \qquad (*).$$

2. Hecke–Clifford superalgebras and some special elements

- Let R be a commutative ring of characteristic not equal to 2.
- Let $\mathcal{C}_r$ denote the **Clifford superalgebra** over R generated by odd elements $c_1, \ldots, c_r$ subject to the relations

$$c_i^2 = -1, \quad c_i c_j = -c_j c_i, \quad 1 \leqslant i \neq j \leqslant r \qquad (*).$$

- Let $q \in R$. The **Hecke-Clifford superalgebra** $\mathcal{H}^c_{r,R}$ is the associative R-superalgebra with the even generators $T_1, \ldots, T_{r-1}$ and the odd generators $c_1, \ldots, c_r$ subject to (*) and the following additional relations:

$$(T_i - q)(T_i + 1) = 0, \quad T_i T_{i'} = T_{i'} T_i, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

$$T_i c_j = c_j T_i, \quad T_i c_i = c_{i+1} T_i, \quad T_i c_{i+1} = c_i T_i - (q-1)(c_i - c_{i+1}).$$

2. Hecke–Clifford superalgebras and some special elements

- Let R be a commutative ring of characteristic not equal to 2.
- Let $\mathcal{C}_r$ denote the **Clifford superalgebra** over R generated by odd elements $c_1, \ldots, c_r$ subject to the relations
$$c_i^2 = -1, \quad c_i c_j = -c_j c_i, \quad 1 \leqslant i \neq j \leqslant r \qquad (*).$$
- Let $q \in R$. The **Hecke-Clifford superalgebra** $\mathcal{H}^c_{r,R}$ is the associative R-superalgebra with the even generators $T_1, \ldots, T_{r-1}$ and the odd generators $c_1, \ldots, c_r$ subject to (*) and the following additional relations:
$$(T_i - q)(T_i + 1) = 0, \quad T_i T_{i'} = T_{i'} T_i, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$
$$T_i c_j = c_j T_i, \quad T_i c_i = c_{i+1} T_i, \quad T_i c_{i+1} = c_i T_i - (q-1)(c_i - c_{i+1}).$$
- Natural basis: $\{c^{\boldsymbol{a}} T_w \mid w \in \mathfrak{S}_r, \boldsymbol{a} \in \mathbb{N}_2^r\}$ form bases for $\mathcal{H}^c_{r,R}$. Here $c^{\boldsymbol{a}} = c_1^{a_1} \cdots c_r^{a_r}$.

2. Hecke–Clifford superalgebras and some special elements

- Let R be a commutative ring of characteristic not equal to 2.
- Let $\mathcal{C}_r$ denote the **Clifford superalgebra** over R generated by odd elements $c_1, \ldots, c_r$ subject to the relations
$$c_i^2 = -1, \quad c_i c_j = -c_j c_i, \quad 1 \leqslant i \neq j \leqslant r \qquad (*).$$
- Let $q \in R$. The **Hecke-Clifford superalgebra** $\mathcal{H}^c_{r,R}$ is the associative R-superalgebra with the even generators $T_1, \ldots, T_{r-1}$ and the odd generators $c_1, \ldots, c_r$ subject to (*) and the following additional relations:
$$(T_i - q)(T_i + 1) = 0, \quad T_i T_{i'} = T_{i'} T_i, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$
$$T_i c_j = c_j T_i, \quad T_i c_i = c_{i+1} T_i, \quad T_i c_{i+1} = c_i T_i - (q-1)(c_i - c_{i+1}).$$
- Natural basis: $\{c^{\boldsymbol{a}} T_w \mid w \in \mathfrak{S}_r, \boldsymbol{a} \in \mathbb{N}_2^r\}$ form bases for $\mathcal{H}^c_{r,R}$. Here $c^{\boldsymbol{a}} = c_1^{a_1} \cdots c_r^{a_r}$.
- Structure constants of generators relative to the natural basis:

Basic structure constants

$$c_i(c^{\boldsymbol{a}} T_w) = \begin{cases} (-1)^{\widetilde{a}_{i-1}} c^{\boldsymbol{a}+\varepsilon_i} T_w, & \text{if } a_i = 0; \\ (-1)^{\widetilde{a}_{i-1}+1} c^{\boldsymbol{a}-\varepsilon_i} T_w, & \text{if } a_i = 1. \end{cases}$$

$$T_i(c^{\boldsymbol{a}} T_w) = \begin{cases} c^{\boldsymbol{a}} T_i T_w, & \text{if } a_i = 0, a_{i+1} = 0; \\ c^{\boldsymbol{a}+\varepsilon_{i+1}} T_i T_w, & \text{if } a_i = 1, a_{i+1} = 0; \\ c^{\boldsymbol{a}+\alpha_i} T_i T_w + (q-1)(c^{\boldsymbol{a}} - c^{\boldsymbol{a}+\alpha_i}) T_w, & \text{if } a_i = 0, a_{i+1} = 1; \\ -c^{\boldsymbol{a}} T_i T_w + (q-1)(c^{\boldsymbol{a}} - c^{\boldsymbol{a}-\varepsilon_i-\varepsilon_{i+1}}) T_w, & \text{if } a_i = 1, a_{i+1} = 1; \end{cases}$$

Basic structure constants

$$c_i(c^{\boldsymbol{a}}T_w) = \begin{cases} (-1)^{\widetilde{a}_{i-1}} c^{\boldsymbol{a}+\varepsilon_i} T_w, & \text{if } a_i = 0; \\ (-1)^{\widetilde{a}_{i-1}+1} c^{\boldsymbol{a}-\varepsilon_i} T_w, & \text{if } a_i = 1. \end{cases}$$

$$T_i(c^{\boldsymbol{a}}T_w) = \begin{cases} c^{\boldsymbol{a}} T_i T_w, & \text{if } a_i = 0, a_{i+1} = 0; \\ c^{\boldsymbol{a}+\varepsilon_{i+1}} T_i T_w, & \text{if } a_i = 1, a_{i+1} = 0; \\ c^{\boldsymbol{a}+\alpha_i} T_i T_w + (q-1)(c^{\boldsymbol{a}} - c^{\boldsymbol{a}+\alpha_i}) T_w, & \text{if } a_i = 0, a_{i+1} = 1; \\ -c^{\boldsymbol{a}} T_i T_w + (q-1)(c^{\boldsymbol{a}} - c^{\boldsymbol{a}-\varepsilon_i-\varepsilon_{i+1}}) T_w, & \text{if } a_i = 1, a_{i+1} = 1; \end{cases}$$

We may further break down into to 8 cases using

$$T_i T_w = \begin{cases} T_{s_i w} \quad {\scriptstyle (s_i = (i, i+1))}, & \text{if } s_i w > w; \\ (q-1) T_w + q T_{s_i w}, & \text{if } s_i w < w. \end{cases}$$

Basic structure constants

$$c_i(c^{\mathbf{a}}T_w) = \begin{cases} (-1)^{\widetilde{a}_{i-1}} c^{\mathbf{a}+\varepsilon_i} T_w, & \text{if } a_i = 0; \\ (-1)^{\widetilde{a}_{i-1}+1} c^{\mathbf{a}-\varepsilon_i} T_w, & \text{if } a_i = 1. \end{cases}$$

$$T_i(c^{\mathbf{a}}T_w) = \begin{cases} c^{\mathbf{a}} T_i T_w, & \text{if } a_i = 0, a_{i+1} = 0; \\ c^{\mathbf{a}+\varepsilon_{i+1}} T_i T_w, & \text{if } a_i = 1, a_{i+1} = 0; \\ c^{\mathbf{a}+\alpha_i} T_i T_w + (q-1)(c^{\mathbf{a}} - c^{\mathbf{a}+\alpha_i}) T_w, & \text{if } a_i = 0, a_{i+1} = 1; \\ -c^{\mathbf{a}} T_i T_w + (q-1)(c^{\mathbf{a}} - c^{\mathbf{a}-\varepsilon_i-\varepsilon_{i+1}}) T_w, & \text{if } a_i = 1, a_{i+1} = 1; \end{cases}$$

We may further break down into to 8 cases using

$$T_i T_w = \begin{cases} T_{s_i w} \quad {\scriptstyle (s_i = (i, i+1))}, & \text{if } s_i w > w; \\ (q-1) T_w + q T_{s_i w}, & \text{if } s_i w < w. \end{cases}$$

We now use this fundamental structure to build the structure of the supergroups $\mathbf{U}_\upsilon(\mathfrak{q}_n)$, following the roadmap mentioned above:

Basic structure constants

$$c_i(c^{\boldsymbol{a}}T_w) = \begin{cases} (-1)^{\widetilde{a}_{i-1}} c^{\boldsymbol{a}+\varepsilon_i} T_w, & \text{if } a_i = 0; \\ (-1)^{\widetilde{a}_{i-1}+1} c^{\boldsymbol{a}-\varepsilon_i} T_w, & \text{if } a_i = 1. \end{cases}$$

$$T_i(c^{\boldsymbol{a}}T_w) = \begin{cases} c^{\boldsymbol{a}} T_i T_w, & \text{if } a_i = 0, a_{i+1} = 0; \\ c^{\boldsymbol{a}+\varepsilon_{i+1}} T_i T_w, & \text{if } a_i = 1, a_{i+1} = 0; \\ c^{\boldsymbol{a}+\alpha_i} T_i T_w + (q-1)(c^{\boldsymbol{a}} - c^{\boldsymbol{a}+\alpha_i}) T_w, & \text{if } a_i = 0, a_{i+1} = 1; \\ -c^{\boldsymbol{a}} T_i T_w + (q-1)(c^{\boldsymbol{a}} - c^{\boldsymbol{a}-\varepsilon_i-\varepsilon_{i+1}}) T_w, & \text{if } a_i = 1, a_{i+1} = 1; \end{cases}$$

We may further break down into to 8 cases using

$$T_i T_w = \begin{cases} T_{s_i w} \quad (s_i = (i, i+1)), & \text{if } s_i w > w; \\ (q-1) T_w + q T_{s_i w}, & \text{if } s_i w < w. \end{cases}$$

We now use this fundamental structure to build the structure of the supergroups $\mathbf{U}_\upsilon(\mathfrak{q}_n)$, following the roadmap mentioned above:

1. Define special elements in $\mathcal{H}^c_{r,R}$: x_λ, y_λ, $c_{q,i,j}$, $c^{\boldsymbol{a}}_\lambda$, $c_{A^\star}$, $T_{A^\star}$,
2. Some commutation relations (CR1), (CR2), and (CR3);
3. ....

Some special elements in $\mathcal{H}^{c}_{r,R}$

The elements x_λ, y_λ

Denote by $\Lambda(n, r) \subset \mathbb{N}^n$ the set of compositions of r with n parts.

Some special elements in $\mathcal{H}^c_{r,R}$

The elements x_λ, y_λ

Denote by $\Lambda(n,r) \subset \mathbb{N}^n$ the set of compositions of r with n parts. Given $\lambda \in \Lambda(n,r)$, elements $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w, \quad y_\lambda = \sum_{w\in\mathfrak{S}_\lambda}(-q^{-1})^{\ell(w)} T_w,$ where $\ell(w)$ is the length of w, to define **queer permutation modules**.

Some special elements in $\mathcal{H}^{c}_{r,R}$

The elements x_λ, y_λ

Denote by $\Lambda(n,r) \subset \mathbb{N}^n$ the set of compositions of r with n parts. Given $\lambda \in \Lambda(n,r)$, elements $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w, \quad y_\lambda = \sum_{w\in\mathfrak{S}_\lambda}(-q^{-1})^{\ell(w)} T_w$, where $\ell(w)$ is the length of w, to define **queer permutation modules**.

The elements $c_{q,i,j}, c^{\mathbf{a}}_\lambda$

For $r \geqslant 1$ and $1 \leqslant i < j \leqslant r$, we set

$$c_{q,i,j} = q^{j-i}c_i + q^{j-i-1}c_{i+1} + \cdots + qc_{j-1} + c_j, \; c'_{q,i,j} = c_i + qc_{i+1} + \cdots + q^{j-i}c_j$$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$ and $\boldsymbol{a} \in \mathbb{N}_2^n$, let $\widetilde{\lambda}_k = \lambda_1 + \cdots + \lambda_k$ and assume $a_k \leqslant \lambda_k$, for $1 \leqslant k \leqslant n$.

Some special elements in $\mathcal{H}^c_{r,R}$

The elements x_λ, y_λ

Denote by $\Lambda(n,r) \subset \mathbb{N}^n$ the set of compositions of r with n parts. Given $\lambda \in \Lambda(n,r)$, elements $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w$, $\quad y_\lambda = \sum_{w\in\mathfrak{S}_\lambda}(-q^{-1})^{\ell(w)} T_w$, where $\ell(w)$ is the length of w, to define **queer permutation modules**.

The elements $c_{q,i,j}, c^{\mathbf{a}}_\lambda$

For $r \geqslant 1$ and $1 \leqslant i < j \leqslant r$, we set

$$c_{q,i,j} = q^{j-i}c_i + q^{j-i-1}c_{i+1} + \cdots + qc_{j-1} + c_j,\ \ c'_{q,i,j} = c_i + qc_{i+1} + \cdots + q^{j-i}c_j$$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$ and $\boldsymbol{a} \in \mathbb{N}_2^n$, let $\widetilde{\lambda}_k = \lambda_1 + \cdots + \lambda_k$ and assume $a_k \leqslant \lambda_k$, for $1 \leqslant k \leqslant n$. Define the following elements in $\mathcal{C}_r$:

$$c^{\boldsymbol{a}}_\lambda := (c_{q,1,\widetilde{\lambda}_1})^{a_1}(c_{q,\widetilde{\lambda}_1+1,\widetilde{\lambda}_2})^{a_2}\cdots(c_{q,\widetilde{\lambda}_{N-1}+1,\widetilde{\lambda}_n})^{a_n},$$
$$(c^{\boldsymbol{a}}_\lambda)' := (c'_{q,1,\widetilde{\lambda}_1})^{a_1}(c'_{q,\widetilde{\lambda}_1+1,\widetilde{\lambda}_2})^{a_2}\cdots(c'_{q,\widetilde{\lambda}_{N-1}+1,\widetilde{\lambda}_n})^{a_n}.$$

Some special elements in $\mathcal{H}^c_{r,R}$

The elements x_λ, y_λ

Denote by $\Lambda(n,r) \subset \mathbb{N}^n$ the set of compositions of r with n parts. Given $\lambda \in \Lambda(n,r)$, elements $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w$, $\quad y_\lambda = \sum_{w\in\mathfrak{S}_\lambda}(-q^{-1})^{\ell(w)} T_w$, where $\ell(w)$ is the length of w, to define **queer permutation modules**.

