Geometric Model for Vector Bundles via Infinite Marked Strips

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1. Background

Background–Geometric models for categories

Geometric models for categories have attracted a lot of interest in recent years. Such as the geometric interpretation of module categories, derived categories, cluster categories and tube categories have been widely studied and obtained rich results.

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The topological structure and combinatorial properties of geometric models play an important role in describing and solving the representation theory problems related to categories.

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We are interested in the category of coherent sheaves over weighted projective lines.

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The study of weighted projective lines has been closely related to many branches of mathematics, such as Lie theory, singularity theory and homological mirror symmetry.

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This model provides a combinatorial description of

- the automorphism group of coherent sheaf category,
- Auslander-Reiten translation,
- the dimension of the space Ext^1,
- the tilting bundles,

and then gives a proof of the connectness of the tilting graph.

Goal

Provide a geometric model for weighted projective line of type $(2, 2, n)$ and present some applications.

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There are two ways with different applications:

- Via group action: Ext^1, Auslander-Reiten translation, Auslander-Reiten sequences, tilting objects, the connectedness of tilting graph, $\cdots$ In preparation
- Construct a new geometric model: the slope of vector bundles, the Picard group actions, vector bundles duality, projective covers, injective hulls, $\cdots$

  Jianmin Chen, Shiquan Ruan, Jinfeng Zhang, Geometric model for vector bundles via infinite marked strips, arXiv: 2405.07793v2

Weighted projective line of type $(2, 2, n)$

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- $R := k[x_1, x_2, x_3] / \left(x_1^2 + x_2^2 + x_3^n\right)$ $\mathbb{L}$-graded algebra by setting deg $x_i := \vec{x}_i (i = 1, 2, 3)$.

The category coh-$\mathbb{X}$ of coherent sheaves over $\mathbb{X}$

- By [Geigle-Lenzing] , there has an equivalence

$$\frac{\mathrm{mod}^{\mathbb{L}}\text{-}R}{\mathrm{mod}_0^{\mathbb{L}}\text{-}R} \xrightarrow{\sim} \mathrm{coh}\text{-}\mathbb{X}.$$

[Geigle-Lenzing] W. Geigle, and H. Lenzing. A class of weighted projective curves arising in representation theory of finite dimensional algebras, in: Singularities, Representations of Algebras and Vector Bundles, Lecture Notes in Math. **1273**, 265–297, Springer, 1987.

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- Let $\mathcal{O}$ be the image of $R \in \mathrm{mod}^{\mathbb{L}}\text{-}R$ in $\mathrm{mod}^{\mathbb{L}}\text{-}R/\mathrm{mod}_0^{\mathbb{L}}\text{-}R$. Then $\mathcal{O}$ serves as the *structure sheaf* of coh-$\mathbb{X}$, and the Picard group $\mathbb{L}$ acts on coh-$\mathbb{X}$ by degree shift.

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  - hereditary
  - abelian
  - Hom-finite
  - k-linear
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It satisfies Serre duality in the form

$$D\operatorname{Ext}^1(X,Y) \cong \operatorname{Hom}(Y, X(\vec{\omega})),$$

where $D = \operatorname{Hom}_k(-,k)$ and $\vec{\omega} = \vec{x}_1 - \vec{x}_2 + \vec{x}_3$.

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$$\text{coh-}\mathbb{X} = \text{vect-}\mathbb{X} \vee \text{coh}_0\text{-}\mathbb{X},$$

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where:

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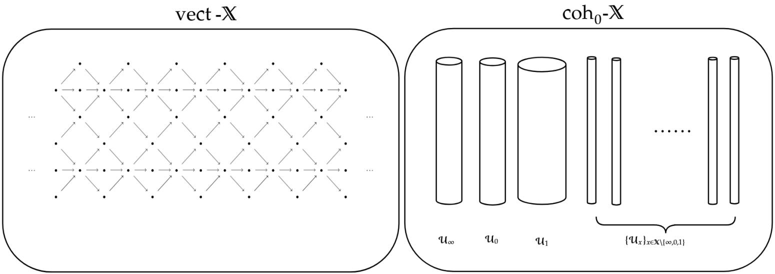
- coh$_0$-$\mathbb{X}$: full subcategory of coh-$\mathbb{X}$ consists of coherent sheaves of finite length.

There are no non-zero morphisms from coh$_0$-$\mathbb{X}$ to vect-$\mathbb{X}$.



