Cotorsion pairs and model structures on Morita rings

Jian Cui

Shanghai Jiao Tong University

2024.8.6

Contents

Definition

Let f and g be morphisms in a category $\mathcal{M}$. The morphism f is said to be a *retract* of g, if there exists two morphisms $\varphi : f \longrightarrow g$ and $\psi : g \longrightarrow f$ in the morphism category $\mathrm{Mor}(\mathcal{M})$, such that $\psi\varphi = \mathrm{Id}_f$. That is, there exists the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\varphi_1} & X' & \xrightarrow{\psi_1} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow{\varphi_2} & Y' & \xrightarrow{\psi_2} & Y \end{array}$$

such that $\psi_1\varphi_1 = \mathrm{Id}_X, \quad \psi_2\varphi_2 = \mathrm{Id}_Y$.

Closed model structures

Definition (Daniel Quillen, 1967)

A closed model structure on a category $\mathcal{M}$ is a triple $(\mathrm{Cofib}(\mathcal{M}),\ \mathrm{Fib}(\mathcal{M}),\ \mathrm{Weq}(\mathcal{M}))$ of classes of morphisms, where the morphisms in the three classes are respectively called cofibrations (usually denoted by $\hookrightarrow$), fibrations (usually denoted by $\twoheadrightarrow$), and weak equivalences, satisfying the following conditions (CM1) - (CM4):

(CM1) (2 out of 3) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in $\mathcal{M}$. If two of the morphisms $f,\ g,\ gf$ are weak equivalences, then so is the third.

(CM2) (closed under retracts) If f is a retract of g, and g is a cofibration (fibration, weak equivalence), then so is f.

Definition

(CM3) (lifting) Given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\ a\ } & X \\ {\scriptstyle i}\big\downarrow & \overset{s}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{\ b\ } & Y \end{array}$$

where $i \in \mathrm{Cofib}(\mathcal{M})$ and $p \in \mathrm{Fib}(\mathcal{M})$, if either $i \in \mathrm{Weq}(\mathcal{M})$ or $p \in \mathrm{Weq}(\mathcal{M})$, then there exists a morphism $s : B \longrightarrow X$ such that $a = si, \quad b = ps$.

(CM4) (factorization) Any morphism $f : X \longrightarrow Y$ has two factorizations:

(i) $f = pi$, where $i \in \mathrm{Cofib}(\mathcal{M}) \cap \mathrm{Weq}(\mathcal{M}), \quad p \in \mathrm{Fib}(\mathcal{M})$;

(ii) $f = p'i'$, where $i' \in \mathrm{Cofib}(\mathcal{M}), \quad p' \in \mathrm{Fib}(\mathcal{M}) \cap \mathrm{Weq}(\mathcal{M})$.

Definition

Let $(\mathrm{Cofib}(\mathcal{M}), \mathrm{Fib}(\mathcal{M}), \mathrm{W}(\mathcal{M}))$ be a closed model structure on a category $\mathcal{M}$ with 0 object.

(1) *trivial cofibrations*: $\mathrm{Cofib}(\mathcal{M}) \cap \mathrm{W}(\mathcal{M})$.

(2) *trivial fibrations*: $\mathrm{Fib}(\mathcal{M}) \cap \mathrm{W}(\mathcal{M})$.

(3) *cofibrant objects*: $\mathcal{C} = \{X \in \mathcal{M} \mid 0 \longrightarrow X$ is a cofibration$\}$.

(4) *fibrant objects*: $\mathcal{F} = \{Y \in \mathcal{M} \mid Y \longrightarrow 0$ is a fibration$\}$.

(5) *trivial objects*: $\mathcal{W} = \{W \in \mathcal{M} \mid 0 \longrightarrow W$ is a weak equivalence$\} = \{X \in \mathcal{M} \mid W \longrightarrow 0$ is a weak equivalence$\}$.

(6) *trivial cofibrant objects*: $\mathcal{C} \cap \mathcal{W}$.

(7) *trivial fibrant objects*: $\mathcal{F} \cap \mathcal{W}$.

Definition

A category $\mathcal{M}$ endowed with a closed model structure is called *a closed model category*, if

(CM0) $\mathcal{M}$ is closed under finite projective and inductive limits.