The elements $c_{q,i,j}, c^{\mathbf{a}}_\lambda$

For $r \geqslant 1$ and $1 \leqslant i < j \leqslant r$, we set

$$c_{q,i,j} = q^{j-i}c_i + q^{j-i-1}c_{i+1} + \cdots + qc_{j-1} + c_j,\ \ c'_{q,i,j} = c_i + qc_{i+1} + \cdots + q^{j-i}c_j$$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$ and $\boldsymbol{a} \in \mathbb{N}_2^n$, let $\widetilde{\lambda}_k = \lambda_1 + \cdots + \lambda_k$ and assume $a_k \leqslant \lambda_k$, for $1 \leqslant k \leqslant n$. Define the following elements in $\mathcal{C}_r$:

$$c^{\boldsymbol{a}}_\lambda := (c_{q,1,\widetilde{\lambda}_1})^{a_1}(c_{q,\widetilde{\lambda}_1+1,\widetilde{\lambda}_2})^{a_2}\cdots(c_{q,\widetilde{\lambda}_{N-1}+1,\widetilde{\lambda}_n})^{a_n},$$

$$(c^{\boldsymbol{a}}_\lambda)' := (c'_{q,1,\widetilde{\lambda}_1})^{a_1}(c'_{q,\widetilde{\lambda}_1+1,\widetilde{\lambda}_2})^{a_2}\cdots(c'_{q,\widetilde{\lambda}_{N-1}+1,\widetilde{\lambda}_n})^{a_n}.$$

Commutation relations: $x_\lambda c^{\boldsymbol{a}}_\lambda = (c^{\boldsymbol{a}}_\lambda)' x_\lambda$

[4] J. Du and J. Wan, *The queer q-Schur superalgebra*, J. AustMS, **105** (2018), 316–346.

Elements defined by matrices

Let

$$M_n(\mathbb{N}|\mathbb{N}_2) := \{A^\star = (A^{\bar 0}|A^{\bar 1}) \mid A^{\bar 0} \in M_n(\mathbb{N}), A^{\bar 1} \in M_n(\mathbb{N}_2)\}$$

and let $M_n(\mathbb{N}|\mathbb{N}_2)_r$ be the subset consisting of $(A^{\bar 0}|A^{\bar 1})$ with $|A^{\bar 0} + A^{\bar 1}| = r$.

Elements defined by matrices

Let

$$M_n(\mathbb{N}|\mathbb{N}_2) := \{A^\star = (A^{\bar 0}|A^{\bar 1}) \mid A^{\bar 0} \in M_n(\mathbb{N}), A^{\bar 1} \in M_n(\mathbb{N}_2)\}$$

and let $M_n(\mathbb{N}|\mathbb{N}_2)_r$ be the subset consisting of $(A^{\bar 0}|A^{\bar 1})$ with $|A^{\bar 0} + A^{\bar 1}| = r$.

1. Given $A^\star = (A^{\bar 0}|A^{\bar 1}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, define the base of $A^\star$ to be $A = A^{\bar 0} + A^{\bar 1}$.

Elements defined by matrices

Let

$$M_n(\mathbb{N}|\mathbb{N}_2) := \{A^\star = (A^{\bar 0}|A^{\bar 1}) \mid A^{\bar 0} \in M_n(\mathbb{N}), A^{\bar 1} \in M_n(\mathbb{N}_2)\}$$

and let $M_n(\mathbb{N}|\mathbb{N}_2)_r$ be the subset consisting of $(A^{\bar 0}|A^{\bar 1})$ with $|A^{\bar 0} + A^{\bar 1}| = r$.

1. Given $A^\star = (A^{\bar 0}|A^{\bar 1}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, define the base of $A^\star$ to be $A = A^{\bar 0} + A^{\bar 1}$.
2. For A, define "double coset" $(\mathrm{ro}(A), d_A, \mathrm{co}(A))$, where $d_A \in \mathfrak{S}_{\mathrm{ro}(A)} d_A \mathfrak{S}_{\mathrm{co}(A)}$ has minimal length.

Elements defined by matrices

Let

$$M_n(\mathbb{N}|\mathbb{N}_2) := \{A^\star = (A^{\bar 0}|A^{\bar 1}) \mid A^{\bar 0} \in M_n(\mathbb{N}), A^{\bar 1} \in M_n(\mathbb{N}_2)\}$$

and let $M_n(\mathbb{N}|\mathbb{N}_2)_r$ be the subset consisting of $(A^{\bar 0}|A^{\bar 1})$ with $|A^{\bar 0} + A^{\bar 1}| = r$.

1. Given $A^\star = (A^{\bar 0}|A^{\bar 1}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, define the base of $A^\star$ to be $A = A^{\bar 0} + A^{\bar 1}$.
2. For A, define "double coset" $(\mathrm{ro}(A), d_A, \mathrm{co}(A))$, where $d_A \in \mathfrak{S}_{\mathrm{ro}(A)} d_A \mathfrak{S}_{\mathrm{co}(A)}$ has minimal length.
3. Associated with A and $A^{\bar 1}$, let

$$\nu = \nu_A := (a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, \ldots, a_{1n}, \ldots, a_{nn}) \in \mathbb{N}^{n^2}$$

Elements defined by matrices

Let

$$M_n(\mathbb{N}|\mathbb{N}_2) := \{A^\star = (A^{\bar 0}|A^{\bar 1}) \mid A^{\bar 0} \in M_n(\mathbb{N}), A^{\bar 1} \in M_n(\mathbb{N}_2)\}$$

and let $M_n(\mathbb{N}|\mathbb{N}_2)_r$ be the subset consisting of $(A^{\bar 0}|A^{\bar 1})$ with $|A^{\bar 0} + A^{\bar 1}| = r$.

1. Given $A^\star = (A^{\bar 0}|A^{\bar 1}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, define the base of $A^\star$ to be $A = A^{\bar 0} + A^{\bar 1}$.
2. For A, define "double coset" $(\mathrm{ro}(A), d_A, \mathrm{co}(A))$, where $d_A \in \mathfrak{S}_{\mathrm{ro}(A)} d_A \mathfrak{S}_{\mathrm{co}(A)}$ has minimal length.
3. Associated with A and $A^{\bar 1}$, let

$$\nu = \nu_A := (a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, \ldots, a_{1n}, \ldots, a_{nn}) \in \mathbb{N}^{n^2}$$
$$\alpha = \nu_{A^{\bar 1}} = (a^{\bar 1}_{11}, \ldots, a^{\bar 1}_{n1}, \ldots, a^{\bar 1}_{1n}, \ldots, a^{\bar 1}_{nn}) \in (\mathbb{N}_2)^{n^2}.$$

Elements defined by matrices

Let

$$M_n(\mathbb{N}|\mathbb{N}_2) := \{A^\star = (A^{\bar 0}|A^{\bar 1}) \mid A^{\bar 0} \in M_n(\mathbb{N}), A^{\bar 1} \in M_n(\mathbb{N}_2)\}$$

and let $M_n(\mathbb{N}|\mathbb{N}_2)_r$ be the subset consisting of $(A^{\bar 0}|A^{\bar 1})$ with $|A^{\bar 0} + A^{\bar 1}| = r$.

1. Given $A^\star = (A^{\bar 0}|A^{\bar 1}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, define the base of $A^\star$ to be $A = A^{\bar 0} + A^{\bar 1}$.
2. For A, define "double coset" $(\mathrm{ro}(A), d_A, \mathrm{co}(A))$, where $d_A \in \mathfrak{S}_{\mathrm{ro}(A)} d_A \mathfrak{S}_{\mathrm{co}(A)}$ has minimal length.
3. Associated with A and $A^{\bar 1}$, let

$$\nu = \nu_A := (a_{11}, \dots, a_{n1}, a_{12}, \dots, a_{n2}, \dots, a_{1n}, \dots, a_{nn}) \in \mathbb{N}^{n^2}$$
$$\alpha = \nu_{A^{\bar 1}} = (a^{\bar 1}_{11}, \dots, a^{\bar 1}_{n1}, \dots, a^{\bar 1}_{1n}, \dots, a^{\bar 1}_{nn}) \in (\mathbb{N}_2)^{n^2}.$$

Since $a^{\bar 1}_{i,j} \leqslant a_{i,j}$ (i.e., $\alpha \leqslant \nu$), $c_{A^\star} := c^\alpha_\nu \in \mathcal{C}_r$ is well-defined.

Elements defined by matrices

Let

$$M_n(\mathbb{N}|\mathbb{N}_2) := \{A^\star = (A^{\bar 0}|A^{\bar 1}) \mid A^{\bar 0} \in M_n(\mathbb{N}), A^{\bar 1} \in M_n(\mathbb{N}_2)\}$$

and let $M_n(\mathbb{N}|\mathbb{N}_2)_r$ be the subset consisting of $(A^{\bar 0}|A^{\bar 1})$ with $|A^{\bar 0} + A^{\bar 1}| = r$.

1. Given $A^\star = (A^{\bar 0}|A^{\bar 1}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, define the base of $A^\star$ to be $A = A^{\bar 0} + A^{\bar 1}$.
2. For A, define "double coset" $(\mathrm{ro}(A), d_A, \mathrm{co}(A))$, where $d_A \in \mathfrak{S}_{\mathrm{ro}(A)} d_A \mathfrak{S}_{\mathrm{co}(A)}$ has minimal length.
3. Associated with A and $A^{\bar 1}$, let

$$\nu = \nu_A := (a_{11}, \dots, a_{n1}, a_{12}, \dots, a_{n2}, \dots, a_{1n}, \dots, a_{nn}) \in \mathbb{N}^{n^2}$$
$$\alpha = \nu_{A^{\bar 1}} = (a^{\bar 1}_{11}, \dots, a^{\bar 1}_{n1}, \dots, a^{\bar 1}_{1n}, \dots, a^{\bar 1}_{nn}) \in (\mathbb{N}_2)^{n^2}.$$

Since $a^{\bar 1}_{i,j} \leqslant a_{i,j}$ (i.e., $\alpha \leqslant \nu$), $c_{A^\star} := c^\alpha_\nu \in \mathcal{C}_r$ is well-defined. Define

$$T_{A^\star} := x_\lambda T_{d_A} c_{A^\star} \sum_{\sigma \in \mathcal{D}_{\nu_A} \cap \mathfrak{S}_\mu} T_\sigma = x_\lambda T_{d_A} c_{A^\star} \Sigma_A.$$

Key commutation relations in $\mathcal{H}^{c}_{r,R}$: (CR1),(CR2)&(CR3)

For $A = (a_{i,j}) \in M_n(\mathbb{N})$, $1 \leqslant h \leqslant n-1$ and $1 \leqslant k \leqslant n$, let

$$A^{+}_{h,k} := A + E_{h,k} - E_{h+1,k}, \text{ if } a_{h+1,k} > 0 \text{ (move 1 up a row);}$$

$$A^{-}_{h,k} := A - E_{h,k} + E_{h+1,k}, \text{ if } a_{h,k} > 0 \text{ (move 1 down a row);}$$

Key commutation relations in $\mathcal{H}^c_{r,R}$: (CR1),(CR2)&(CR3)

For $A = (a_{i,j}) \in M_n(\mathbb{N})$, $1 \leqslant h \leqslant n-1$ and $1 \leqslant k \leqslant n$, let

$$A^+_{h,k} := A + E_{h,k} - E_{h+1,k}, \text{ if } a_{h+1,k} > 0 \text{ (move 1 up a row)};$$

$$A^-_{h,k} := A - E_{h,k} + E_{h+1,k}, \text{ if } a_{h,k} > 0 \text{ (move 1 down a row)};$$

(1) If $a_{h+1,k} > 0$, then (CR1)

$$\sum_{j=\overleftarrow{\mathbf{r}}^{k}_{h+1}}^{\overleftarrow{\mathbf{r}}^{k+1}_{h+1}-1} T_{\widetilde{\lambda}_h+1} T_{\widetilde{\lambda}_h+2} \cdots T_{\widetilde{\lambda}_h+j} T_{d_A} = T_{\widetilde{\lambda}_h} T_{\widetilde{\lambda}_h-1} \cdots T_{\widetilde{\lambda}_h-\overrightarrow{\mathbf{r}}^{k}_{h}+1} T_{d_{A^+_{h,k}}} T^{\triangleleft}_{(\widetilde{a}_{h,k}+1,\widetilde{a}_{h,k}+a_{h+1,k}-1)}$$

Key commutation relations in $\mathcal{H}^{c}_{r,R}$: (CR1),(CR2)&(CR3)

For $A = (a_{i,j}) \in M_n(\mathbb{N})$, $1 \leqslant h \leqslant n-1$ and $1 \leqslant k \leqslant n$, let

$$A^+_{h,k} := A + E_{h,k} - E_{h+1,k}, \text{ if } a_{h+1,k} > 0 \text{ (move 1 up a row)};$$

$$A^-_{h,k} := A - E_{h,k} + E_{h+1,k}, \text{ if } a_{h,k} > 0 \text{ (move 1 down a row)};$$

(1) If $a_{h+1,k} > 0$, then (CR1)

$$\sum_{j=\overleftarrow{\mathbf{r}}^{k}_{h+1}}^{\overleftarrow{\mathbf{r}}^{k+1}_{h+1}-1} T_{\widetilde{\lambda}_h+1} T_{\widetilde{\lambda}_h+2} \cdots T_{\widetilde{\lambda}_h+j} T_{d_A} = T_{\widetilde{\lambda}_h} T_{\widetilde{\lambda}_h-1} \cdots T_{\widetilde{\lambda}_h-\overrightarrow{\mathbf{r}}^{k}_{h}+1} T_{d_{A^+_{h,k}}} T^{\triangleleft}_{(\widetilde{a}_{h,k}+1,\widetilde{a}_{h,k}+a_{h+1,k}-1)}$$

(CR2) $T^{\triangleleft}_{(\widetilde{a}_{h,k}+1,\widetilde{a}_{h,k}+a_{h+1,k}-1)} \Sigma_A = T^{\triangleright}_{(\widetilde{a}_{h,k},\widetilde{a}_{h,k}-a_{h,k}+1)} \Sigma_{A^+_{h,k}}$,

Key commutation relations in $\mathcal{H}^{c}_{r,R}$: (CR1),(CR2)&(CR3)

For $A = (a_{i,j}) \in M_n(\mathbb{N})$, $1 \leqslant h \leqslant n-1$ and $1 \leqslant k \leqslant n$, let

$$A^{+}_{h,k} := A + E_{h,k} - E_{h+1,k}, \text{ if } a_{h+1,k} > 0 \text{ (move 1 up a row)};$$

$$A^{-}_{h,k} := A - E_{h,k} + E_{h+1,k}, \text{ if } a_{h,k} > 0 \text{ (move 1 down a row)};$$

(1) If $a_{h+1,k} > 0$, then (CR1)

$$\sum_{j=\overleftarrow{\mathbf{r}}^{k}_{h+1}}^{\overleftarrow{\mathbf{r}}^{k+1}_{h+1}-1} T_{\widetilde{\lambda}_h+1} T_{\widetilde{\lambda}_h+2} \cdots T_{\widetilde{\lambda}_h+j} T_{d_A} = T_{\widetilde{\lambda}_h} T_{\widetilde{\lambda}_h-1} \cdots T_{\widetilde{\lambda}_h-\overrightarrow{\mathbf{r}}^{k}_{h}+1} T_{d_{A^{+}_{h,k}}} T^{\lhd}_{(\widetilde{a}_{h,k}+1,\widetilde{a}_{h,k}+a_{h+1,k}-1)}$$

(CR2) $T^{\lhd}_{(\widetilde{a}_{h,k}+1,\widetilde{a}_{h,k}+a_{h+1,k}-1)} \Sigma_A = T^{\rhd}_{(\widetilde{a}_{h,k},\widetilde{a}_{h,k}-a_{h,k}+1)} \Sigma_{A^{+}_{h,k}}$,

where $T^{\lhd}_{(i,j)} = 1 + T_i + T_i T_{i+1} + \cdots + T_i T_{i+1} \cdots T_j$, $T^{\rhd}_{(j,i)} = 1 + T_j + T_j T_{j-1} + \cdots + T_j T_{j-1} \cdots T_i$, for $i \leqslant j$.