The AR quiver $\Gamma(\text{vect-}\mathbb{X})$ has the form $\mathbb{Z}\tilde{D}_{n+2}$ and the AR quiver $\Gamma(\text{coh}_0\text{-}\mathbb{X})$ consists of tubes.

vect-$\mathbb{X}$

Definition (Kussin-Lenzing-Meltzer)

A sequence $0 \to X' \xrightarrow{u} X \xrightarrow{v} X'' \to 0$ in vect-$\mathbb{X}$ is called *distinguished exact* if for each line bundle L the induced sequence

$$0 \to \operatorname{Hom}\left(L, X'\right) \to \operatorname{Hom}(L, X) \to \operatorname{Hom}\left(L, X''\right) \to 0$$

is exact.

Proposition (Kussin-Lenzing-Meltzer)

The distinguished exact sequences define an exact structure on vect-$\mathbb{X}$ *which is Frobenius, such that the indecomposable projectives (resp. injectives) are exactly the line bundles. Moreover,* vect-$\mathbb{X}$ *is equivalent to* $\mathrm{CM}^{\mathbb{L}}$-R *as Frobenius category.*

[Kussin-Lenzing-Meltzer] D. Kussin, H. Lenzing, and H. Meltzer. Triangle singularities, ADE-chains, and weighted projective lines, *Adv. Math.*, **237**, 194–251, 2013.

Vector Bundles

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- The number r is called the *rank* of E, denote by $\mathrm{rk} E = r$.

Vector bundles

- The *degree function*

$$\deg : K_0(\mathbb{X}) \to \mathbb{Z}$$

is uniquely determined by setting $\deg \mathcal{O}(\vec{x}) = \delta(\vec{x})$, where

$$\delta(\vec{x}_1) = \delta(\vec{x}_2) = \frac{\mathrm{l.c.m}(2, \mathrm{n})}{2} \text{ and } \delta(\vec{x}_3) = \frac{\mathrm{l.c.m}(2, \mathrm{n})}{n}.$$

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- For each none zero vector bundle E, define the slope

$$\mu(E) = \frac{\deg E}{\mathrm{rk} E} \in \mathbb{Q}.$$

Extension bundles

Define $\vec{\delta} := (n-2)\vec{x}_3$ as the *dominant element* of $\mathbb{L}$.

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$$\mathrm{Ext}^1(L(\vec{x}), L(\vec{\omega})) \cong D\,\mathrm{Hom}(L, L(\vec{x})) \cong k.$$

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The indecomposable middle term E in the non-split exact sequence:

$$0 \longrightarrow L(\vec{\omega}) \longrightarrow E \longrightarrow L(\vec{x}) \longrightarrow 0$$

is uniquely determined up to isomorphism. Denote E by $E_L\langle\vec{x}\rangle$ and call it the *extension bundle* associated with L and $\vec{x}$.

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Vector bundles

Remark:

Each indecomposable bundle in vect-$\mathbb{X}$ is either a line bundle or a extension bundle.

An infinite marked strip under a specific group action

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- ∂ the upper boundary of $\widetilde{\mathcal{S}}$.
- ∂' the lower boundary of $\widetilde{\mathcal{S}}$.
- Two bijections on the strip $\widetilde{\mathcal{S}}$:
  - σ_n translates all points on $\widetilde{\mathcal{S}}$ along the positive x-axis by n units:

$$\sigma_n : \widetilde{\mathcal{S}} \to \widetilde{\mathcal{S}} \quad (x, y) \mapsto (x + n, y);$$

  - θ reflects all points on $\widetilde{\mathcal{S}}$ with respect to the point $(0, \frac{1}{2})$:

$$\theta : \widetilde{\mathcal{S}} \to \widetilde{\mathcal{S}} \quad (x, y) \mapsto (-x, 1 - y).$$

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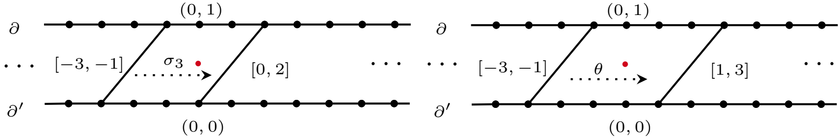
- $\mathrm{Seg}(M)$ the collection of line segments $\{[i,j] | i,j \in \mathbb{Z}\}$.
- The maps σ_n and θ naturally induce two bijections on $\mathrm{Seg}(M)$, also denoted by σ_n and θ. Precisely,

$$\sigma_n : \mathrm{Seg}(M) \to \mathrm{Seg}(M) \qquad [i,j] \mapsto [i+n, j+n];$$
$$\theta : \mathrm{Seg}(M) \to \mathrm{Seg}(M) \qquad [i,j] \mapsto [-j,-i].$$



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- Following, we will give a categorical interpretation of $\widetilde{\mathrm{Seg}(M)}$ in term of a full subcategory of vect-$\mathbb{X}$.

