Resolving dualities and applications to semi-derived Ringel-Hall algebras

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Contents

Equivalences induced by tilting modules

This talk is based on a joint work with Hongxing Chen (Capital Normal University).

In the talk:

A: Artin algebra (e.g. finite-dim. k-algebra over a field k);

A-mod: the category of finitely generated (left) A-modules;

D: the usual duality over A-mod (e.g. $D = \mathrm{Hom}_k(-, k)$).

Tilting modules over algebras

Definition

A module ${}_AT \in A\text{-mod}$ *is called* ***n-tilting*** $(n \in \mathbb{N})$ *if the following hold:*

$(T1)$ $\text{proj.dim}\,({}_AT) \leqslant n$;

$(T2)$ $\text{Ext}^i_A(T,T) = 0 \;\; \forall i \geqslant 1$;

$(T3)$ $\exists$ *exact sequence* $0 \to A \to T_0 \to T_1 \to \cdots \to T_n \to 0$ *in* A-mod *with* $T_i \in \text{add}\,(T)$ *for all* $i \geqslant 0$.

T is also called a **tilting A-B-bimodule** with $B := \text{End}_A(T)$.

Equivalences induced by tilting modules

- Brenner-Butler tilting theorem [Brenner and Butler, 1980]
- one-to-one correspondence between basic tilting modules and contravariant finite resolving subcategories of modules with finite projective dimension [Auslander and Reiten, 1991]
- derived equivalences between the bounded derived category of an algebra and the one of the endomorphism algebra of a tilting module over the algebra [Happel, 1988]
- homological invariants or dimensions
- ......

How about "dualities" of categories induced by (tilting) modules?

Dualities of categories

Definition

Contravariant functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ between categories $\mathcal{C}$ and $\mathcal{D}$ are called inverse dualities if both $G \circ F$ and $F \circ G$ are isomorphic to the **identity** *functors.*

Natural conditions are added on F and G whenever $\mathcal{C}$ and $\mathcal{D}$ are endowed with more structures.

e.g. if $\mathcal{C}$ and $\mathcal{D}$ are (additive, exact, triangulated) categories, then F and G are required to be (additive, exact, triangulated) functors.

Morita duality

R, S: rings.

Theorem (Morita)

Let $\mathcal{C} \subseteq R\text{-Mod}$ *and* $\mathcal{D} \subseteq S^{op}\text{-Mod}$ *be full subcategories closed under isomorphisms. Suppose* ${}_RR \in \mathcal{C}$, $S_S \in \mathcal{D}$, *and* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{C}$ *are inverse dualities.*

$\Longrightarrow$ $\exists$ *faithfully balanced bimodule* ${}_RM_S$ *s.t.*

(a) $F \cong \mathrm{Hom}_R(-, M)|_{\mathcal{C}}$ *and* $G \cong \mathrm{Hom}_{S^{op}}(-, M)|_{\mathcal{D}}$;

(b) $\mathcal{C} \subseteq \{X \in R\text{-Mod} \mid X \simeq \mathrm{Hom}_{S^{op}}(\mathrm{Hom}_R(X, M), M)\}$ *and* $\mathcal{D} \subseteq \{Y \in S^{op}\text{-Mod} \mid Y \simeq \mathrm{Hom}_R(\mathrm{Hom}_{S^{op}}(Y, M), M)\}$.

Moreover, the bimodule M *defines a Morita duality* $\Leftrightarrow$ ${}_RM$ *and* M_S *are injective cogenerators.*

Def: ${}_RM_S$ is **faithfully balanced** if $S \cong \mathrm{End}_R(M)$ and $R^{op} \cong \mathrm{End}_{S^{op}}(M)$.

Modules of finite projective dimensions

$$\mathcal{P}^{<\infty}(A) := \{X \in A\text{-mod} \mid \text{proj.dim}\,({}_AX) < \infty\}.$$

For $T \in A$-mod, define

$$^{\perp}({}_AT) := \{M \in A\text{-mod} \mid \text{Ext}^i_A(M,T) = 0,\ \forall i \geqslant 1\}.$$

M. Auslander and I. Reiten, Applications of contravariantly finite subcategories, *Adv. Math.* 86 (1991) 111-152.

Definition (Auslander and Reiten)

A full subcategory of A-mod *is resolving if it contains projective modules and is closed under isomorphisms, extensions and kernels of epimorphisms.*

Miyashita's duality

Y. Miyashita, Tilting modules of finite projective dimension, *Math. Z.* 193 (1986) 113-146.

Theorem (Miyashita)