Key commutation relations in $\mathcal{H}^c_{r,R}$: (CR1),(CR2)&(CR3)

For $A = (a_{i,j}) \in M_n(\mathbb{N})$, $1 \leqslant h \leqslant n-1$ and $1 \leqslant k \leqslant n$, let

$$A^+_{h,k} := A + E_{h,k} - E_{h+1,k},\ \text{if } a_{h+1,k} > 0 \text{ (move 1 up a row)};$$

$$A^-_{h,k} := A - E_{h,k} + E_{h+1,k},\ \text{if } a_{h,k} > 0 \text{ (move 1 down a row)};$$

(1) If $a_{h+1,k} > 0$, then (CR1)

$$\sum_{j=\overleftarrow{\mathbf{r}}^{k}_{h+1}}^{\overleftarrow{\mathbf{r}}^{k+1}_{h+1}-1} T_{\widetilde{\lambda}_h+1} T_{\widetilde{\lambda}_h+2} \cdots T_{\widetilde{\lambda}_h+j} T_{d_A} = T_{\widetilde{\lambda}_h} T_{\widetilde{\lambda}_h-1} \cdots T_{\widetilde{\lambda}_h-\overrightarrow{\mathbf{r}}^{k}_{h}+1} T_{d_{A^+_{h,k}}} T^{\lhd}_{(\widetilde{a}_{h,k}+1,\widetilde{a}_{h,k}+a_{h+1,k}-1)}$$

(CR2) $T^{\lhd}_{(\widetilde{a}_{h,k}+1,\widetilde{a}_{h,k}+a_{h+1,k}-1)} \Sigma_A = T^{\rhd}_{(\widetilde{a}_{h,k},\widetilde{a}_{h,k}-a_{h,k}+1)} \Sigma_{A^+_{h,k}}$,

where $T^{\lhd}_{(i,j)} = 1 + T_i + T_i T_{i+1} + \cdots + T_i T_{i+1} \cdots T_j$, $T^{\rhd}_{(j,i)} = 1 + T_j + T_j T_{j-1} + \cdots + T_j T_{j-1} \cdots T_i$, for $i \leqslant j$.

(2) Let $A = (a_{i,j}) \in M_n(\mathbb{N})$. If $a_{h,k} > 0$ and

$$(CR3)\quad c_{\widetilde{a}^r_{h,k-1}+p} T_{d_A} = T_{d_A} c_{\widetilde{a}^c_{h-1,k}+p} \qquad (\text{in } \mathcal{H}^c_{r,R})$$

for each $p \in [1, a_{h,k}]$, then A is said to satisfy the **semi-direct product** (SDP) *condition* at (h,k).

The SDP commutation condition: (CR3)

Definition

Let $A = (a_{i,j}) \in M_n(\mathbb{N})$. If $a_{h,k} > 0$ and

$$c_{\widetilde{a}^r_{h,k-1}+p} T_{d_A} = T_{d_A} c_{\widetilde{a}^c_{h-1,k}+p} \qquad (\text{in } \mathcal{H}^{\mathsf{c}}_{r,R})$$

for each $p \in [1, a_{h,k}]$, then A is said to satisfy the **semi-direct product** (SDP) *condition* at (h, k).

The SDP commutation condition: (CR3)

Definition

Let $A = (a_{i,j}) \in M_n(\mathbb{N})$. If $a_{h,k} > 0$ and

$$c_{\widetilde{a}^r_{h,k-1}+p} T_{d_A} = T_{d_A} c_{\widetilde{a}^c_{h-1,k}+p} \qquad (\text{in } \mathcal{H}^{\mathsf{c}}_{r,R})$$

for each $p \in [1, a_{h,k}]$, then A is said to satisfy the **semi-direct product** (SDP) *condition* at (h, k).

If A satisfies the SDP condition at (h, k) for every $k \in [1, n]$ (resp., $h \in [1, n]$) with $a_{h,k} > 0$, then A is said to satisfy the SDP condition on the hth row (resp., kth column).

The SDP commutation condition: (CR3)

Definition

Let $A = (a_{i,j}) \in M_n(\mathbb{N})$. If $a_{h,k} > 0$ and

$$c_{\widetilde{a}^r_{h,k-1}+p} T_{d_A} = T_{d_A} c_{\widetilde{a}^c_{h-1,k}+p} \qquad (\text{in } \mathcal{H}^c_{r,R})$$

for each $p \in [1, a_{h,k}]$, then A is said to satisfy the **semi-direct product** (SDP) *condition* at (h, k).

If A satisfies the SDP condition at (h, k) for every $k \in [1, n]$ (resp., $h \in [1, n]$) with $a_{h,k} > 0$, then A is said to satisfy the SDP condition on the hth row (resp., kth column).

Theorem

Let $A \in M_n(\mathbb{N})$ and $h, k \in [1, n]$. Then A satisfies the SDP condition at (h, k) if and only if $a_{h,k} > 0$ and $a_{i,j} = 0$, for $i > h$ and $j < k$ (i.e., $a_{h,k} > 0$ and the lower left corner matrix $A^{h,k}_{\neg}$ at (h, k) is 0).

The SDP commutation condition: (CR3)

Definition

Let $A = (a_{i,j}) \in M_n(\mathbb{N})$. If $a_{h,k} > 0$ and

$$c_{\widetilde{a}^r_{h,k-1}+p} T_{d_A} = T_{d_A} c_{\widetilde{a}^c_{h-1,k}+p} \qquad (\text{in } \mathcal{H}^{\mathrm{c}}_{r,R})$$

for each $p \in [1, a_{h,k}]$, then A is said to satisfy the **semi-direct product** (SDP) *condition* at (h, k).

If A satisfies the SDP condition at (h, k) for every $k \in [1, n]$ (resp., $h \in [1, n]$) with $a_{h,k} > 0$, then A is said to satisfy the SDP condition on the hth row (resp., kth column).

Theorem

Let $A \in M_n(\mathbb{N})$ and $h, k \in [1, n]$. Then A satisfies the SDP condition at (h, k) if and only if $a_{h,k} > 0$ and $a_{i,j} = 0$, for $i > h$ and $j < k$ (i.e., $a_{h,k} > 0$ and the lower left corner matrix $A^{h,k}_{\urcorner}$ at (h, k) is 0).

Corollary

Every $A = (a_{i,j}) \in M_n(\mathbb{N})$ satisfies the SDP condition on the 1st column or nth row.

3. The queer q-Schur superalgebra (using x_λ, y_λ) and its standardisation

3. The queer q-Schur superalgebra (using x_λ, y_λ) and its standardisation

As a super analog of the q-Schur algebra or a quantum analog of the Schur superalgebra of type Q, define the **queer q-Schur superalgebra**:

$$\mathcal{Q}_q(n, r; R) := \mathrm{End}_{\mathcal{H}^c_{r,R}}\Big(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda \mathcal{H}^c_{r,R}\Big)$$

$$\cong \mathrm{End}_{\mathcal{H}^c_{r,R}}\Big(\bigoplus_{\lambda\in\Lambda(n,r)} y_\lambda \mathcal{H}^c_{r,R}\Big).$$

3. The queer q-Schur superalgebra (using x_λ, y_λ) and its standardisation

As a super analog of the q-Schur algebra or a quantum analog of the Schur superalgebra of type Q, define the **queer q-Schur superalgebra**:

$$\mathcal{Q}_{\boldsymbol{q}}(n,r;R) := \mathrm{End}_{\mathcal{H}^c_{r,R}}\Big(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda \mathcal{H}^c_{r,R}\Big) \cong \mathrm{End}_{\mathcal{H}^c_{r,R}}\Big(\bigoplus_{\lambda\in\Lambda(n,r)} y_\lambda \mathcal{H}^c_{r,R}\Big).$$

In particular, for indeterminate $\boldsymbol{q} = \boldsymbol{v}^2$, we write
$\mathcal{Q}_{\boldsymbol{q}}(n,r) := \mathcal{Q}_{\boldsymbol{q}}(n,r;\mathbb{Z}[\boldsymbol{q}])$ and $\mathcal{Q}_{\boldsymbol{v}}(n,r) := \mathcal{Q}_{\boldsymbol{q}}(n,r;\mathcal{Z})$.

3. The queer q-Schur superalgebra (using x_λ, y_λ) and its standardisation

As a super analog of the q-Schur algebra or a quantum analog of the Schur superalgebra of type Q, define the **queer q-Schur superalgebra**:

$$\mathcal{Q}_{\boldsymbol{q}}(n,r;R) := \mathrm{End}_{\mathcal{H}^c_{r,R}}\Big(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda \mathcal{H}^c_{r,R}\Big) \cong \mathrm{End}_{\mathcal{H}^c_{r,R}}\Big(\bigoplus_{\lambda\in\Lambda(n,r)} y_\lambda \mathcal{H}^c_{r,R}\Big).$$

In particular, for indeterminate $\boldsymbol{q} = \boldsymbol{v}^2$, we write
$\mathcal{Q}_{\boldsymbol{q}}(n,r) := \mathcal{Q}_{\boldsymbol{q}}(n,r;\mathbb{Z}[\boldsymbol{q}])$ and $\mathcal{Q}_{\boldsymbol{v}}(n,r) := \mathcal{Q}_{\boldsymbol{q}}(n,r;\mathcal{Z})$.

Aim: Construct the natural basis for $\mathcal{Q}_q(n,r;R)$.

Bases for $x_\lambda \mathcal{H}^{c}_{r,R} \cap \mathcal{H}^{c}_{r,R} x_\mu$ and $\mathcal{Q}_q(n, r; R)$

Bases for $x_\lambda\mathcal{H}^{c}_{r,R} \cap \mathcal{H}^{c}_{r,R}x_\mu$ and $\mathcal{Q}_q(n,r;R)$

Proposition

Suppose $\lambda, \mu \in \Lambda(n,r)$. *Then the intersection* $x_\lambda\mathcal{H}^{c}_{r,R} \cap \mathcal{H}^{c}_{r,R}x_\mu$ *is a free R-module with basis*

$$\{T_{A^\star} \mid A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_{\lambda,\mu}\}.$$

Bases for $x_\lambda \mathcal{H}^{c}_{r,R} \cap \mathcal{H}^{c}_{r,R} x_\mu$ and $\mathcal{Q}_q(n, r; R)$

Proposition

Suppose $\lambda, \mu \in \Lambda(n, r)$. Then the intersection $x_\lambda \mathcal{H}^{c}_{r,R} \cap \mathcal{H}^{c}_{r,R} x_\mu$ is a free R-module with basis

$$\{T_{A^\star} \mid A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_{\lambda,\mu}\}.$$

Theorem (D.-Wan, 2018 JAustMS)

Let R be a commutative ring of characteristic not equal to 2. Then the algebra $\mathcal{Q} = \mathcal{Q}_q(n, r; R)$ is a free R-module with a basis given by the set

$$\{\phi_{A^\star} \mid A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r\},$$

where $\phi_{A^\star}(x_\mu h) = \delta_{\mu,\mathrm{co}(A)} T_{A^\star} h$.

Bases for $x_\lambda \mathcal{H}^c_{r,R} \cap \mathcal{H}^c_{r,R} x_\mu$ and $\mathcal{Q}_q(n,r;R)$

Proposition

Suppose $\lambda, \mu \in \Lambda(n,r)$. Then the intersection $x_\lambda \mathcal{H}^c_{r,R} \cap \mathcal{H}^c_{r,R} x_\mu$ is a free R-module with basis

$$\{T_{A^\star} \mid A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_{\lambda,\mu}\}.$$

Theorem (D.-Wan, 2018 JAustMS)

Let R be a commutative ring of characteristic not equal to 2. Then the algebra $\mathcal{Q} = \mathcal{Q}_q(n,r;R)$ is a free R-module with a basis given by the set

$$\{\phi_{A^\star} \mid A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r\},$$

where $\phi_{A^\star}(x_\mu h) = \delta_{\mu,\mathrm{co}(A)} T_{A^\star} h$. In particular, if R is an $\mathbb{Z}[\boldsymbol{q}]$-algebra via $\boldsymbol{q} \mapsto q$, then $\mathcal{Q}_q(n,r;R) \cong \mathcal{Q}_{\boldsymbol{q}}(n,r)_R := \mathcal{Q}_{\boldsymbol{q}}(n,r) \otimes_{\mathcal{A}} R$ (base change p'ty).

Bases for $x_\lambda \mathcal{H}^c_{r,R} \cap \mathcal{H}^c_{r,R} x_\mu$ and $\mathcal{Q}_q(n,r;R)$

Proposition

Suppose $\lambda, \mu \in \Lambda(n,r)$. Then the intersection $x_\lambda \mathcal{H}^c_{r,R} \cap \mathcal{H}^c_{r,R} x_\mu$ is a free R-module with basis

$$\{T_{A^\star} \mid A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_{\lambda,\mu}\}.$$

Theorem (D.-Wan, 2018 JAustMS)

Let R be a commutative ring of characteristic not equal to 2. Then the algebra $\mathcal{Q} = \mathcal{Q}_q(n,r;R)$ is a free R-module with a basis given by the set

$$\{\phi_{A^\star} \mid A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r\},$$

where $\phi_{A^\star}(x_\mu h) = \delta_{\mu,\mathrm{co}(A)} T_{A^\star} h$. In particular, if R is an $\mathbb{Z}[\boldsymbol{q}]$-algebra via $\boldsymbol{q} \mapsto q$, then $\mathcal{Q}_q(n,r;R) \cong \mathcal{Q}_{\boldsymbol{q}}(n,r)_R := \mathcal{Q}_{\boldsymbol{q}}(n,r) \otimes_{\mathcal{A}} R$ (base change p'ty).

The basis is called the **natural basis** for $\mathcal{Q}_q(n,r;R)$.