Generalized extension bundles

- F the degree shift automorphism by $\vec{x}_1 - \vec{x}_2$ on vect-$\mathbb{X}$.

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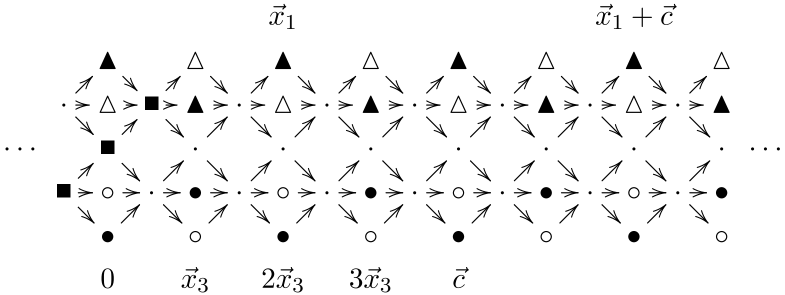
- From now on, we fix a line bundle L.

Generalized extension bundle

For example, let $n = 4$. The Auslander-Reiten quiver $\Gamma(\text{vect-}\mathbb{X})$ is illustrated below:



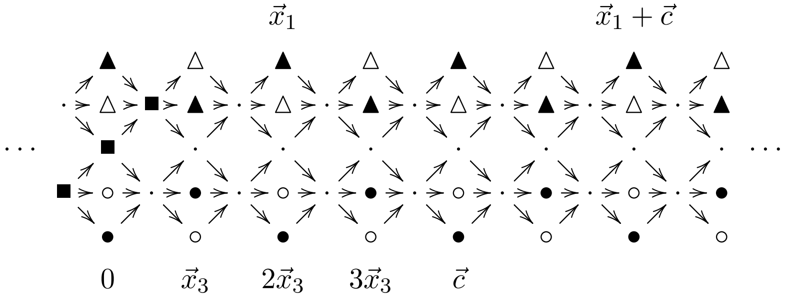
■ extension bundles $E_L\langle\vec{x}\rangle$, where $0 \leq \vec{x} \leq \vec{\delta}$.

$\bullet$ $L(j\vec{x}_3)$ $\circ$ $L^*(j\vec{x}_3)$

▲ $L(\vec{x}_1 + j\vec{x}_3)$ △ $L^*(\vec{x}_1 + j\vec{x}_3)$, where $j \in \mathbb{Z}$.

Generalized extension bundle

σ_F exchanges the ▲ and △ on the same vertical lines, as well as the ○ and ●, while fixing all other vertices.



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Proposition

There exists a bijection

$$\begin{aligned} \phi : \widetilde{\mathrm{Seg}(M)} &\to \mathrm{ind}(\mathrm{vect}^{F}\text{-}\mathbb{X}) \\ \widetilde{[i,j]} &\mapsto \mathsf{E}_{L(-i\vec{x}_3)}\langle (i+j-1)\vec{x}_3 \rangle. \end{aligned}$$

Refinement of the bijection ϕ

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$\iff i + j \equiv 0 (\mod n)$

$\iff \phi(\widetilde{[i, j]})$ is formed by the direct sum of two line bundles.

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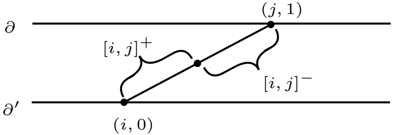

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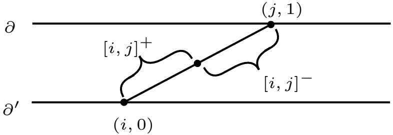

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- $\mathrm{Seg}^*(M) = (\mathrm{Seg}(M) \setminus \mathrm{Seg}_0(M)) \cup \mathrm{Seg}_0^*(M)$.