For a tilting bimodule ${}_AT_B$, *let*

$$\mathcal{C} := {}^{\perp}({}_AT) \cap \mathcal{P}^{<\infty}(A) \text{ and } \mathcal{D} := {}^{\perp}(T_B) \cap \mathcal{P}^{<\infty}(B^{op}).$$

Then $\operatorname{Hom}_A(-,T)|_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ *and* $\operatorname{Hom}_{B^{op}}(-,T)|_{\mathcal{D}} : \mathcal{D} \to \mathcal{C}$ *are inverse dualities.*

Huisgen-Zimmermann's correspondence

B. Huisgen-Zimmermann, Dualities from iterated tilting, *Isr. J. Math.* 243 (2021) 315-353.

Theorem (Huisgen-Zimmermann)

Let $\mathcal{C} \subseteq A\text{-mod}$ *and* $\mathcal{D} \subseteq B^{op}\text{-mod}$ *be resolving subcategories. Suppose* $\mathcal{C} \subseteq \mathcal{P}^{<\infty}(A)$, $\mathcal{D} \subseteq \mathcal{P}^{<\infty}(B^{op})$, *and* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{C}$ *are inverse dualities which are* **strictly exact**.

$\Longrightarrow$ $\exists$ *tilting bimodule* ${}_AT_B$ *s.t.*

(a) $F \cong \mathrm{Hom}_A(-, T)|_{\mathcal{C}}$ *and* $G \cong \mathrm{Hom}_{B^{op}}(-, T)|_{\mathcal{D}}$;

(b) $\mathcal{C} = {}^{\perp}({}_AT) \cap \mathcal{P}^{<\infty}(A)$ *and* $\mathcal{D} = {}^{\perp}(T_B) \cap \mathcal{P}^{<\infty}(B^{op})$.

Full subcategories cogenerated by modules

For $T \in A$-mod, define $\mathrm{cogen}^*({}_AT) \subseteq A$-mod:
$M \in \mathrm{cogen}^*({}_AT) \Leftrightarrow \exists$ exact sequence in A-mod

$$0 \to M \to T_0 \to T_1 \to T_2 \to \cdots$$

with all $T_i \in \mathrm{add}({}_AT)$ for $i \geqslant 0$ s.t. the sequence

$$\cdots \to \mathrm{Hom}_A(T_2, T) \to \mathrm{Hom}_A(T_1, T) \to \mathrm{Hom}_A(T_0, T) \to \mathrm{Hom}_A(M, T) \to 0$$

is exact.
Let

$$\mathcal{W}({}_AT) := {}^{\perp}({}_AT) \cap \mathrm{cogen}^*({}_AT).$$

Then $T \in \mathcal{W}({}_AT) \Leftrightarrow \mathrm{Ext}^i_A(T, T) = 0,\ \forall i \geqslant 1.$

Wakamatsu tilting modules

Definition (Wakamatsu,1988)

An A-module T is called a Wakamatsu tilting module if $A \oplus T \in \mathcal{W}({}_AT)$.
This is equivalent to the following two conditions:
(1) $\mathrm{End}_{B^{\mathrm{op}}}(T) \cong A^{op}$, *where* $B := \mathrm{End}_A(T)$;
(2) $\mathrm{Ext}^i_A(T,T) = 0 = \mathrm{Ext}^i_{B^{\mathrm{op}}}(T,T)$ *for all* $i \geqslant 1$.

- If ${}_AT$ is Wakamatsu tilting, then T_B is Wakamatsu tilting. So, T is also called a Wakamatsu tilting A-B-bimodule.
- A Wakamatsu tilting A-module T is tilting $\Leftrightarrow \max\{\mathrm{proj.dim}\,({}_AT), \mathrm{proj.dim}\,(T_B)\} < \infty$.

Wakamatsu tilting conjecture [Beligiannis and Reiten, 2007]:

If ${}_AT$ is Wakamatsu tilting and $\mathrm{proj.dim}\,({}_AT) < \infty$, then it is tilting.

Wakamatsu's duality

T. Wakamatsu, Tilting modules and Auslander's Gorenstein property, *J. Algebra* 275 (2004) 3-39.

E.L. Green, I. Reiten and Ø. Solberg, Dualities on Generalized Koszul Algebras, *Mem. Amer. Math. Soc.* vol. 159, 2002.