Bases for $x_\lambda \mathcal{H}^c_{r,R} \cap \mathcal{H}^c_{r,R} x_\mu$ and $\mathcal{Q}_q(n,r;R)$

Proposition

Suppose $\lambda, \mu \in \Lambda(n,r)$. Then the intersection $x_\lambda \mathcal{H}^c_{r,R} \cap \mathcal{H}^c_{r,R} x_\mu$ is a free R-module with basis

$$\{T_{A^\star} \mid A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_{\lambda,\mu}\}.$$

Theorem (D.-Wan, 2018 JAustMS)

Let R be a commutative ring of characteristic not equal to 2. Then the algebra $\mathcal{Q} = \mathcal{Q}_q(n,r;R)$ is a free R-module with a basis given by the set

$$\{\phi_{A^\star} \mid A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r\},$$

where $\phi_{A^\star}(x_\mu h) = \delta_{\mu,\mathrm{co}(A)} T_{A^\star} h$. In particular, if R is an $\mathbb{Z}[\boldsymbol{q}]$-algebra via $\boldsymbol{q} \mapsto q$, then $\mathcal{Q}_q(n,r;R) \cong \mathcal{Q}_{\boldsymbol{q}}(n,r)_R := \mathcal{Q}_{\boldsymbol{q}}(n,r) \otimes_{\mathcal{A}} R$ (base change p'ty).

The basis is called the **natural basis** for $\mathcal{Q}_q(n,r;R)$. To study the regular module ${}_{\mathcal{Q}}\mathcal{Q}$, it is natural to compute $\phi_{B^\star}\phi_{A^\star}$ for some "generators" $\phi_{B^\star}$.

Key ingredients for deriving multiplication formulas $\phi_{B^\star}\phi_{A^\star}$

Key ingredients for deriving multiplication formulas $\phi_{B^\star}\phi_{A^\star}$

For $A^\star, B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, let $\Sigma_A = \sum_{\sigma\in\mathcal{D}_{\nu_A}\cap\mathfrak{S}_\mu} T_\sigma$ be the "tail term" in the elements: $T_{A^\star} = x_{\mathrm{co}(A)} T_{d_A} c_{A^\star} \Sigma_A$.

Key ingredients for deriving multiplication formulas $\phi_{B^\star}\phi_{A^\star}$

For $A^\star, B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, let $\Sigma_A = \sum_{\sigma\in\mathcal{D}_{\nu_A}\cap\mathfrak{S}_\mu} T_\sigma$ be the "tail term" in the elements: $T_{A^\star} = x_{\mathrm{co}(A)} T_{d_A} c_{A^\star} \Sigma_A$. Then

$$\phi_{B^\star}\phi_{A^\star}(x_{\mathrm{co}(A)}) = x_{\mathrm{co}(B)} T_{d_B} c_{B^\star} \Sigma_B T_{d_A} c_{A^\star} \Sigma_A$$

Key ingredients for deriving multiplication formulas $\phi_{B^\star}\phi_{A^\star}$

For $A^\star, B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, let $\Sigma_A = \sum_{\sigma\in\mathcal{D}_{\nu_A}\cap\mathfrak{S}_\mu} T_\sigma$ be the "tail term" in the elements: $T_{A^\star} = x_{\mathrm{co}(A)}T_{d_A}c_{A^\star}\Sigma_A$. Then

$$\phi_{B^\star}\phi_{A^\star}(x_{\mathrm{co}(A)}) = x_{\mathrm{co}(B)}T_{d_B}c_{B^\star}\Sigma_B T_{d_A}c_{A^\star}\Sigma_A = \sum_{M^\star\in M_n(\mathbb{N}|\mathbb{N}_2)_r}\gamma_{B^\star,A^\star}^{M^\star}T_{M^\star}.$$

Key ingredients for deriving multiplication formulas $\phi_{B^\star}\phi_{A^\star}$

For $A^\star, B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, let $\Sigma_A = \sum_{\sigma\in\mathcal{D}_{\nu_A}\cap\mathfrak{S}_\mu} T_\sigma$ be the "tail term" in the elements: $T_{A^\star} = x_{\mathrm{co}(A)} T_{d_A} c_{A^\star}\Sigma_A$. Then

$$\phi_{B^\star}\phi_{A^\star}(x_{\mathrm{co}(A)}) = x_{\mathrm{co}(B)} T_{d_B} c_{B^\star}\Sigma_B T_{d_A} c_{A^\star}\Sigma_A = \sum_{M^\star\in M_n(\mathbb{N}|\mathbb{N}_2)_r} \gamma^{M^\star}_{B^\star,A^\star} T_{M^\star}.$$

In general, this computation is too complicated.

1. We take $B^\star$ to be simple enough (e.g., $\sim$ simple roots) such that $d_B = 1$ and $B^\star$ is related to the generators the queer quantum supergroup.

Key ingredients for deriving multiplication formulas $\phi_{B^\star}\phi_{A^\star}$

For $A^\star, B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, let $\Sigma_A = \sum_{\sigma\in\mathcal{D}_{\nu_A}\cap\mathfrak{S}_\mu} T_\sigma$ be the "tail term" in the elements: $T_{A^\star} = x_{\mathrm{co}(A)} T_{d_A} c_{A^\star} \Sigma_A$. Then

$$\phi_{B^\star}\phi_{A^\star}(x_{\mathrm{co}(A)}) = x_{\mathrm{co}(B)} T_{d_B} c_{B^\star} \Sigma_B T_{d_A} c_{A^\star} \Sigma_A = \sum_{M^\star\in M_n(\mathbb{N}|\mathbb{N}_2)_r} \gamma^{M^\star}_{B^\star,A^\star} T_{M^\star}.$$

In general, this computation is too complicated.

1. We take $B^\star$ to be simple enough (e.g., $\sim$ simple roots) such that $d_B = 1$ and $B^\star$ is related to the generators the queer quantum supergroup.
2. We then require some commutation relations in $\mathcal{H}^{c}_{r,R}$:

(CR1) Commuting the tail term Σ_B with T_{d_A} (so $M = A^{\pm}_{h,k}$ occurs);

Key ingredients for deriving multiplication formulas $\phi_{B^\star}\phi_{A^\star}$

For $A^\star, B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, let $\Sigma_A = \sum_{\sigma\in\mathcal{D}_{\nu_A}\cap\mathfrak{S}_\mu} T_\sigma$ be the "tail term" in the elements: $T_{A^\star} = x_{\mathrm{co}(A)} T_{d_A} c_{A^\star} \Sigma_A$. Then

$$\phi_{B^\star}\phi_{A^\star}(x_{\mathrm{co}(A)}) = x_{\mathrm{co}(B)} T_{d_B} c_{B^\star} \Sigma_B T_{d_A} c_{A^\star} \Sigma_A = \sum_{M^\star\in M_n(\mathbb{N}|\mathbb{N}_2)_r} \gamma_{B^\star,A^\star}^{M^\star} T_{M^\star}.$$

In general, this computation is too complicated.

1. We take $B^\star$ to be simple enough (e.g., $\sim$ simple roots) such that $d_B = 1$ and $B^\star$ is related to the generators the queer quantum supergroup.
2. We then require some commutation relations in $\mathcal{H}^{c}_{r,R}$:

(CR1) Commuting the tail term Σ_B with T_{d_A} (so $M = A^{\pm}_{h,k}$ occurs);
(CR2) Reorganising the tail term Σ_A to Σ_M;

Key ingredients for deriving multiplication formulas $\phi_{B^\star}\phi_{A^\star}$

For $A^\star, B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, let $\Sigma_A = \sum_{\sigma\in\mathcal{D}_{\nu_A}\cap\mathfrak{S}_\mu} T_\sigma$ be the "tail term" in the elements: $T_{A^\star} = x_{\mathrm{co}(A)} T_{d_A} c_{A^\star} \Sigma_A$. Then

$$\phi_{B^\star}\phi_{A^\star}(x_{\mathrm{co}(A)}) = x_{\mathrm{co}(B)} T_{d_B} c_{B^\star} \Sigma_B T_{d_A} c_{A^\star} \Sigma_A = \sum_{M^\star\in M_n(\mathbb{N}|\mathbb{N}_2)_r} \gamma_{B^\star,A^\star}^{M^\star} T_{M^\star}.$$

In general, this computation is too complicated.

1. We take $B^\star$ to be simple enough (e.g., $\sim$ simple roots) such that $d_B = 1$ and $B^\star$ is related to the generators the queer quantum supergroup.
2. We then require some commutation relations in $\mathcal{H}^c_{r,R}$:

(CR1) Commuting the tail term Σ_B with T_{d_A} (so $M = A^{\pm}_{h,k}$ occurs);
(CR2) Reorganising the tail term Σ_A to Σ_M;
(CR3) Commuting $c_{B^\star}$ (in the odd case) with $T_{d_A^{\pm}}$—the SDP condition.

Key ingredients for deriving multiplication formulas $\phi_{B^\star}\phi_{A^\star}$

For $A^\star, B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, let $\Sigma_A = \sum_{\sigma\in\mathcal{D}_{\nu_A}\cap\mathfrak{S}_\mu} T_\sigma$ be the "tail term" in the elements: $T_{A^\star} = x_{\mathrm{co}(A)} T_{d_A} c_{A^\star} \Sigma_A$. Then

$$\phi_{B^\star}\phi_{A^\star}(x_{\mathrm{co}(A)}) = x_{\mathrm{co}(B)} T_{d_B} c_{B^\star} \Sigma_B T_{d_A} c_{A^\star} \Sigma_A = \sum_{M^\star\in M_n(\mathbb{N}|\mathbb{N}_2)_r} \gamma_{B^\star,A^\star}^{M^\star} T_{M^\star}.$$

In general, this computation is too complicated.

1. We take $B^\star$ to be simple enough (e.g., $\sim$ simple roots) such that $d_B = 1$ and $B^\star$ is related to the generators the queer quantum supergroup.
2. We then require some commutation relations in $\mathcal{H}^c_{r,R}$:

(CR1) Commuting the tail term Σ_B with T_{d_A} (so $M = A^\pm_{h,k}$ occurs);
(CR2) Reorganising the tail term Σ_A to Σ_M;
(CR3) Commuting $c_{B^\star}$ (in the odd case) with $T_{d_A^\pm}$—the SDP condition.

For the above goals, we need the following:

- The permutation d_A;
- A reduced expression of d_A.

The quantum queer supergroup and selections of $B^{\star}$

The quantum queer supergroup and selections of $B^\star$

The queer quantum supergroup $\mathbf{U}_v(\mathfrak{q}_n)$ is a Hopf superalgebra over $\mathbb{Q}(v)$ whose unital associative superalgebra is generated by

$$\text{even generators: } K_i^{\pm 1}, E_j, F_j; \quad \text{odd generators: } K_{\bar{i}}, E_{\bar{j}}, F_{\bar{j}},$$

for $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1$, subject to some ∼40 relations.

The quantum queer supergroup and selections of $B^{\star}$

The queer quantum supergroup $\mathbf{U}_{\upsilon}(\mathfrak{q}_n)$ is a Hopf superalgebra over $\mathbb{Q}(\upsilon)$ whose unital associative superalgebra is generated by

even generators: $K_i^{\pm 1}, E_j, F_j$; odd generators: $K_{\bar{i}}, E_{\bar{j}}, F_{\bar{j}}$,

for $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1$, subject to some ∼40 relations.

These generators correspond to the generators:

$$(E_{j,j}|O), (E_{h,h+1}|O), (E_{h+1,h}|O); \quad (O|E_{j,j}), (O|E_{h,h+1}), (O|E_{h+1,h}).$$

for the queer Lie superalgebra

$$\mathfrak{q}_n = \left\{ A^{\star} = (A^{\bar{0}}|A^{\bar{1}}) := \begin{pmatrix} A^{\bar{0}} & A^{\bar{1}} \\ A^{\bar{1}} & A^{\bar{0}} \end{pmatrix} \mid A, B \in M_n(\mathbb{C}) \right\}$$

The quantum queer supergroup and selections of $B^\star$

The queer quantum supergroup $\mathbf{U}_v(\mathfrak{q}_n)$ is a Hopf superalgebra over $\mathbb{Q}(v)$ whose unital associative superalgebra is generated by

$$\text{even generators: } K_i^{\pm 1}, E_j, F_j; \quad \text{odd generators: } K_{\bar{i}}, E_{\bar{j}}, F_{\bar{j}},$$

for $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1$, subject to some ∼40 relations.

These generators correspond to the generators:

$$(E_{j,j}|O), (E_{h,h+1}|O), (E_{h+1,h}|O); \quad (O|E_{j,j}), (O|E_{h,h+1}), (O|E_{h+1,h}).$$

for the queer Lie superalgebra

$$\mathfrak{q}_n = \left\{ A^\star = (A^{\bar{0}}|A^{\bar{1}}) := \begin{pmatrix} A^{\bar{0}} & A^{\bar{1}} \\ A^{\bar{1}} & A^{\bar{0}} \end{pmatrix} \mid A, B \in M_n(\mathbb{C}) \right\}$$

Thus, we compute $\phi_{B^\star}\phi_{A^\star}$ with $A^\star$ arbitrary and $B^\star$ being one of the following matrices:

The even cases: $(E_{j,j}|O), (E_{h,h+1}|O), (E_{h+1,h}|O)$;

The odd cases: $(O|E_{j,j}), (O|E_{h,h+1}), (O|E_{h+1,h})$.

fMFs—the even case (almost done in 2014)

Let

$D^{\star}_{\mu} := (\mu|O),\ E^{\star}_{h,\lambda} := (\lambda - E_{h+1,h+1} + E_{h,h+1}|O),\ F^{\star}_{h,\lambda} := (\lambda - E_{h,h} + E_{h+1,h}|O).$

fMFs—the even case (almost done in 2014)

Let

$D^\star_\mu := (\mu|O)$, $E^\star_{h,\lambda} := (\lambda - E_{h+1,h+1} + E_{h,h+1}|O)$, $F^\star_{h,\lambda} := (\lambda - E_{h,h} + E_{h+1,h}|O)$.