Quillen's homotopy category

By localizing the model structure $(\mathrm{Cofib}(\mathcal{M}),\ \mathrm{Fib}(\mathcal{M}),\ \mathrm{W}(\mathcal{M}))$ on the category $\mathcal{M}$ using $\mathrm{W}(\mathcal{M})$, we obtain the homotopy category $\mathrm{Ho}\mathcal{M}$, which makes weak equivalence become isomorphisms and has universal property for it.

Theorem (D. Quillen; A. Beligiannis, I. Reiten)

Let $(\mathrm{Cofib}(\mathcal{M}),\ \mathrm{Fib}(\mathcal{M}),\ \mathrm{W}(\mathcal{M}))$ be a model structure on an additive category $\mathcal{M}$. Then the homotopy category $\mathrm{Ho}\mathcal{M}$ is pretriangulated.

Abelian model structures

Definition (M. Hovey; A. Beligiannis, I. Reiten)

A model structure $(\mathrm{Cofib}(\mathcal{A}),\ \mathrm{Fib}(\mathcal{A}),\ \mathrm{W}(\mathcal{A}))$ on an Abelian category $\mathcal{A}$ is called an abelian model structure, if:

(1) $\mathrm{Fib}(\mathcal{A}) = \{$epimorphism $f \mid \mathrm{Ker}\, f$ is a fibrant object$\}$;

(2) $\mathrm{Cofib}(\mathcal{A}) = \{$monomorphism $f \mid \mathrm{Coker}\, f$ is a cofibrant object$\}$.

Cotorsion pairs

Definition

Let $\mathcal{A}$ be an Abelian category.

(1) A pair of classes of objects $(\mathcal{C}, \mathcal{F})$ in $\mathcal{A}$ is called a *cotorsion pair*, if:

$$\mathcal{C} = {}^{\perp}\mathcal{F} \ = \{X \in \mathcal{A} \mid \mathrm{Ext}^1(X, \mathcal{F}) = 0\}, \qquad \mathcal{F} = \mathcal{C}^{\perp}.$$

(2) A cotorsion pair $(\mathcal{C}, \mathcal{F})$ is called *hereditary*, if:

- $\mathcal{C}$ is closed under the kernel of epimorphisms, and
- $\mathcal{F}$ is closed under cokernel of monomorphisms.

(3) A cotorsion pair $(\mathcal{C}, \mathcal{F})$ is called *complete*, if there exists short exact sequence $\forall X \in \mathcal{A}$

$$0 \to F \to C \to X \to 0, \quad 0 \to X \to F' \to C' \to 0$$

where $C,\ C' \in \mathcal{C},\ F,\ F' \in \mathcal{F}$.

proposition

Let $\mathcal{A}$ be an Abelian category with enough projective and injective objects. Then the two conditions in the hereditary of a cotorsion pair above are equivalent, and each of them is equivalent to $\mathrm{Ext}^2_{\mathcal{A}}(\mathcal{C}, \mathcal{F}) = 0$; and also equivalent to $\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{C}, \mathcal{F}) = 0$ for $i \geq 1$.

proposition

Let $\mathcal{A}$ be an Abelian category with enough projective and injective objects. Then the two conditions in the completeness of a cotorsion pair above are equivalent.

Hovey triples

Definition

Let $\mathcal{A}$ be an Abelian category. A triple of classes of objects $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is called a *Hovey triple*, if:

(1) the class of objects $\mathcal{Z}$ is a *thick subcategory*, that is, $\mathcal{Z}$ is closed under direct summands, and for a short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathcal{A}$, if there are two terms in $\mathcal{Z}$, then so is the third term;

(2) $(\mathcal{X},\ \mathcal{Y} \cap \mathcal{Z})$ and $(\mathcal{X} \cap \mathcal{Z},\ \mathcal{Y})$ are complete cotorsion pairs.