Theorem (Wakamatsu)

For a Wakamatsu tilting bimodule ${}_AT_B$, *the functors* $\mathrm{Hom}_A(-,T) : A\text{-mod} \to B^{op}\text{-mod}$ *and* $\mathrm{Hom}_{B^{op}}(-,T) : B^{op}\text{-mod} \to A\text{-mod}$ *can be restricted to inverse dualities*

$$\mathcal{W}({}_AT) \simeq \mathcal{W}(T_B).$$

Resolving dualities

$\mathcal{A}$, $\mathcal{B}$: abelian categories with enough projective objects;
$\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{D} \subseteq \mathcal{B}$: full subcategories.

Definition

Contravariant additive functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are called inverse resolving dualities if the following hold:

1. *$\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{D} \subseteq \mathcal{B}$ are* **resolving** *subcategories.*
   ($\mathcal{C}$ and $\mathcal{D}$ contains all projective objects and are closed under isomorphisms, extensions and kernels of epimorphisms.)
2. *Both of compositions $G \circ F$ and $F \circ G$ are isomorphic to the* **identity** *functors.*
3. *F and G are* **exact** *functors between $\mathcal{C}$ and $\mathcal{D}$.*
   ($\mathcal{C}$ and $\mathcal{D}$ are regarded as fully exact subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively).

Gorenstein projective modules

Definition

A module $X \in A\text{-mod}$ *is called* *Gorenstein projective* *if* $\exists$ *exact complex of projective* A*-modules*

$$P^\bullet : \quad \cdots \to P^{-2} \to P^{-1} \to P^0 \xrightarrow{d^0} P^1 \to P^2 \to \cdots$$

s.t. $X \cong \mathrm{Im}\,(d^0)$ *and the complex* $\mathrm{Hom}_A^\bullet(P^\bullet, A)$ *is also exact.*

Definition

A module $X \in A\text{-mod}$ *is called* *semi-Gorenstein-projective* *if* $\mathrm{Ext}_A^i(X, A) = 0,\ \forall i \geqslant 1.$

C.M. Ringel and P. Zhang, Gorenstein-projective and semi-Gorenstein-projective modules, *Algebra & Number Theory* 14 (2020) 1-36.

Resolving subcategories of module categories

For each $n \in \mathbb{N}$, define three resolving subcats. of A-mod:

- $\mathcal{P}^{\leqslant n}(A)$: modules with projective dimension $\leqslant n$,
- $\mathcal{GP}^{\leqslant n}(A)$: modules with Gorenstein-projective dimension $\leqslant n$,
- $\mathcal{SGP}^{\leqslant n}(A)$: modules with semi-Gorenstein-projective dimension $\leqslant n$.

$\mathcal{GP}(A) := \mathcal{GP}^{\leqslant 0}(A)$: cat. of Gorenstein-projective A-modules;
$\mathcal{SGP}(A) := \mathcal{SGP}^{\leqslant 0}(A)$: cat. of semi-Gorenstein-projective A-modules.
In general, $\mathcal{GP}(A) \subseteq \mathcal{SGP}(A)$.

Characterization of resolving dualities

A, B: Artin algebras;
$\mathcal{C} \subseteq A$-mod, $\mathcal{D} \subseteq B^{op}$-mod: full subcategories.

Theorem

Suppose that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are inverse resolving dualities. Then:

(1) $\exists$ *a* **Wakamatsu tilting bimodule** ${}_AT_B$ *s.t.*

(a) $F \cong \mathrm{Hom}_A(-, T)|_{\mathcal{C}}$ *and* $G \cong \mathrm{Hom}_{B^{op}}(-, T)|_{\mathcal{D}}$;

(b) $\mathcal{C} \subseteq \mathcal{W}({}_AT)$ *and* $\mathcal{D} \subseteq \mathcal{W}(T_B)$.

(2) *The bimodule* ${}_AT_B$ *is* **tilting** $\Leftrightarrow$ F *and* G *can be restricted to any one of inverse dualities of the following types:*

$\mathcal{C} \cap \mathcal{P}^{<\infty}(A) \simeq \mathcal{D} \cap \mathcal{P}^{<\infty}(B^{op})$, $\mathcal{C} \cap \mathcal{P}^{\leqslant n}(A) \simeq \mathcal{D} \cap \mathcal{P}^{\leqslant m}(B^{op})$,
$\mathcal{C} \cap \mathcal{GP}^{\leqslant n}(A) \simeq \mathcal{D} \cap \mathcal{GP}^{\leqslant m}(B^{op})$, $\mathcal{C} \cap \mathcal{SGP}^{\leqslant n}(A) \simeq \mathcal{D} \cap \mathcal{SGP}^{\leqslant m}(B^{op})$,
where n and m are some natural numbers.