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Refinement of the bijection ϕ

Proposition

The bijection ϕ induces a bijection $\widehat{\phi} : \widetilde{\mathrm{Seg}^*(M)} \to \mathrm{ind}(\text{vect-}\mathbb{X})$, which can be explicitly described by the following table:

G-orbits in $\widetilde{\mathrm{Seg}^*(M)}$	***Indecomposable objects in*** vect-$\mathbb{X}$
$\widetilde{[i,-i]^+}$	$L^*(-(i+1)\vec{x}_3)$
$\widetilde{[i,-i]^-}$	$L(-(i+1)\vec{x}_3)$
$\widetilde{[i,n-i]^+}$	$L(\vec{x}_1-(i+1)\vec{x}_3)$
$\widetilde{[i,n-i]^-}$	$L^*(\vec{x}_1-(i+1)\vec{x}_3)$
$\widetilde{[i,k-i]}$	$E_{L(-i\vec{x}_3)}\langle(k-1)\vec{x}_3\rangle$

where i, k are integers with $1 \leq k \leq n-1$.

A geometric model for vector bundles

Remark:

The above proposition gives a geometric model for the category of vector bundles over weighted projective lines of type $(2, 2, n)$.

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By using the model, we provide the intuitive combinatorial interpretation for several intrinsic geometric properties of vect-$\mathbb{X}$, such as

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The group $\mathbb{L}/\mathbb{Z}(\vec{x}_1 - \vec{x}_2)$ is isomorphic to $\mathcal{MG}(\mathcal{S})$, where $\mathcal{S}$ is the orbit space of $\widetilde{\mathcal{S}}$ under the G-action and $\mathcal{MG}(\mathcal{S})$ is the mapping class group of $\mathcal{S}$.

Applications

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Proposition

The slope μX *of* X *is given by* $\mu X = (j - i - 2) \times \frac{\bar{p}}{2n} + \mu L$, *where* $\bar{p} := \mathrm{l.\,c.\,m}\,(2, n)$ *and* $\frac{1}{\infty} = 0$ *is defined.*

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Proposition

The $\mathbb{L}$-action on vect-$\mathbb{X}$ is determined by

- $X(\vec{x}_1) = \widehat{\phi}(\widetilde{[i, j+n]^{\varrho(*)}})$;
- $X(\vec{x}_2) = \widehat{\phi}(\widetilde{[i, j+n]^*})$;
- $X(\vec{x}_3) = \widehat{\phi}(\widetilde{[i-1, j+1]^*})$.

Applications

Recall that the vector bundle duality

$$^\vee : \text{vect-}\mathbb{X} \to \text{vect-}\mathbb{X}, X \mapsto \mathcal{H}om(X, \mathcal{O}),$$

sends line bundles to line bundles, and preserves distinguished exact sequences.

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Proposition

Assume $X = \widehat{\phi}(\widetilde{[i,j]^*})$. *Then we have*

$$X^\vee = \widehat{\phi}(\widetilde{[j,i]^*}).$$

Moreover, X *is fixed under* $^\vee$ *if and only if* $i = j$.

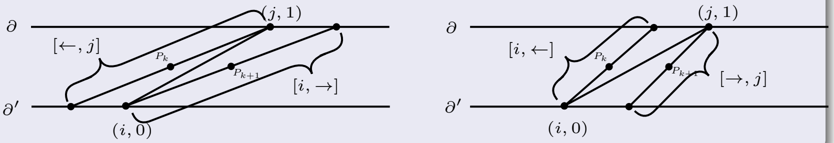
Applications

Proposition

Assume $X = \widehat{\phi}(\widetilde{[i,j]})$ *is an extension bundle in* vect-$\mathbb{X}$. *Then*

- *the* *projective cover* $P(X) = \widehat{\phi}(\widetilde{[i,\leftarrow]}) \oplus \widehat{\phi}(\widetilde{[\rightarrow,j]})$;
- *the* *injective hull* $I(X) = \widehat{\phi}(\widetilde{[i,\rightarrow]}) \oplus \widehat{\phi}(\widetilde{[\leftarrow,j]})$,

which can be illustrated below:



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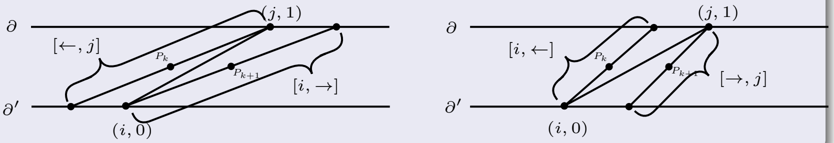

Remark: Let Y be the kernel of the projective cover $\pi : P(X) \to X$. From a graphical perspective, we have $I(Y)=P(X)$.

Thank you!