Hovey correspondence

Theorem (M. Hovey, 2002)

There is a one-to-one correspondence between abelian model structures $(\mathrm{Cofib}(\mathcal{A}),\ \mathrm{Fib}(\mathcal{A}),\ \mathrm{W}(\mathcal{A}))$ with cofibrant objects $\mathcal{C}$, fibrant objects $\mathcal{F}$ and trivial objects $\mathcal{W}$ on Abelian category $\mathcal{A}$ and Hovey triples:

$$(\mathrm{Cofib}(\mathcal{A}),\ \mathrm{Fib}(\mathcal{A}),\ \mathrm{W}(\mathcal{A})) \mapsto (\mathcal{C}, \mathcal{F}, \mathcal{W})$$

and its inverse:

$$\text{Hovey triple } (\mathcal{C}, \mathcal{F}, \mathcal{W}) \mapsto (\mathrm{Cofib}(\mathcal{A}),\ \mathrm{Fib}(\mathcal{A}),\ \mathrm{W}(\mathcal{A}))$$

where

$$\mathrm{Cofib}(\mathcal{A}) = \{\text{monomorphism } f \mid \mathrm{Coker}\, f \in \mathcal{C}\},$$

$$\mathrm{Fib}(\mathcal{A}) = \{\text{epimorphism } f \mid \mathrm{Ker}\, f \in \mathcal{F}\},$$

$$\mathrm{W}(\mathcal{A}) = \{pi \mid i \text{ monic with } \mathrm{Coker}\, i \in \mathcal{C} \cap \mathcal{W},\ p \text{ epic with } \mathrm{Ker}\, p \in \mathcal{F} \cap \mathcal{W}\}.$$

Compatible cotorsion pairs

Definition

Two cotorsion pairs $(\Phi,\ \Phi^{\perp})$ and $({}^{\perp}\Psi,\ \Psi)$ in an Abelian category $\mathcal{A}$ are called *compatible*, if:

(1) $\mathrm{Ext}^1_{\mathcal{A}}(\Phi, \Psi) = 0$, that is, $\Phi \subseteq {}^{\perp}\Psi, \ \ \Psi \subseteq \Phi^{\perp}$;

(2) $\Phi \cap \Phi^{\perp} = {}^{\perp}\Psi \cap \Psi$.

Notice that the compatibility depends on the order of two cotorsion pairs.

Theorem (H. Becker 2014; J. Gillespie, 2015)

Let $(\Phi,\ \Phi^{\perp})$ and $({}^{\perp}\Psi,\ \Psi)$ be two hereditary, complete and compatible cotorsion pairs in an Abelian category $\mathcal{A}$. Then $({}^{\perp}\Psi,\ \Phi^{\perp},\ \mathcal{W})$ is a hereditary Hovey triple, where

$$\begin{aligned}\mathcal{W} &= \{\, Y \in \mathcal{A} \mid \exists \text{ exact sequence } 0 \to P \to F \to Y \to 0,\ P \in \Psi,\ F \in \Phi\} \\ &= \{\, Y \in \mathcal{A} \mid \exists \text{ exact sequence } 0 \to Y \to P' \to F' \to 0,\ P' \in \Psi,\ F' \in \Phi\}.\end{aligned}$$

That is, ${}^{\perp}\Psi \cap \mathcal{W} = \Phi,\ \ \Phi^{\perp} \cap\ \mathcal{W} = \Psi$, and $\mathcal{W}$ is a thick subcategory. Thus the corresponding model structure is:

- $\mathrm{Cofib}(\mathcal{A}) = \{\text{monomorphism } f \mid \mathrm{Coker}\, f \in {}^{\perp}\Psi\}$;
- $\mathrm{Fib}(\mathcal{A}) = \{\text{epimorphism } f \mid \mathrm{Ker}\, f \in \Phi^{\perp}\}$;
- $\mathrm{W}(\mathcal{A}) = \{pi \mid i \text{ monic with } \mathrm{Coker}\, i \in \Phi,\ p \text{ epic with } \mathrm{Ker}\, p \in \Psi\}$;
- $\mathrm{Ho}(\mathcal{A}) = ({}^{\perp}\Psi \cap \Phi^{\perp})/({}^{\perp}\Psi \cap \Psi) = ({}^{\perp}\Psi \cap \Phi^{\perp})/(\Phi \cap \Phi^{\perp})$.