A key lemma

$$\mathcal{W}({}_AT) := {}^{\perp}({}_AT) \cap \mathrm{cogen}^*({}_AT)$$

Lemma

If ${}_AT$ is a tilting module of projective dimension ℓ, then

$$\mathcal{W}({}_AT) = {}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant \ell}(A) = {}^{\perp}({}_AT) \cap \mathcal{GP}^{<\infty}(A),$$

$$\mathcal{W}(T_B) = {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant \ell}(B^{op}) = {}^{\perp}(T_B) \cap \mathcal{GP}^{<\infty}(B^{op}).$$

Clearly, $\mathcal{W}({}_AA) = \mathcal{GP}(A)$.

Gorenstein version of Miyachita's duality and Huisgen-Zimmermann's correspondence

Corollary

(1) *If* ${}_AT_B$ *is a tilting bimodule with* $\ell := \mathrm{proj.dim}\,({}_AT)$, *then* $\mathrm{Hom}\,_A(-, T)$ *and* $\mathrm{Hom}\,_{B^{op}}(-, T)$ *can be restricted to inverse resolving dualities*

$$^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant \ell}(A) \simeq {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant \ell}(B^{op}).$$

(2) *Let* $\mathcal{C} \subseteq A\text{-mod}$ *and* $\mathcal{D} \subseteq B^{op}\text{-mod}$ *be full subcategories. If there are inverse resolving dualities* $\mathcal{C} \simeq \mathcal{D}$ *s.t.* $\mathcal{GP}(A) \subseteq \mathcal{C} \subseteq \mathcal{GP}^{\leqslant n}(A)$ *and* $\mathcal{GP}(B^{op}) \subseteq \mathcal{D} \subseteq \mathcal{GP}^{\leqslant m}(B^{op})$ *for some natural numbers* n *and* m, *then* $\exists$ *a tilting bimodule* ${}_AT_B$ *s.t.*

$$\mathcal{C} = {}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant \ell}(A) \quad \textit{and} \quad \mathcal{D} = {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant \ell}(B^{op})$$

where $\ell := \mathrm{proj.dim}\,({}_AT)$.

Applications of resolving dualities

Corollary

Let ${}_AT_B$ be a tilting bimodule. Then:

1. *There is a triangle equivalence* $\mathscr{D}(\mathcal{GP}(A)) \simeq \mathscr{D}(\mathcal{GP}(B))$ *which restrictes to an equivalence* $\mathscr{D}^*(\mathcal{GP}(A)) \simeq \mathscr{D}^*(\mathcal{GP}(B))$ *for any* $* \in \{+, -, b\}$.
2. $K_n(\mathcal{GP}(A)) \simeq K_n(\mathcal{GP}(B))$ *for any* $n \in \mathbb{N}$.

Applications of resolving dualities

A: a finite-dimensional algebra A over a finite field.

Corollary

If ${}_AT_B$ is a 1-tilting bimodule, then there is an isomorphism between the semi-derived Ringel-Hall algebra of $\mathcal{GP}(A)$ and the one of $\mathcal{GP}(B)$:

$$\mathcal{SDH}(\mathcal{GP}(A)) \cong \mathcal{SDH}(\mathcal{GP}(B)).$$

Corollary [LW, Corollary A23]

If ${}_AT_B$ is a 1-tilting bimodule over 1-Gorenstein algebras A and B, then $\mathcal{SDH}(\mathcal{GP}(A)) \cong \mathcal{SDH}(\mathcal{GP}(B))$ as algebras.

(Semi-)derived Ringel-Hall algebras

[B] T. Bridgeland, Quantum groups via Hall algebras of complexes, *Ann. of Math.* 177 (2013) 739-759.
[G] M. Gorsky, Semi-derived and derived Hall algebras for stable categories, *Int. Math. Res. Not.* 2018 (2018), 138-159.
[LP] M. Lu and L.G. Peng, Semi-derived Ringel-Hall algebras and Drinfeld double, *Adv. Math.* 383 (2021) 107668.
[LW] M. Lu and W.Q. Wang, Hall algebras and quantum symmetric pairs I: foundations, *Proc. Lond. Math. Soc.* 124 (2022) 1-82.

Semi-derived Ringel-Hall algebras of weakly 1-Gorenstein exact categories

Let k be a finite field and $\mathcal{A}$ a small exact category linear over k.