Theorem

Let $h \in [1, n-1]$ and $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$. Assume $A = A^{\bar 0} + A^{\bar 1}$ and $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. Then, for $\lambda, \mu \in \Lambda(n, r)$ and $\varepsilon = \delta_{\lambda,\mathrm{ro}(A)}$, the following multiplication formulas hold in $\mathcal{Q}_q(n, r; R)$:

fMFs—the even case (almost done in 2014)

Let

$D^\star_\mu := (\mu|O),\ E^\star_{h,\lambda} := (\lambda - E_{h+1,h+1} + E_{h,h+1}|O),\ F^\star_{h,\lambda} := (\lambda - E_{h,h} + E_{h+1,h}|O).$

Theorem

Let $h \in [1, n-1]$ and $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$. Assume $A = A^{\bar 0} + A^{\bar 1}$ and $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. Then, for $\lambda, \mu \in \Lambda(n, r)$ and $\varepsilon = \delta_{\lambda,\mathrm{ro}(A)}$, the following multiplication formulas hold in $\mathcal{Q}_q(n, r; R)$:

$$(1)\ \ \phi_{D^\star_\mu}\phi_{A^\star} = \delta_{\mu,\mathrm{ro}(A)}\phi_{A^\star}, \qquad \phi_{A^\star}\phi_{D^\star_\mu} = \delta_{\mu,\mathrm{co}(A)}\phi_{A^\star} \quad (D^\star_\mu := (\mu|O)).$$

fMFs—the even case (almost done in 2014)

Let

$D^\star_\mu := (\mu|O),\ E^\star_{h,\lambda} := (\lambda - E_{h+1,h+1} + E_{h,h+1}|O),\ F^\star_{h,\lambda} := (\lambda - E_{h,h} + E_{h+1,h}|O).$

Theorem

Let $h \in [1, n-1]$ and $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$. Assume $A = A^{\bar 0} + A^{\bar 1}$ and $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. Then, for $\lambda, \mu \in \Lambda(n, r)$ and $\varepsilon = \delta_{\lambda,\mathrm{ro}(A)}$, the following multiplication formulas hold in $\mathcal{Q}_q(n, r; R)$:

$$(1)\quad \phi_{D^\star_\mu}\phi_{A^\star} = \delta_{\mu,\mathrm{ro}(A)}\phi_{A^\star}, \qquad \phi_{A^\star}\phi_{D^\star_\mu} = \delta_{\mu,\mathrm{co}(A)}\phi_{A^\star} \quad (D^\star_\mu := (\mu|O)).$$

$$(2)\quad \phi_{E^\star_{h,\lambda}}\phi_{A^\star} = \varepsilon\sum_{k=1}^{n}\Big\{q^{\overrightarrow{\mathbf{r}}^k_h + a^{\bar 1}_{h+1,k}}[\![a^{\bar 0}_{h,k}+1]\!]_q\,\phi_{(A^{\bar 0}+E_{h,k}-E_{h+1,k}|A^{\bar 1})}$$
$$+\, q^{\overrightarrow{\mathbf{r}}^k_h}\phi_{(A^{\bar 0}|A^{\bar 1}+E_{h,k}-E_{h+1,k})}$$
$$+\, q^{\overrightarrow{\mathbf{r}}^k_h - 1}[\![a_{h,k}+1]\!]_{q,q^2}\,\phi_{(A^{\bar 0}+2E_{h,k}|A^{\bar 1}-E_{h,k}-E_{h+1,k})}\Big\}.$$

fMFs—the even case (almost done in 2014)

Let

$D^\star_\mu := (\mu|O)$, $E^\star_{h,\lambda} := (\lambda - E_{h+1,h+1} + E_{h,h+1}|O)$, $F^\star_{h,\lambda} := (\lambda - E_{h,h} + E_{h+1,h}|O)$.

Theorem

Let $h \in [1, n-1]$ and $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$. Assume $A = A^{\bar 0} + A^{\bar 1}$ and $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. Then, for $\lambda, \mu \in \Lambda(n,r)$ and $\varepsilon = \delta_{\lambda,\mathrm{ro}(A)}$, the following multiplication formulas hold in $\mathfrak{Q}_q(n,r;R)$:

(1) $\phi_{D^\star_\mu}\phi_{A^\star} = \delta_{\mu,\mathrm{ro}(A)}\phi_{A^\star}, \qquad \phi_{A^\star}\phi_{D^\star_\mu} = \delta_{\mu,\mathrm{co}(A)}\phi_{A^\star} \quad (D^\star_\mu := (\mu|O))$.

(2)
$$\phi_{E^\star_{h,\lambda}}\phi_{A^\star} = \varepsilon\sum_{k=1}^{n}\Big\{q^{\overrightarrow{\mathbf{r}}^k_h + a^{\bar 1}_{h+1,k}}[\![a^{\bar 0}_{h,k}+1]\!]_q\phi_{(A^{\bar 0}+E_{h,k}-E_{h+1,k}|A^{\bar 1})} + q^{\overrightarrow{\mathbf{r}}^k_h}\phi_{(A^{\bar 0}|A^{\bar 1}+E_{h,k}-E_{h+1,k})} + q^{\overrightarrow{\mathbf{r}}^k_h-1}[\![a_{h,k}+1]\!]_{q,q^2}\phi_{(A^{\bar 0}+2E_{h,k}|A^{\bar 1}-E_{h,k}-E_{h+1,k})}\Big\}.$$

(3)
$$\phi_{F^\star_{h,\lambda}}\phi_{A^\star} = \varepsilon\sum_{k=1}^{n}\Big\{q^{\overleftarrow{\mathbf{r}}^k_{h+1}}[\![a^{\bar 0}_{h+1,k}+1]\!]_q\phi_{(A^{\bar 0}-E_{h,k}+E_{h+1,k}|A^{\bar 1})} + \cdots\Big\}.$$

fMFs—the odd case

Theorem (The Cartan case)

fMFs—the odd case

Theorem (The Cartan case)

For $h \in [1, n]$ *and* $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, *let* $A = A^{\bar 0} + A^{\bar 1}$, $\lambda = \mathrm{ro}(A)$, *and* $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. *Let* $D^\star_{\bar h} = (\lambda - E_{h,h}|E_{h,h})$.

fMFs—the odd case

Theorem (The Cartan case)

For $h \in [1, n]$ and $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, let $A = A^{\bar 0} + A^{\bar 1}$, $\lambda = \mathrm{ro}(A)$, and $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. Let $D^\star_{\bar h} = (\lambda - E_{h,h}|E_{h,h})$.

1. *Assume that A satisfies the SDP condition on the h-th row if $h < n$. Then we have in $\mathcal{Q}_q(n, r; R)$*

$$\phi_{D^\star_{\bar h}} \phi_{A^\star} = \sum_{k=1}^{n} (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}} q^{\overrightarrow{\mathbf{r}}^k_h} \Big\{ \phi_{(A^{\bar 0} - E_{h,k}|A^{\bar 1} + E_{h,k})} - [\![a_{h,k}]\!]_{q^2} \phi_{(A^{\bar 0} + E_{h,k}|A^{\bar 1} - E_{h,k})} \Big\} =: {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{K}}.$$

fMFs—the odd case

Theorem (The Cartan case)

For $h \in [1, n]$ *and* $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$, *let* $A = A^{\bar 0} + A^{\bar 1}$, $\lambda = \mathrm{ro}(A)$, *and* $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. *Let* $D^\star_{\bar h} = (\lambda - E_{h,h}|E_{h,h})$.

1. *Assume that* A *satisfies the SDP condition on the* h*-th row if* $h < n$. *Then we have in* $\mathfrak{Q}_q(n, r; R)$

$$\phi_{D^\star_{\bar h}} \phi_{A^\star} = \sum_{k=1}^{n} (-1)^{\tilde a^{\bar 1}_{h-1,k}} q^{\overrightarrow{\mathbf{r}}^k_h} \Big\{ \phi_{(A^{\bar 0} - E_{h,k}|A^{\bar 1} + E_{h,k})} - [\![a_{h,k}]\!]_{q^2} \phi_{(A^{\bar 0} + E_{h,k}|A^{\bar 1} - E_{h,k})} \Big\} =: {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{K}}.$$

2. *In general, we have*

$$\phi_{D^\star_{\bar h}} \phi_{A^\star} = {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{K}} + \sum_{\substack{B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r \\ \lfloor B^\star \rfloor \prec A}} f^{D^\star_{\bar h}, A^\star}_{B^\star} \phi_{B^\star} \ \ (f^{D^\star_{\bar h}, A^\star}_{B^\star} \in R).$$

The odd positive simple root case

The odd positive simple root case

Theorem

Let $h \in [1, n-1]$ *and* $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$ *with base* A, $\lambda = \mathrm{ro}(A)$, *and* $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. *Let* $E^\star_{\bar h} = (\lambda - E_{h+1,h+1}|E_{h,h+1})$.

The odd positive simple root case

Theorem

Let $h \in [1, n-1]$ *and* $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$ *with base* A, $\lambda = \mathrm{ro}(A)$, *and* $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. *Let* $E^\star_{\bar h} = (\lambda - E_{h+1,h+1}|E_{h,h+1})$.

1. *Suppose that, for every* $k \in [1, n]$ *such that* $a_{h+1,k} > 0$, A *satisfies the SDP condition at* (h, k) *if* $a_{h,k} > 0$ *and satisfies* $A^{h,k}_{\urcorner} = 0$ *if* $a_{h,k} = 0$. *Then we have in* $\mathcal{Q}_q(n, r; R)$

$$\begin{aligned}\phi_{E^\star_{\bar h}}\phi_{A^\star} &= \sum_{k=1}^{n}\Big\{(-1)^{\tilde a^{\bar 1}_{h-1,k}} q^{\overrightarrow{\mathbf{r}}^k_h + a^{\bar 1}_{h+1,k}}\phi_{(A^{\bar 0}-E_{h+1,k}|A^{\bar 1}+E_{h,k})} \\ &+ (-1)^{\tilde a^{\bar 1}_{h-1,k}+1-a^{\bar 1}_{h,k}} q^{\overrightarrow{\mathbf{r}}^k_h}[\![a^{\bar 0}_{h,k}+1]\!]_q \phi_{(A^{\bar 0}+E_{h,k}|A^{\bar 1}-E_{h+1,k})} \\ &+ (-1)^{\tilde a^{\bar 1}_{h-1,k}} q^{\overrightarrow{\mathbf{r}}^k_h - 1 + a^{\bar 1}_{h+1,k}}[\![a_{h,k}+1]\!]_{q^2,q}\phi_{(A^{\bar 0}+2E_{h,k}-E_{h+1,k}|A^{\bar 1}-E_{h,k})}\Big\} \\ &=: {}_{\mathrm{SDP}}\mathrm{H\overline{E}}.\end{aligned}$$

The odd positive simple root case

Theorem

Let $h \in [1, n-1]$ *and* $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$ *with base* A, $\lambda = \mathrm{ro}(A)$, *and* $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. *Let* $E^\star_{\bar h} = (\lambda - E_{h+1,h+1}|E_{h,h+1})$.

1. *Suppose that, for every* $k \in [1, n]$ *such that* $a_{h+1,k} > 0$, A *satisfies the SDP condition at* (h, k) *if* $a_{h,k} > 0$ *and satisfies* $A^{h,k}_{\neg} = 0$ *if* $a_{h,k} = 0$. *Then we have in* $\mathcal{Q}_q(n, r; R)$

$$\phi_{E^\star_{\bar h}}\phi_{A^\star} = \sum_{k=1}^{n}\Big\{(-1)^{\widetilde{a}^{\bar 1}_{h-1,k}} q^{\overrightarrow{\mathbf{r}}^k_h + a^{\bar 1}_{h+1,k}} \phi_{(A^{\bar 0}-E_{h+1,k}|A^{\bar 1}+E_{h,k})}$$
$$+ (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}+1-a^{\bar 1}_{h,k}} q^{\overrightarrow{\mathbf{r}}^k_h} [\![a^{\bar 0}_{h,k}+1]\!]_q \phi_{(A^{\bar 0}+E_{h,k}|A^{\bar 1}-E_{h+1,k})}$$
$$+ (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}} q^{\overrightarrow{\mathbf{r}}^k_h - 1 + a^{\bar 1}_{h+1,k}} [\![a_{h,k}+1]\!]_{q^2,q} \phi_{(A^{\bar 0}+2E_{h,k}-E_{h+1,k}|A^{\bar 1}-E_{h,k})}\Big\}$$
$$=: {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{E}}.$$

2. *In general, we have* $\phi_{E^\star_{\bar h}}\phi_{A^\star} = {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{E}} + \sum_{\substack{B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r \\ \exists k,\, \lfloor B^\star \rfloor \prec A^+_{h.k}}} f^{E^\star_{\bar h}, A^\star}_{B^\star} \phi_{B^\star}$.

The odd positive simple root case

Theorem

Let $h \in [1, n-1]$ and $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$ with base A, $\lambda = \mathrm{ro}(A)$, and $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. Let $E^\star_{\bar h} = (\lambda - E_{h+1,h+1}|E_{h,h+1})$.

1. *Suppose that, for every $k \in [1, n]$ such that $a_{h+1,k} > 0$, A satisfies the SDP condition at (h, k) if $a_{h,k} > 0$ and satisfies $A^{h,k}_{\urcorner} = 0$ if $a_{h,k} = 0$. Then we have in $\mathcal{Q}_q(n, r; R)$*

$$\phi_{E^\star_{\bar h}}\phi_{A^\star} = \sum_{k=1}^{n}\Big\{(-1)^{\widetilde{a}^{\bar 1}_{h-1,k}} q^{\overrightarrow{\mathbf{r}}^k_h + a^{\bar 1}_{h+1,k}}\phi_{(A^{\bar 0}-E_{h+1,k}|A^{\bar 1}+E_{h,k})}$$

$$+ (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}+1-a^{\bar 1}_{h,k}} q^{\overrightarrow{\mathbf{r}}^k_h}[\![a^{\bar 0}_{h,k}+1]\!]_q \phi_{(A^{\bar 0}+E_{h,k}|A^{\bar 1}-E_{h+1,k})}$$

$$+ (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}} q^{\overrightarrow{\mathbf{r}}^k_h - 1 + a^{\bar 1}_{h+1,k}}[\![a_{h,k}+1]\!]_{q^2,q}\phi_{(A^{\bar 0}+2E_{h,k}-E_{h+1,k}|A^{\bar 1}-E_{h,k})}\Big\}$$

$$=: {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{E}}.$$

2. *In general, we have* $\phi_{E^\star_{\bar h}}\phi_{A^\star} = {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{E}} + \sum\limits_{\substack{B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r \\ \exists k,\, \lfloor B^\star \rfloor \prec A^+_{h,k}}} f^{E^\star_{\bar h}, A^\star}_{B^\star}\phi_{B^\star}.$

The odd negative simple root case is a bit more complicated.