Morita rings

A, B: rings, ${}_BM_A$, ${}_AN_B$: bimodules, $\phi : M \otimes_A N \to B$, $\psi : N \otimes_B M \to A$: bimodule maps, s.t. $\forall\ m, m' \in M$, $n, n' \in N$

$$m'\psi(n \otimes_B m) = \phi(m' \otimes_A n)m, \qquad n'\phi(m \otimes_A n) = \psi(n' \otimes_B m)n.$$

A *Morita ring* is $\Lambda = \Lambda_{(\phi,\psi)} := \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$, with componentwise addition, and multiplication

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' + \psi(n \otimes_B m') & an' + nb' \\ ma' + bm' & \phi(m \otimes_A n') + bb' \end{pmatrix}.$$

E. Green: A Λ-module: $\binom{X}{Y}_{f,g}$, $f \in \mathrm{Hom}_B(M \otimes_A X, Y)$ and $g \in \mathrm{Hom}_A(N \otimes_B Y, X)$, s.t. $g(1 \otimes f) = 0 = f(1 \otimes g)$. A Λ-map is $\binom{a}{b} : \binom{X}{Y}_{f,g} \to \binom{X'}{Y'}_{f',g'}$, with a and b an A-map and a B-map, respectively, s.t. the diagrams commute:

$$\begin{array}{ccc} M \otimes_A X & \xrightarrow{1_M \otimes a} & M \otimes_A X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{\quad b \quad} & Y' \end{array} \qquad \begin{array}{ccc} N \otimes_B Y & \xrightarrow{1_N \otimes b} & N \otimes_B Y' \\ g \downarrow & & \downarrow g' \\ X & \xrightarrow{\quad a \quad} & X'. \end{array}$$

Adjoint pair: $A\text{-Mod} \underset{\mathrm{Hom}_B(M,-)}{\overset{M \otimes_A -}{\rightleftarrows}} B\text{-Mod}$ with adjoint isomorphism:

$$f \in \mathrm{Hom}_B(M \otimes_A X, Y) \xrightarrow{\cong} \mathrm{Hom}_A(X, \mathrm{Hom}_B(M, Y)) \ni \widetilde{f}$$

Adjoint pair: $B\text{-Mod} \underset{\mathrm{Hom}_A(N,-)}{\overset{N \otimes_B -}{\rightleftarrows}} A\text{-Mod}$ with adjoint isomorphism:

$$g \in \mathrm{Hom}_A(N \otimes_B Y, X) \xrightarrow{\cong} \mathrm{Hom}_B(Y, \mathrm{Hom}_A(N, X)) \ni \widetilde{g}.$$

12 functors

- $\mathrm{T}_A : A\text{-Mod} \to \Lambda\text{-Mod}, \ \ X \mapsto \left(\begin{smallmatrix} X \\ M\otimes_A X \end{smallmatrix}\right)_{1,0}.$
- $\mathrm{T}_B : B\text{-Mod} \to \Lambda\text{-Mod}, \ \ Y \mapsto \left(\begin{smallmatrix} N\otimes_B Y \\ Y \end{smallmatrix}\right)_{0,1}.$
- $\mathrm{U}_A : \Lambda\text{-Mod} \to A\text{-Mod}, \ \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_{f,g} \mapsto X.$
- $\mathrm{U}_B : \Lambda\text{-Mod} \to B\text{-Mod}, \ \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_{f,g} \mapsto Y.$
- $\mathrm{H}_A : A\text{-Mod} \to \Lambda\text{-Mod}, \ X \mapsto \left(\begin{smallmatrix} X \\ (N,X) \end{smallmatrix}\right)_{0,\ \epsilon'_X}.$
- $\mathrm{H}_B : B\text{-Mod} \to \Lambda\text{-Mod}, \ Y \mapsto \left(\begin{smallmatrix} (M,Y) \\ Y \end{smallmatrix}\right)_{\epsilon_Y,0}.$
- $\mathrm{C}_A : \Lambda\text{-Mod} \to A\text{-Mod}, \ \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_{f,g} \mapsto \mathrm{Coker}\, g.$
- $\mathrm{C}_B : \Lambda\text{-Mod} \to B\text{-Mod}, \ \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_{f,g} \mapsto \mathrm{Coker}\, f.$
- $\mathrm{K}_A : \Lambda\text{-Mod} \to A\text{-Mod}, \ \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_{f,g} \mapsto \mathrm{Ker}\, \widetilde{f}.$
- $\mathrm{K}_B : \Lambda\text{-Mod} \to B\text{-Mod}, \ \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_{f,g} \mapsto \mathrm{Ker}\, \widetilde{g}.$
- $\mathrm{Z}_A : A\text{-Mod} \to \Lambda\text{-Mod}, \ \ X \mapsto \left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right)_{0,0}.$
- $\mathrm{Z}_B : B\text{-Mod} \to \Lambda\text{-Mod}, \ \ Y \mapsto \left(\begin{smallmatrix} 0 \\ Y \end{smallmatrix}\right)_{0,0}.$