Definition (Lu and Wang, 2022)

$\mathcal{A}$ is called weakly Gorenstein if $\mathcal{P}^{<\infty}(\mathcal{A}) = \mathcal{I}^{<\infty}(\mathcal{A})$; weakly d-Gorenstein if it is weakly Gorenstein and $\mathcal{P}^{<\infty}(\mathcal{A}) = \mathcal{P}^{\leqslant d}(\mathcal{A}) = \mathcal{I}^{\leqslant d}(\mathcal{A})$.

In case $\mathcal{A}$ is a weakly 1-Gorenstein exact category with finite morphism spaces and finite extension spaces, the *semi-derived Ringel-Hall algebra* of $\mathcal{A}$ was defined in [LW] (see also [B,G,LP] for some cases) and denoted by $\mathcal{SDH}(\mathcal{A})$.

Based on Lu-Wang's construction, we introduce a new definition for $\mathcal{SDH}(\mathcal{A})$ (up to isomorphism of algebras) which behaves better under dualities.

- $\mathcal{GP}(A)$ for A: finite-dim. k-algebra;
- A-mod for A: finite-dim. **1-Gorenstein** k-algebra.

Applications of resolving dualities to establish isomorphisms of semi-derived Ringel-Hall algebras

Set $\mathcal{A} := {}^{\perp}({}_{A}T) \cap \mathcal{GP}^{\leqslant 1}(A)$ and $\mathcal{B} := {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant 1}(B^{op})$.

$$\begin{array}{ccccc} & & \mathcal{GP}(A) & \xrightarrow{\subseteq} & \mathcal{A} \\ & & & & \big\downarrow{\scriptstyle F} \\ \mathcal{GP}(B) & \underset{\simeq}{\xleftarrow{G}} & \mathcal{GP}(B^{op}) & \xrightarrow{\subseteq} & \mathcal{B}. \end{array} \tag{1}$$

$$\begin{array}{ccc} \mathcal{SDH}(\mathcal{GP}(A)) & \xrightarrow{\widetilde{\phi}_A} & \mathcal{SDH}(\mathcal{A}) \\ & & \big\downarrow{\scriptstyle \Upsilon_F} \\ & & (\mathcal{SDH}(\mathcal{B}))^{op} \\ & & \big\downarrow{\scriptstyle (\widetilde{\psi}_{B^{op}})^{op}} \\ \mathcal{SDH}(\mathcal{GP}(B)) & \xleftarrow{(\Upsilon_G)^{op}} & (\mathcal{SDH}(\mathcal{GP}(B^{op}))^{op} \end{array} \tag{2}$$

Different methods

$\mathcal{X} = \{X \in A\text{-mod} \mid \mathrm{Ext}^1_A(T, X) = 0\}, \mathcal{Y} = \{Y \in B\text{-mod} \mid \mathrm{Tor}_1^B(T, Y) = 0.$

Note: $\mathcal{X} \cong \mathcal{Y}$ of exact categories by the Brenner-Butler tilting theorem.

The proof of [LW, Corollary A23]:

$$\mathcal{SDH}(\mathcal{GP}(A)) \cong \mathcal{SDH}(A\text{-mod}) \cong \mathcal{SDH}(\mathcal{X}) \cong \mathcal{SDH}(\mathcal{Y}) \cong \mathcal{SDH}(B\text{-mod}) \cong \mathcal{SDH}(\mathcal{GP}(B))$$

where $\mathcal{X}$ and $\mathcal{Y}$ are weakly 1-Gorenstein exact categories.

Note: for a general algebra A, the category $\mathcal{X}$ may not be weakly 1-Gorenstein.

Our proof:

$$\mathcal{SDH}(\mathcal{GP}(A)) \cong \mathcal{SDH}(\mathcal{A}) \cong (\mathcal{SDH}(\mathcal{B}))^{op} \cong (\mathcal{SDH}(\mathcal{GP}(B^{op}))^{op} \cong \mathcal{SDH}(\mathcal{GP}(B)).$$

where $\mathcal{A} := {}^{\perp}({}_AT) \cap \mathcal{GP}^{\leqslant 1}(A)$ and $\mathcal{B} := {}^{\perp}(T_B) \cap \mathcal{GP}^{\leqslant 1}(B^{op})$.

More details

- H.X. Chen and J.S. Hu, Resolving dualities and applications to homological invariants, to appear in Canad. J. Math., arXiv:2209.11627.

Thank you very much!