Standardisation of everything over $\Bbbk = \mathbb{Q}(\boldsymbol{v})$

1. Replacing the endo-algebra $\mathfrak{Q}_{\boldsymbol{v}}(n,r) = \mathrm{End}_{\mathcal{H}_r^c}(T_\Bbbk(n,r))$ by the superendo-algebra $\mathfrak{Q}_{\boldsymbol{v}}(n,r)$, consisting of $f : T_\Bbbk(n,r) \to T_\Bbbk(n,r)$ s.t. $f(mh) = (-1)^{\wp(f)\wp(h)} f(m)h$:

Standardisation of everything over $\Bbbk = \mathbb{Q}(\boldsymbol{v})$

1. Replacing the endo-algebra $\mathfrak{Q}_{\boldsymbol{v}}(n,r) = \mathrm{End}_{\mathcal{H}_r^c}(T_\Bbbk(n,r))$ by the superendo-algebra $\mathfrak{Q}_{\boldsymbol{v}}(n,r)$, consisting of $f : T_\Bbbk(n,r) \to T_\Bbbk(n,r)$ s.t. $f(mh) = (-1)^{\wp(f)\wp(h)} f(m)h$:
2. The natural basis $\phi_{A^\star}$ is replaced by the natural basis of superhom.
$$\Phi_{A^\star} : x_\mu h \mapsto (-1)^{\wp(A)\cdot\wp(h)} \delta_{\mu,\mathrm{co}(A)} T_A \cdot h.$$

Standardisation of everything over $\Bbbk = \mathbb{Q}(\boldsymbol{v})$

1. Replacing the endo-algebra $\mathfrak{Q}_{\boldsymbol{v}}(n,r) = \mathrm{End}_{\mathcal{H}_r^c}(T_\Bbbk(n,r))$ by the superendo-algebra $\mathfrak{Q}_{\boldsymbol{v}}(n,r)$, consisting of $f : T_\Bbbk(n,r) \to T_\Bbbk(n,r)$ s.t. $f(mh) = (-1)^{\wp(f)\wp(h)} f(m)h$:
2. The natural basis $\phi_{A^\star}$ is replaced by the natural basis of superhom.
$$\Phi_{A^\star} : x_\mu h \mapsto (-1)^{\wp(A)\cdot\wp(h)} \delta_{\mu,\mathrm{co}(A)} T_A \cdot h.$$
3. Standardise the elements $c_{q,i,j}$ and $c'_{q,i,j}$ ($q = \boldsymbol{v}^2$):
$$c_{q,i,j} = \boldsymbol{v}^{2(j-i)} c_i + \boldsymbol{v}^{2(j-i-1)} c_{i+1} + \cdots + \boldsymbol{v}^2 c_{j-1} + c_j = \boldsymbol{v}^{j-i} o_{\boldsymbol{v},i,j}$$

Standardisation of everything over $\Bbbk = \mathbb{Q}(\boldsymbol{v})$

1. Replacing the endo-algebra $\mathfrak{Q}_{\boldsymbol{v}}(n,r) = \mathrm{End}_{\mathcal{H}_r^c}(T_\Bbbk(n,r))$ by the superendo-algebra $\mathfrak{Q}_{\boldsymbol{v}}(n,r)$, consisting of $f : T_\Bbbk(n,r) \to T_\Bbbk(n,r)$ s.t. $f(mh) = (-1)^{\wp(f)\wp(h)} f(m)h$:
2. The natural basis $\phi_{A^\star}$ is replaced by the natural basis of superhom.
$$\Phi_{A^\star} : x_\mu h \mapsto (-1)^{\wp(A)\cdot\wp(h)} \delta_{\mu,\mathrm{co}(A)} T_A \cdot h.$$
3. Standardise the elements $c_{q,i,j}$ and $c'_{q,i,j}$ $(q = \boldsymbol{v}^2)$:
$$c_{q,i,j} = \boldsymbol{v}^{2(j-i)} c_i + \boldsymbol{v}^{2(j-i-1)} c_{i+1} + \cdots + \boldsymbol{v}^2 c_{j-1} + c_j = \boldsymbol{v}^{j-i} o_{\boldsymbol{v},i,j}$$
4. Define $o_\lambda^{\boldsymbol{a}}$ similarly. Then $c_{A^\star} = \boldsymbol{v}^{A^{\bar 0}\cdot A^{\bar 1}} o_{A^\star}$ $(A^{\bar 0}\cdot A^{\bar 1} = \sum_{i,j} a^{\bar 0}_{i,j} a^{\bar 1}_{i,j})$.

Standardisation of everything over $\Bbbk = \mathbb{Q}(\boldsymbol{v})$

1. Replacing the endo-algebra $\mathfrak{Q}_{\boldsymbol{v}}(n,r) = \mathrm{End}_{\mathcal{H}_r^c}(T_\Bbbk(n,r))$ by the superendo-algebra $\mathfrak{Q}_{\boldsymbol{v}}(n,r)$, consisting of $f : T_\Bbbk(n,r) \to T_\Bbbk(n,r)$ s.t. $f(mh) = (-1)^{\wp(f)\wp(h)} f(m)h$:
2. The natural basis $\phi_{A^\star}$ is replaced by the natural basis of superhom.
$$\Phi_{A^\star} : x_\mu h \mapsto (-1)^{\wp(A)\cdot\wp(h)} \delta_{\mu,\mathrm{co}(A)} T_A \cdot h.$$
3. Standardise the elements $c_{q,i,j}$ and $c'_{q,i,j}$ ($q = \boldsymbol{v}^2$):
$$c_{q,i,j} = \boldsymbol{v}^{2(j-i)} c_i + \boldsymbol{v}^{2(j-i-1)} c_{i+1} + \cdots + \boldsymbol{v}^2 c_{j-1} + c_j = \boldsymbol{v}^{j-i} o_{\boldsymbol{v},i,j}$$
4. Define $o_\lambda^{\boldsymbol{a}}$ similarly. Then $c_{A^\star} = \boldsymbol{v}^{A^{\bar 0}\cdot A^{\bar 1}} o_{A^\star}$ $(A^{\bar 0}\cdot A^{\bar 1} = \sum_{i,j} a^{\bar 0}_{i,j} a^{\bar 1}_{i,j})$.
5. Standardise the natural basis $\Phi_{A^\star}$ to the standard (or normalised) basis
$$[A^\star] := \boldsymbol{v}^{-\partial(A^\star)} \Phi_{A^\star}$$
where $\partial(A^\star) = \ell(d_A^+) - \ell(w_{0,\mathrm{co}(A)}) + A^{\bar 0}\cdot A^{\bar 1}$.
Note that $\ell(d_A^+) - \ell(w_{0,\mathrm{co}(A)}) = \dim \mathcal{O}_A = \sum_{i\geqslant k, j<l} a_{i,j} a_{k,l}$.

4. Standard multiplication formulas and their expansions

Recall

$D^\star_\mu := (\mu|O),\ E^\star_{h,\lambda} := (\lambda - E_{h+1,h+1} + E_{h,h+1}|O),\ F^\star_{h,\lambda} := (\lambda - E_{h,h} + E_{h+1,h}|O).$

4. Standard multiplication formulas and their expansions

Recall

$D^\star_\mu := (\mu|O),\ E^\star_{h,\lambda} := (\lambda - E_{h+1,h+1} + E_{h,h+1}|O),\ F^\star_{h,\lambda} := (\lambda - E_{h,h} + E_{h+1,h}|O).$

Theorem (The even case)

Let $h \in [1, n-1]$ and $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$. Assume $A = A^{\bar 0} + A^{\bar 1}$ and $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. Then, for $\lambda, \mu \in \Lambda(n,r)$ and $\varepsilon = \delta_{\lambda,\mathrm{ro}(A)}$, the following multiplication formulas hold in $\mathcal{Q}_q(n,r;R)$:

4. Standard multiplication formulas and their expansions

Recall

$D^\star_\mu := (\mu|O)$, $E^\star_{h,\lambda} := (\lambda - E_{h+1,h+1} + E_{h,h+1}|O)$, $F^\star_{h,\lambda} := (\lambda - E_{h,h} + E_{h+1,h}|O)$.

Theorem (The even case)

Let $h \in [1, n-1]$ and $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$. Assume $A = A^{\bar 0} + A^{\bar 1}$ and $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. Then, for $\lambda, \mu \in \Lambda(n, r)$ and $\varepsilon = \delta_{\lambda,\mathrm{ro}(A)}$, the following multiplication formulas hold in $\mathfrak{Q}_q(n, r; R)$:

(1) $[D^\star_\mu][A^\star] = \delta_{\mu,\mathrm{ro}(A)}[A^\star]$, $\quad [A^\star][D^\star_\mu] = \delta_{\mu,\mathrm{co}(A)}[A^\star]$.

(2)
$$[E^\star_{h,\lambda}][A^\star] = \varepsilon \sum_{k=1}^{n} \boldsymbol{\upsilon}^{g_h(A^\star,k)} \Big\{ \boldsymbol{\upsilon}^{a^{\bar 1}_{h+1,k}} [a^{\bar 0}_{h,k} + 1][(A^{\bar 0} + E_{h,k} - E_{h+1,k}|A^{\bar 1})]$$
$$+ \boldsymbol{\upsilon}^{-a^{\bar 0}_{h+1,k}} [(A^{\bar 0}|A^{\bar 1} + E_{h,k} - E_{h+1,k})]$$
$$- (\boldsymbol{\upsilon} - \boldsymbol{\upsilon}^{-1}) \boldsymbol{\upsilon}^{-a^{\bar 0}_{h+1,k}} \begin{bmatrix} a_{h,k} + 1 \\ 2 \end{bmatrix} [(A^{\bar 0} + 2E_{h,k}|A^{\bar 1} - E_{h,k} - E_{h+1,k})] \Big\}.$$

(3)
$$[F^\star_{h,\lambda}][A^\star] = \varepsilon \sum_{k=1}^{n} \boldsymbol{\upsilon}^{f_h(A^\star,k)} \Big\{ \boldsymbol{\upsilon}^{-a^{\bar 1}_{h,k}} [a^{\bar 0}_{h+1,k} + 1][(A^{\bar 0} - E_{h,k} + E_{h+1,k}|A^{\bar 1})]$$

Standard multiplication formulas—the odd case

Standard multiplication formulas—the odd case

We now want the multiplication formulas $\phi_{X^\star}\phi_{A^\star}$, where, for $\lambda = \mathrm{ro}(A)$, $X^\star$ is one of the matrices

$$D^\star_{\bar{h},\lambda} := (\lambda - E_{h,h}|E_{h,h}),$$
$$E^\star_{\bar{h},\lambda} := (\lambda - E_{h+1,h+1}|E_{h,h+1}),$$
$$F^\star_{\bar{h},\lambda} := (\lambda - E_{h,h}|E_{h+1,h}).$$

Standard multiplication formulas—the odd case

We now want the multiplication formulas $\phi_{X^\star}\phi_{A^\star}$, where, for $\lambda = \mathrm{ro}(A)$, $X^\star$ is one of the matrices

$$D^\star_{\bar{h},\lambda} := (\lambda - E_{h,h}|E_{h,h}),$$
$$E^\star_{\bar{h},\lambda} := (\lambda - E_{h+1,h+1}|E_{h,h+1}),$$
$$F^\star_{\bar{h},\lambda} := (\lambda - E_{h,h}|E_{h+1,h}).$$

It is still impossible to find a complete multiplication formula for $[X^\star][A^\star]$ for each $X^\star$ above. However, we are able to determine the "head part"! In other word, we have

$$[X^\star][A^\star] = {}_{\mathrm{SDP}}\mathrm{Hd} + \text{an undetermined big tail.}$$

Standard multiplication formulas—the odd case

We now want the multiplication formulas $\phi_{X^\star}\phi_{A^\star}$, where, for $\lambda = \mathrm{ro}(A)$, $X^\star$ is one of the matrices

$$D^\star_{\bar{h},\lambda} := (\lambda - E_{h,h}|E_{h,h}),$$
$$E^\star_{\bar{h},\lambda} := (\lambda - E_{h+1,h+1}|E_{h,h+1}),$$
$$F^\star_{\bar{h},\lambda} := (\lambda - E_{h,h}|E_{h+1,h}).$$

It is still impossible to find a complete multiplication formula for $[X^\star][A^\star]$ for each $X^\star$ above. However, we are able to determine the "head part"! In other word, we have

$$[X^\star][A^\star] = {}_{\mathrm{SDP}}\mathrm{Hd} + \text{an undetermined big tail.}$$

Perhaps, AI can do it in the near future!

Theorem (The odd case for positive simple roots)

Let $h \in [1, n-1]$ *and* $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$ *with base* $A = A^{\bar 0} + A^{\bar 1}$ *and* $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. *Let* $E^\star_{\bar h} = (\mathrm{ro}(A) - E_{h+1,h+1}|E_{h,h+1})$.

Theorem (The odd case for positive simple roots)

Let $h \in [1, n-1]$ and $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$ with base $A = A^{\bar 0} + A^{\bar 1}$ and $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. Let $E^\star_{\bar h} = (\mathrm{ro}(A) - E_{h+1,h+1}|E_{h,h+1})$.

1. *Suppose that, for every $k \in [1, n]$ such that $a_{h+1,k} > 0$, A satisfies the SDP condition at (h, k) if $a_{h,k} > 0$ and satisfies $A^{h,k}_\neg = 0$ if $a_{h,k} = 0$. (OK, if $h = n$.)*

Theorem (The odd case for positive simple roots)

Let $h \in [1, n-1]$ *and* $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$ *with base* $A = A^{\bar 0} + A^{\bar 1}$ *and* $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. *Let* $E^\star_{\bar h} = (\mathrm{ro}(A) - E_{h+1,h+1}|E_{h,h+1})$.

1. *Suppose that, for every* $k \in [1, n]$ *such that* $a_{h+1,k} > 0$, A *satisfies the SDP condition at* (h, k) *if* $a_{h,k} > 0$ *and satisfies* $A^{h,k}_{\neg} = 0$ *if* $a_{h,k} = 0$. *(OK, if* $h = n$*.) Then we have in* $\mathfrak{Q}^s_{\boldsymbol{\upsilon}}(n, r)$

$$\begin{aligned}[E^\star_{\bar h}][A^\star] = (-1)^{p(A^\star)} \sum_{k=1}^{n} \boldsymbol{\upsilon}^{g_h(A^\star,k)} \Big\{ & (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}} \boldsymbol{\upsilon}^{a^{\bar 1}_{h+1,k}} [A^{\bar 0} - E_{h+1,k}|A^{\bar 1} + E_{h,k}] \\ & + (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}+1-a^{\bar 1}_{h,k}} \boldsymbol{\upsilon}^{-a^{\bar 0}_{h+1,k}} [a^{\bar 0}_{h,k} + 1][A^{\bar 0} + E_{h,k}|A^{\bar 1} - E_{h+1,k}] \\ & + (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}} \boldsymbol{\upsilon}^{a^{\bar 1}_{h+1,k}} (\boldsymbol{\upsilon} - \boldsymbol{\upsilon}^{-1}) \begin{bmatrix} a_{h,k} + 1 \\ 2 \end{bmatrix} [A^{\bar 0} + 2E_{h,k} - E_{h+1,k}|A^{\bar 1} - E_{h,k}] \Big\} \\ =: {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{E}} \end{aligned}$$

Theorem (The odd case for positive simple roots)

Let $h \in [1, n-1]$ and $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$ with base $A = A^{\bar 0} + A^{\bar 1}$ and $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. Let $E^\star_{\bar h} = (\mathrm{ro}(A) - E_{h+1,h+1}|E_{h,h+1})$.