2 rerecollements

There are two recollements of abelian categories (E. L. Green - C. Psaroudakis):

$$A\text{-Mod} \;\underset{\mathrm{K}_A}{\overset{\mathrm{C}_A}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathrm{Z}_A}}}}}\; \Lambda_{(0,0)}\text{-Mod} \;\underset{\mathrm{H}_B}{\overset{\mathrm{T}_B}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathrm{U}_B}}}}}\; B\text{-Mod}$$

and

$$B\text{-Mod} \;\underset{\mathrm{K}_B}{\overset{\mathrm{C}_B}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathrm{Z}_B}}}}}\; \Lambda_{(0,0)}\text{-Mod} \;\underset{\mathrm{H}_A}{\overset{\mathrm{T}_A}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathrm{U}_A}}}}}\; A\text{-Mod}.$$

(Hereditary) cotorsion pairs (ctp) in Morita rings: Series I

For a class $\mathcal{X}$ of A-modules and a class $\mathcal{Y}$ of B-modules, define:

$\binom{\mathcal{X}}{\mathcal{Y}} := \{ \binom{X}{Y}_{f,g} \in \Lambda\text{-Mod} \mid X \in \mathcal{X}, \quad Y \in \mathcal{Y}\}$

$\Delta(\mathcal{X},\mathcal{Y}) := \{ \binom{L_1}{L_2}_{f,g} \in \Lambda\text{-Mod} \mid f\colon M\otimes_A L_1 \longrightarrow L_2$ and $g\colon N\otimes_B L_2 \longrightarrow L_1$ are monomorphisms, $\operatorname{Coker} f \in \mathcal{Y}, \quad \operatorname{Coker} g \in \mathcal{X}\}$

$\nabla(\mathcal{X},\mathcal{Y}) := \{ \binom{L_1}{L_2}_{f,g} \in \Lambda\text{-Mod} \mid \widetilde{f}\colon L_1 \longrightarrow \operatorname{Hom}_B(M,L_2)$ and $\widetilde{g}\colon L_2 \longrightarrow \operatorname{Hom}_A(N,L_1)$ are epimorphisms, $\operatorname{Ker}\widetilde{f} \in \mathcal{X}, \ \operatorname{Ker}\widetilde{g} \in \mathcal{Y}\}$

$\operatorname{Mon}(\Lambda) := \Delta(A\text{-Mod}, B\text{-Mod}) = \{ \binom{L_1}{L_2}_{f,g} \in \Lambda\text{-Mod} \mid$ f and g are monomorphisms$\}$

$\operatorname{Epi}(\Lambda) := \nabla(A\text{-Mod}, B\text{-Mod}) = \{ \binom{L_1}{L_2}_{f,g} \in \Lambda\text{-Mod} \mid$ $\widetilde{f}$ and $\widetilde{g}$ are epimorphisms$\}$.

(Hereditary) cotorsion pairs (ctp) in Morita rings: Series I

Theorem

Let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be a Morita ring with $\phi = 0 = \psi$, and $(\mathcal{U}, \mathcal{X})$, $(\mathcal{V}, \mathcal{Y})$ cotorsion pairs in A-Mod and B-Mod respectively. Then:

(1) If $\mathrm{Tor}_1(M, \mathcal{U}) = 0 = \mathrm{Tor}_1(N, \mathcal{V})$, then $({}^{\perp}\begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix}, \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix})$ is a cotorsion pair. It is hereditary if and only if $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{V}, \mathcal{Y})$ are hereditary.