1. *Suppose that, for every $k \in [1, n]$ such that $a_{h+1,k} > 0$, A satisfies the SDP condition at (h, k) if $a_{h,k} > 0$ and satisfies $A^{h,k}_{\neg} = 0$ if $a_{h,k} = 0$. (OK, if $h = n$.) Then we have in $\mathfrak{Q}^s_{\boldsymbol{\upsilon}}(n, r)$*

$$[E^\star_{\bar h}][A^\star] = (-1)^{p(A^\star)} \sum_{k=1}^{n} \boldsymbol{\upsilon}^{g_h(A^\star,k)} \Big\{ (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}} \boldsymbol{\upsilon}^{a^{\bar 1}_{h+1,k}} [A^{\bar 0} - E_{h+1,k}|A^{\bar 1} + E_{h,k}]$$
$$+ (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}+1-a^{\bar 1}_{h,k}} \boldsymbol{\upsilon}^{-a^{\bar 0}_{h+1,k}} [a^{\bar 0}_{h,k} + 1][A^{\bar 0} + E_{h,k}|A^{\bar 1} - E_{h+1,k}]$$
$$+ (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}} \boldsymbol{\upsilon}^{a^{\bar 1}_{h+1,k}} (\boldsymbol{\upsilon} - \boldsymbol{\upsilon}^{-1}) \begin{bmatrix} a_{h,k}+1 \\ 2 \end{bmatrix} [A^{\bar 0} + 2E_{h,k} - E_{h+1,k}|A^{\bar 1} - E_{h,k}] \Big\}$$
$$=: {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{E}}$$

2. *In general, we have $[E^\star_{\bar h}][A^\star] = {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{E}} + \sum_{\substack{B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r \\ \exists k,\, B \prec A^+_{h.k}}} f^{E^\star_{\bar h}, A^\star}_{B^\star} [B^\star]$.*

Theorem (The odd case for positive simple roots)

Let $h \in [1, n-1]$ and $A^\star = (A^{\bar 0}|A^{\bar 1}) = (a^{\bar 0}_{i,j}|a^{\bar 1}_{i,j}) \in M_n(\mathbb{N}|\mathbb{N}_2)_r$ with base $A = A^{\bar 0} + A^{\bar 1}$ and $\overrightarrow{\mathbf{r}}^k_h = \overrightarrow{\mathbf{r}}^k_h(A)$. Let $E^\star_{\bar h} = (\mathrm{ro}(A) - E_{h+1,h+1}|E_{h,h+1})$.

1. *Suppose that, for every $k \in [1, n]$ such that $a_{h+1,k} > 0$, A satisfies the SDP condition at (h, k) if $a_{h,k} > 0$ and satisfies $A^{h,k}_{\neg} = 0$ if $a_{h,k} = 0$. (OK, if $h = n$.) Then we have in $\mathfrak{Q}^s_{\boldsymbol{\upsilon}}(n, r)$*

$$[E^\star_{\bar h}][A^\star] = (-1)^{p(A^\star)} \sum_{k=1}^{n} \boldsymbol{\upsilon}^{g_h(A^\star,k)} \Big\{ (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}} \boldsymbol{\upsilon}^{a^{\bar 1}_{h+1,k}} [A^{\bar 0} - E_{h+1,k}|A^{\bar 1} + E_{h,k}]$$
$$+ (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}+1-a^{\bar 1}_{h,k}} \boldsymbol{\upsilon}^{-a^{\bar 0}_{h+1,k}} [a^{\bar 0}_{h,k} + 1][A^{\bar 0} + E_{h,k}|A^{\bar 1} - E_{h+1,k}]$$
$$+ (-1)^{\widetilde{a}^{\bar 1}_{h-1,k}} \boldsymbol{\upsilon}^{a^{\bar 1}_{h+1,k}} (\boldsymbol{\upsilon} - \boldsymbol{\upsilon}^{-1}) \begin{bmatrix} a_{h,k} + 1 \\ 2 \end{bmatrix} [A^{\bar 0} + 2E_{h,k} - E_{h+1,k}|A^{\bar 1} - E_{h,k}] \Big\}$$
$$=: {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{E}}$$

2. *In general, we have $[E^\star_{\bar h}][A^\star] = {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{E}} + \sum_{\substack{B^\star \in M_n(\mathbb{N}|\mathbb{N}_2)_r \\ \exists k,\, B \prec A^+_{h.k}}} f^{E^\star_{\bar h}, A^\star}_{B^\star} [B^\star]$.*

For the negative case, $[F^\star_{\bar h}][A^\star] = {}_{\mathrm{SDP}}\mathrm{H}\overline{\mathrm{F}} + (\boldsymbol{\upsilon} - \boldsymbol{\upsilon}^{-1})\mathrm{H}\overline{\mathrm{H}}\overline{\mathrm{F}}$ + lower terms.

Expansions to long elements in $\mathfrak{Q}_{\upsilon}^{s}(n, r)$

Expansions to long elements in $\mathfrak{Q}_{\boldsymbol{\upsilon}}^s(n,r)$

For $A^\star = (A^{\bar 0}|A^{\bar 1}) \in M_n(\mathbb{N}|\mathbb{N}_2)^{\pm}$, $\boldsymbol{j} \in \mathbb{Z}^n$, we define the following elements in $\mathfrak{Q}_{\boldsymbol{\upsilon}}^s(n,r)$:

$$A^\star(\boldsymbol{j}, r) = \begin{cases} \sum\limits_{\lambda \in \Lambda(n, r-|A|)} \boldsymbol{\upsilon}^{\lambda\cdot\boldsymbol{j}}[A^{\bar 0}+\lambda|A^{\bar 1}], & \text{if } |A| \leqslant r; \\ 0, & \text{otherwise.} \end{cases} \tag{4.0.1}$$

where $\lambda \cdot \boldsymbol{j} = \sum_{i=1}^n \lambda_i j_i$.

Expansions to long elements in $\mathfrak{Q}_{\boldsymbol{\upsilon}}^s(n, r)$

For $A^\star = (A^{\bar{0}}|A^{\bar{1}}) \in M_n(\mathbb{N}|\mathbb{N}_2)^{\pm}$, $\boldsymbol{j} \in \mathbb{Z}^n$, we define the following elements in $\mathfrak{Q}_{\boldsymbol{\upsilon}}^s(n, r)$:

$$A^\star(\boldsymbol{j}, r) = \begin{cases} \sum_{\lambda\in\Lambda(n,r-|A|)} \boldsymbol{\upsilon}^{\lambda\cdot\boldsymbol{j}}[A^{\bar{0}} + \lambda|A^{\bar{1}}], & \text{if } |A| \leqslant r; \\ 0, & \text{otherwise.} \end{cases} \tag{4.0.1}$$

where $\lambda \cdot \boldsymbol{j} = \sum_{i=1}^n \lambda_i j_i$.

We now lift the short MFs to some long multiplication formulas (LMFs). For example, the formula for $[E_{h,\lambda}^\star][A^\star]$ has three summations which result in three even bigger summations.

Expansions to long elements in $\mathfrak{Q}_{\upsilon}^{s}(n, r)$

For $A^{\star} = (A^{\bar{0}}|A^{\bar{1}}) \in M_n(\mathbb{N}|\mathbb{N}_2)^{\pm}$, $\boldsymbol{j} \in \mathbb{Z}^n$, we define the following elements in $\mathfrak{Q}_{\boldsymbol{\upsilon}}^{s}(n, r)$:

$$A^{\star}(\boldsymbol{j}, r) = \begin{cases} \sum_{\lambda \in \Lambda(n, r-|A|)} \boldsymbol{\upsilon}^{\lambda \cdot \boldsymbol{j}} [A^{\bar{0}} + \lambda | A^{\bar{1}}], & \text{if } |A| \leqslant r; \\ 0, & \text{otherwise.} \end{cases} \tag{4.0.1}$$

where $\lambda \cdot \boldsymbol{j} = \sum_{i=1}^{n} \lambda_i j_i$.

We now lift the short MFs to some long multiplication formulas (LMFs). For example, the formula for $[E_{h,\lambda}^{\star}][A^{\star}]$ has three summations which result in three even bigger summations.

Proposition

Let $h \in [1, n-1]$. For any $A^{\star} \in M_n(\mathbb{N}|\mathbb{N}_2)^{\pm}$, the following multiplication formulas hold in $\mathfrak{Q}_{\boldsymbol{\upsilon}}^{s}(n, r)$ for all $r \geqslant |A|$:

$$(E_{h,h+1}|\mathrm{O})(\mathbf{0}, r) \cdot A^{\star}(\boldsymbol{j}, r) = (I) + (II) + (III),$$

where

The long multiplication formulas (cont'd)

$$(I) = \sum_{k<h} \boldsymbol{\upsilon}^{g_h(A^\star,k)+a^{\bar 1}_{h+1,k}}[a^{\bar 0}_{h,k}+1](A^{\bar 0}-E_{h+1,k}+E_{h,k}|A^{\bar 1})(\boldsymbol{j}+\boldsymbol{\epsilon}_h-\boldsymbol{\epsilon}_{h+1},r)$$

$$+\boldsymbol{\upsilon}^{g_h(A^\star,h)+a^{\bar 1}_{h+1,h}-j_h}\frac{1}{\boldsymbol{\upsilon}-\boldsymbol{\upsilon}^{-1}}\Big\{[A^{\bar 0}-E_{h+1,h}|A^{\bar 1}](\boldsymbol{j}+\boldsymbol{\epsilon}_h-\boldsymbol{\epsilon}_{h+1},r)$$

$$-[A^{\bar 0}-E_{h+1,h}|A^{\bar 1}](\boldsymbol{j}-\boldsymbol{\epsilon}_h-\boldsymbol{\epsilon}_{h+1},r)\Big\}$$

$$+\boldsymbol{\upsilon}^{g_h(A^\star,h+1)+a^{\bar 1}_{h+1,h+1}+j_{h+1}}[a^{\bar 0}_{h,h+1}+1](A^{\bar 0}+E_{h,h+1}|A^{\bar 1})(\boldsymbol{j},r)$$

$$+\sum_{k>h+1}\boldsymbol{\upsilon}^{g_h(A^\star,k)+a^{\bar 1}_{h+1,k}}[a^{\bar 0}_{h,k}+1](A^{\bar 0}-E_{h+1,k}+E_{h,k}|A^{\bar 1})(\boldsymbol{j},r)$$

$$(II) = \sum_{k<h}\boldsymbol{\upsilon}^{g_h(A^\star,k)-a^{\bar 0}_{h+1,k}}(A^{\bar 0}|A^{\bar 1}-E_{h+1,k}+E_{h,k})(\boldsymbol{j}+\boldsymbol{\epsilon}_h-\boldsymbol{\epsilon}_{h+1},r)$$

$$+\boldsymbol{\upsilon}^{g_h(A^\star,h)-a^{\bar 0}_{h+1,h}}(A^{\bar 0}|A^{\bar 1}-E_{h+1,h}+E_{h,h})(\boldsymbol{j}-\boldsymbol{\epsilon}_{h+1},r)$$

$$+\boldsymbol{\upsilon}^{g_h(A^\star,h+1)}(A^{\bar 0}|A^{\bar 1}-E_{h+1,h+1}+E_{h,h+1})(\boldsymbol{j}-\boldsymbol{\epsilon}_{h+1},r)$$

$$+\sum_{k>h+1}\boldsymbol{\upsilon}^{g_h(A^\star,k)-a^{\bar 0}_{h+1,k}}(A^{\bar 0}|A^{\bar 1}-E_{h+1,k}+E_{h,k})(\boldsymbol{j},r)$$

The long multiplication formulas (cont'd)

$(III) =$

$$\sum_{k<h} \boldsymbol{v}^{g_h(A^\star,k)+a^{\bar{0}}_{h+1,k}} \begin{bmatrix} a_{h,k}+1 \\ 2 \end{bmatrix} (A^{\bar{0}} + 2E_{h,k}|A^{\bar{1}} - E_{h,k} - E_{h+1,k})(\boldsymbol{j} + \epsilon_h - \epsilon_{h+1}, r)$$

$$+ \frac{\boldsymbol{v}^{g_h(A^\star,h)+a^{\bar{0}}_{h+1,h}-2j_h}}{(\boldsymbol{v}-\boldsymbol{v}^{-1})} \Big\{ \frac{\boldsymbol{v}^{-1}}{[2]} (A^{\bar{0}}|A^{\bar{1}} - E_{h,h} - E_{h+1,h})(\boldsymbol{j} + \epsilon_h - \epsilon_{h+1}, r)$$

$$- \frac{\boldsymbol{v}}{[2]} (A^{\bar{0}}|A^{\bar{1}} - E_{h,h} - E_{h+1,h})(\boldsymbol{j} - \epsilon_h - \epsilon_{h+1}, r)$$

$$- (A^{\bar{0}}|A^{\bar{1}} - E_{h,h} - E_{h+1,h})(\boldsymbol{j} - \epsilon_{h+1}, r) \Big\}$$

$$+ \boldsymbol{v}^{g_h(A^\star,h+1)}(\boldsymbol{v}-\boldsymbol{v}^{-1}) \begin{bmatrix} a_{h,h+1}+1 \\ 2 \end{bmatrix} (A^{\bar{0}} + 2E_{h,h+1}|A^{\bar{1}} - E_{h,h+1} - E_{h+1,h+1})(\boldsymbol{j} + \epsilon_{h+1}, r)$$

$$+ \sum_{k>h+1} \boldsymbol{v}^{g_h(A^\star,k)}(\boldsymbol{v}-\boldsymbol{v}^{-1}) \begin{bmatrix} a_{h,k}+1 \\ 2 \end{bmatrix} (A^{\bar{0}} + 2E_{h,k}|A^{\bar{1}} - E_{h,k} - E_{h+1,k})(\boldsymbol{j}, r)$$

The long multiplication formulas (cont'd)

$(III) =$

$$\sum_{k<h} \boldsymbol{v}^{g_h(A^\star,k)+a^{\bar{0}}_{h+1,k}} \begin{bmatrix} a_{h,k}+1 \\ 2 \end{bmatrix} (A^{\bar{0}} + 2E_{h,k}|A^{\bar{1}} - E_{h,k} - E_{h+1,k})(\boldsymbol{j} + \boldsymbol{\epsilon}_h - \boldsymbol{\epsilon}_{h+1}, r)$$

$$+ \frac{\boldsymbol{v}^{g_h(A^\star,h)+a^{\bar{0}}_{h+1,h}-2j_h}}{(\boldsymbol{v}-\boldsymbol{v}^{-1})} \Big\{ \frac{\boldsymbol{v}^{-1}}{[2]} (A^{\bar{0}}|A^{\bar{1}} - E_{h,h} - E_{h+1,h})(\boldsymbol{j} + \boldsymbol{\epsilon}_h - \boldsymbol{\epsilon}_{h+1}, r)$$

$$- \frac{\boldsymbol{v}}{[2]} (A^{\bar{0}}|A^{\bar{1}} - E_{h,h} - E_{h+1,h})(\boldsymbol{j} - \boldsymbol{\epsilon}_h - \boldsymbol{\epsilon}_{h+1}, r)$$

$$- (A^{\bar{0}}|A^{\bar{1}} - E_{h,h} - E_{h+1,h})(\boldsymbol{j} - \boldsymbol{\epsilon}_{h+1}, r) \Big\}$$

$$+ \boldsymbol{v}^{g_h(A^\star,h+1)}(\boldsymbol{v}-\boldsymbol{v}^{-1}) \begin{bmatrix} a_{h,h+1}+1 \\ 2 \end{bmatrix} (A^{\bar{0}} + 2E_{h,h+1}|A^{\bar{1}} - E_{h,h+1} - E_{h+1,h+1})(\boldsymbol{j} + \boldsymbol{\epsilon}_{h+1}, r)$$

$$+ \sum_{k>h+1} \boldsymbol{v}^{g_h(A^\star,k)}(\boldsymbol{v}-\boldsymbol{v}^{-1}) \begin{bmatrix} a_{h,k}+1 \\ 2 \end{bmatrix} (A^{\bar{0}} + 2E_{h,k}|A^{\bar{1}} - E_{h,k} - E_{h+1,k})(\boldsymbol{j}, r)$$

There are also explicit formulas for $(E_{h+1,h}|\mathrm{O})(\mathbf{0}, r) \cdot (A^{\bar{0}}|A^{\bar{1}})(\boldsymbol{j}, r)$, and for $B^\star(\mathbf{0}, r) \cdot (A^{\bar{0}}|A^{\bar{1}})(\boldsymbol{j}, r)$, for $B^\star \in \{(O|E_{h,h}), (O|E_{h+1,h}), (O|E_{h,h+1})\}$ under the SDP condition.

The long multiplication formulas (cont'd)

$(III) =$

$$\sum_{k<h} \boldsymbol{v}^{g_h(A^\star,k)+a^{\bar 0}_{h+1,k}} \begin{bmatrix} a_{h,k}+1 \\ 2 \end{bmatrix} (A^{\bar 0}+2E_{h,k}|A^{\bar 1}-E_{h,k}-E_{h+1,k})(\boldsymbol{j}+\boldsymbol{\epsilon}_h-\boldsymbol{\epsilon}_{h+1},r)$$

$$+\frac{\boldsymbol{v}^{g_h(A^\star,h)+a^{\bar 0}_{h+1,h}-2j_h}}{(\boldsymbol{v}-\boldsymbol{v}^{-1})}\Big\{\frac{\boldsymbol{v}^{-1}}{[2]}(A^{\bar 0}|A^{\bar 1}-E_{h,h}-E_{h+1,h})(\boldsymbol{j}+\boldsymbol{\epsilon}_h-\boldsymbol{\epsilon}_{h+1},r)$$

$$-\frac{\boldsymbol{v}}{[2]}(A^{\bar 0}|A^{\bar 1}-E_{h,h}-E_{h+1,h})(\boldsymbol{j}-\boldsymbol{\epsilon}_h-\boldsymbol{\epsilon}_{h+1},r)$$

$$-(A^{\bar 0}|A^{\bar 1}-E_{h,h}-E_{h+1,h})(\boldsymbol{j}-\boldsymbol{\epsilon}_{h+1},r)\Big\}$$

$$+\boldsymbol{v}^{g_h(A^\star,h+1)}(\boldsymbol{v}-\boldsymbol{v}^{-1})\begin{bmatrix} a_{h,h+1}+1 \\ 2 \end{bmatrix}(A^{\bar 0}+2E_{h,h+1}|A^{\bar 1}-E_{h,h+1}-E_{h+1,h+1})(\boldsymbol{j}+\boldsymbol{\epsilon}_{h+1},r)$$

$$+\sum_{k>h+1}\boldsymbol{v}^{g_h(A^\star,k)}(\boldsymbol{v}-\boldsymbol{v}^{-1})\begin{bmatrix} a_{h,k}+1 \\ 2 \end{bmatrix}(A^{\bar 0}+2E_{h,k}|A^{\bar 1}-E_{h,k}-E_{h+1,k})(\boldsymbol{j},r)$$

There are also explicit formulas for $(E_{h+1,h}|\mathrm{O})(\mathbf{0},r)\cdot(A^{\bar 0}|A^{\bar 1})(\boldsymbol{j},r)$, and for $B^\star(\mathbf{0},r)\cdot(A^{\bar 0}|A^{\bar 1})(\boldsymbol{j},r)$, for $B^\star\in\{(O|E_{h,h}),(O|E_{h+1,h}),(O|E_{h,h+1})\}$ under the SDP condition.

All coefficients, depending on the entries of $A^\star$ & $\boldsymbol{j}$, are independent of r.

5. The regular module for the quantum queer supergroup

Theorem (1)

For any $r > 0$, there is an epimorphism $\pi_{\mathcal{Q}}^{(r)} : \mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n) \to \mathfrak{Q}_{\boldsymbol{\upsilon}}^s(n, r)$ *s.t.*

$$\mathsf{K}_i^{\pm} \mapsto (\mathrm{O}|\mathrm{O})(\pm\boldsymbol{\varepsilon}_i, r),\ \mathsf{E}_j \mapsto (E_{j,j+1}|\mathrm{O})(\mathbf{0}, r),\ \mathsf{F}_j \mapsto (E_{j+1,j}|\mathrm{O})(\mathbf{0}, r),$$
$$\mathsf{K}_{\bar{i}} \mapsto (\mathrm{O}|E_{i,i})(\mathbf{0}, r),\quad \mathsf{E}_{\bar{j}} \mapsto (\mathrm{O}|E_{j,j+1})(\mathbf{0}, r),\ \mathsf{F}_{\bar{j}} \mapsto (\mathrm{O}|E_{j+1,j})(\mathbf{0}, r).$$

with $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1$.

5. The regular module for the quantum queer supergroup

Theorem (1)

For any $r > 0$, there is an epimorphism $\pi_{\mathfrak{Q}}^{(r)} : \mathbf{U}_{\boldsymbol{v}}(\mathfrak{q}_n) \to \mathfrak{Q}_{\boldsymbol{v}}^{s}(n,r)$ s.t.

$$\mathsf{K}_i^{\pm} \mapsto (\mathrm{O}|\mathrm{O})(\pm\boldsymbol{\varepsilon}_i, r),\ \mathsf{E}_j \mapsto (E_{j,j+1}|\mathrm{O})(\mathbf{0}, r),\ \mathsf{F}_j \mapsto (E_{j+1,j}|\mathrm{O})(\mathbf{0}, r),$$
$$\mathsf{K}_{\bar{i}} \mapsto (\mathrm{O}|E_{i,i})(\mathbf{0}, r),\quad \mathsf{E}_{\bar{j}} \mapsto (\mathrm{O}|E_{j,j+1})(\mathbf{0}, r),\ \mathsf{F}_{\bar{j}} \mapsto (\mathrm{O}|E_{j+1,j})(\mathbf{0}, r).$$

with $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1$.

For $A^{\star} \in M_n(\mathbb{N}|\mathbb{N}_2)^{\pm}$, $\boldsymbol{j} \in \mathbb{Z}^n$, define infinite formal series

$$\boxed{A^{\star}(\boldsymbol{j}) = \sum_{\lambda \in \mathbb{N}^n} \boldsymbol{v}^{\lambda \cdot \boldsymbol{j}} [A + \lambda]}$$

5. The regular module for the quantum queer supergroup

Theorem (1)

For any $r > 0$, *there is an epimorphism* $\pi_{\mathfrak{Q}}^{(r)} : \mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n) \to \mathfrak{Q}_{\boldsymbol{\upsilon}}^{s}(n,r)$ *s.t.*

$$\mathsf{K}_i^{\pm} \mapsto (\mathrm{O}|\mathrm{O})(\pm\boldsymbol{\varepsilon}_i, r),\ \mathsf{E}_j \mapsto (E_{j,j+1}|\mathrm{O})(\mathbf{0}, r),\ \mathsf{F}_j \mapsto (E_{j+1,j}|\mathrm{O})(\mathbf{0}, r),$$
$$\mathsf{K}_{\bar{i}} \mapsto (\mathrm{O}|E_{i,i})(\mathbf{0}, r),\quad \mathsf{E}_{\bar{j}} \mapsto (\mathrm{O}|E_{j,j+1})(\mathbf{0}, r),\ \mathsf{F}_{\bar{j}} \mapsto (\mathrm{O}|E_{j+1,j})(\mathbf{0}, r).$$

with $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1$.

For $A^{\star} \in M_n(\mathbb{N}|\mathbb{N}_2)^{\pm}$, $\boldsymbol{j} \in \mathbb{Z}^n$, define infinite formal series

$$\boxed{A^{\star}(\boldsymbol{j}) = \sum_{\lambda\in\mathbb{N}^n} \boldsymbol{\upsilon}^{\lambda\cdot\boldsymbol{j}}[A+\lambda]} = \big(A^{\star}(\boldsymbol{j}, r)\big)_{r\geqslant 0} \in \prod_{r\geqslant 0} \mathfrak{Q}_{\boldsymbol{\upsilon}}^{s}(n,r).$$

5. The regular module for the quantum queer supergroup

Theorem (1)

For any $r > 0$, there is an epimorphism $\pi_{\mathfrak{Q}}^{(r)} : \mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n) \to \mathfrak{Q}_{\boldsymbol{\upsilon}}^s(n, r)$ s.t.

$$\mathsf{K}_i^{\pm} \mapsto (\mathrm{O}|\mathrm{O})(\pm\varepsilon_i, r),\ \mathsf{E}_j \mapsto (E_{j,j+1}|\mathrm{O})(\mathbf{0}, r),\ \mathsf{F}_j \mapsto (E_{j+1,j}|\mathrm{O})(\mathbf{0}, r),$$
$$\mathsf{K}_{\bar{i}} \mapsto (\mathrm{O}|E_{i,i})(\mathbf{0}, r),\quad \mathsf{E}_{\bar{j}} \mapsto (\mathrm{O}|E_{j,j+1})(\mathbf{0}, r),\ \mathsf{F}_{\bar{j}} \mapsto (\mathrm{O}|E_{j+1,j})(\mathbf{0}, r).$$

with $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1$.

For $A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)^{\pm}$, $\boldsymbol{j} \in \mathbb{Z}^n$, define infinite formal series

$$\boxed{A^\star(\boldsymbol{j}) = \sum_{\lambda \in \mathbb{N}^n} \boldsymbol{\upsilon}^{\lambda \cdot \boldsymbol{j}} [A + \lambda]} = \big(A^\star(\boldsymbol{j}, r)\big)_{r \geqslant 0} \in \prod_{r \geqslant 0} \mathfrak{Q}_{\boldsymbol{\upsilon}}^s(n, r).$$

Theorem (2)

These homomorphisms π_r induce a superalgebra monomorphism

$$\pi_{\mathfrak{Q}} : \mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n) \longrightarrow \textstyle\prod_{r>0} \mathfrak{Q}_{\boldsymbol{\upsilon}}^s(n, r)$$

whose image is spanned by $\{A^\star(\boldsymbol{j}) \mid A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)^{\pm}, \boldsymbol{j} \in \mathbb{Z}^n\}$

5. The regular module for the quantum queer supergroup

Theorem (1)

For any $r > 0$, there is an epimorphism $\pi_{\mathfrak{Q}}^{(r)} : \mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n) \to \mathfrak{Q}_{\boldsymbol{\upsilon}}^s(n,r)$ s.t.

$$\mathsf{K}_i^{\pm} \mapsto (\mathrm{O}|\mathrm{O})(\pm\varepsilon_i, r),\ \mathsf{E}_j \mapsto (E_{j,j+1}|\mathrm{O})(\mathbf{0}, r),\ \mathsf{F}_j \mapsto (E_{j+1,j}|\mathrm{O})(\mathbf{0}, r),$$
$$\mathsf{K}_{\bar{i}} \mapsto (\mathrm{O}|E_{i,i})(\mathbf{0}, r),\quad \mathsf{E}_{\bar{j}} \mapsto (\mathrm{O}|E_{j,j+1})(\mathbf{0}, r),\ \mathsf{F}_{\bar{j}} \mapsto (\mathrm{O}|E_{j+1,j})(\mathbf{0}, r).$$

with $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1$.

For $A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)^{\pm}$, $\boldsymbol{j} \in \mathbb{Z}^n$, define infinite formal series

$$\boxed{A^\star(\boldsymbol{j}) = \sum_{\lambda\in\mathbb{N}^n} \boldsymbol{\upsilon}^{\lambda\cdot\boldsymbol{j}}[A+\lambda]} = \big(A^\star(\boldsymbol{j}, r)\big)_{r\geqslant 0} \in \prod_{r\geqslant 0} \mathfrak{Q}_{\boldsymbol{\upsilon}}^s(n,r).$$

Theorem (2)

These homomorphisms π_r induce a superalgebra monomorphism

$$\pi_{\mathfrak{Q}} : \mathbf{U}_{\boldsymbol{\upsilon}}(\mathfrak{q}_n) \longrightarrow \textstyle\prod_{r>0} \mathfrak{Q}_{\boldsymbol{\upsilon}}^s(n,r)$$

whose image is spanned by $\{A^\star(\boldsymbol{j}) \mid A^\star \in M_n(\mathbb{N}|\mathbb{N}_2)^{\pm}, \boldsymbol{j} \in \mathbb{Z}^n\}$ with respect to which we obtain the matrix representation of the regular module ${}_{\mathbf{U}}\mathbf{U}$.

Applications—work in progress

Applications—work in progress

The new construction of the quantum queer supergroup can be used to address the following problems.

Applications—work in progress

The new construction of the quantum queer supergroup can be used to address the following problems.

1. The integral Schur–Olshanski duality.

Applications—work in progress

The new construction of the quantum queer supergroup can be used to address the following problems.

1. The integral Schur–Olshanski duality.
2. Polynomial representations at roots of unity.

Applications—work in progress

The new construction of the quantum queer supergroup can be used to address the following problems.

1. The integral Schur–Olshanski duality.
2. Polynomial representations at roots of unity.
3. The bar involution and canonical basis theory. This involves a lot!

Applications—work in progress

The new construction of the quantum queer supergroup can be used to address the following problems.

1. The integral Schur–Olshanski duality.
2. Polynomial representations at roots of unity.
3. The bar involution and canonical basis theory. This involves a lot! Here is a proposed bar involution $\bar{\upsilon} = \upsilon^{-1}$, $\bar{\mathsf{E}}_i = \mathsf{E}_i$, $\bar{\mathsf{F}}_i = \mathsf{F}_i$, $\bar{\mathsf{K}}_j = \mathsf{K}_j^{-1}$, $\bar{\mathsf{K}}_{\bar{1}} = \mathsf{K}_{\bar{1}}$.

Applications—work in progress

The new construction of the quantum queer supergroup can be used to address the following problems.

1. The integral Schur–Olshanski duality.
2. Polynomial representations at roots of unity.
3. The bar involution and canonical basis theory. This involves a lot! Here is a proposed bar involution $\bar{\upsilon} = \upsilon^{-1}$, $\bar{\mathsf{E}}_i = \mathsf{E}_i$, $\bar{\mathsf{F}}_i = \mathsf{F}_i$, $\bar{\mathsf{K}}_j = \mathsf{K}_j^{-1}$, $\bar{\mathsf{K}}_{\bar{1}} = \mathsf{K}_{\bar{1}}$.
4. The modified quantum queer supergroup and its canonical basis theory.

Applications—work in progress

The new construction of the quantum queer supergroup can be used to address the following problems.

1. The integral Schur–Olshanski duality.
2. Polynomial representations at roots of unity.
3. The bar involution and canonical basis theory. This involves a lot! Here is a proposed bar involution $\bar{\upsilon} = \upsilon^{-1}$, $\bar{\mathsf{E}}_i = \mathsf{E}_i$, $\bar{\mathsf{F}}_i = \mathsf{F}_i$, $\bar{\mathsf{K}}_j = \mathsf{K}_j^{-1}$, $\bar{\mathsf{K}}_{\bar{1}} = \mathsf{K}_{\bar{1}}$.
4. The modified quantum queer supergroup and its canonical basis theory.
5. Semi-simplicity criterion (à la Doty–Nakano, Erdmann–Nakano).


THANK YOU!