(2) If $\mathrm{Ext}^1(N, \mathcal{X}) = 0 = \mathrm{Ext}^1(M, \mathcal{Y})$, then $(\begin{pmatrix} \mathcal{U} \\ \mathcal{V} \end{pmatrix}, \begin{pmatrix} \mathcal{U} \\ \mathcal{V} \end{pmatrix}^{\perp})$ is a cotorsion pair. It is hereditary if and only if $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{V}, \mathcal{Y})$ are hereditary.

(Hereditary) cotorsion pairs (ctp) in Morita rings: Series II

Theorem

Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$, and $(\mathcal{U}, \mathcal{X})$, $(\mathcal{V}, \mathcal{Y})$ cotorsion pairs in $A\text{-Mod}$ and $B\text{-Mod}$ respectively. Then:

(1) $(\Delta(\mathcal{U}, \mathcal{V}),\ \Delta(\mathcal{U}, \mathcal{V})^{\perp})$ is a cotorsion pair in $\Lambda\text{-Mod}$.

If M_A and N_B are flat, then it is hereditary if and only if $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{V}, \mathcal{Y})$ are hereditary.

(2) $({}^{\perp}\nabla(\mathcal{X}, \mathcal{Y}),\ \nabla(\mathcal{X}, \mathcal{Y}))$ is a cotorsion pair in $\Lambda\text{-Mod}$.

If ${}_BM$ and ${}_AN$ are projective, then it is hereditary if and only if $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{V}, \mathcal{Y})$ are hereditary.

The condition "$M \otimes_A N = 0 = N \otimes_B M$" in the theorem can not be weakened as "$\phi = 0 = \psi$".

comparison

Theorem

Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$, and $(\mathcal{U}, \mathcal{X})$, $(\mathcal{V}, \mathcal{Y})$ cotorsion pairs in A-Mod and B-Mod, respectively.
(1) If $\operatorname{Tor}_1^A(M, \mathcal{U}) = 0 = \operatorname{Tor}_1^B(N, \mathcal{V})$, then the cotorsion pairs

$$({}^{\perp}\left(\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right)) \quad \text{and} \quad (\Delta(\mathcal{U}, \mathcal{V}), \Delta(\mathcal{U}, \mathcal{V})^{\perp})$$

in Λ-mod have a relation ${}^{\perp}\left(\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right) \subseteq \Delta(\mathcal{U}, \mathcal{V})$, or equivalently, $\Delta(\mathcal{U}, \mathcal{V})^{\perp} \subseteq \left(\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right)$.
(2) If $\operatorname{Ext}_A^1(N, \mathcal{X}) = 0 = \operatorname{Ext}_B^1(M, \mathcal{Y})$, then the cotorsion pairs

$$(\left(\begin{smallmatrix} \mathcal{U} \\ \mathcal{V} \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathcal{U} \\ \mathcal{V} \end{smallmatrix}\right)^{\perp}) \quad \text{and} \quad ({}^{\perp}\nabla(\mathcal{X}, \mathcal{Y}), \nabla(\mathcal{X}, \mathcal{Y}))$$

in Λ-mod have a relation $\left(\begin{smallmatrix} \mathcal{U} \\ \mathcal{V} \end{smallmatrix}\right)^{\perp} \subseteq \nabla(\mathcal{X}, \mathcal{Y})$, or equivalently, ${}^{\perp}\nabla(\mathcal{X}, \mathcal{Y}) \subseteq \left(\begin{smallmatrix} \mathcal{U} \\ \mathcal{V} \end{smallmatrix}\right)$.

Remark

In the case of $M=0$, theorems above are obtained by R. M. Zhu , Y. Y. Peng, N. Q. Ding, Publ. Math. Debrecen 98(1)(2021), 83-113.

In particular, when $M=0$ we have:

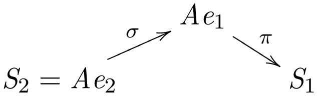
$$({}^{\perp}\binom{\mathcal{X}}{\mathcal{Y}},\ \binom{\mathcal{X}}{\mathcal{Y}}) = (\Delta(\mathcal{U},\mathcal{V}),\ \Delta(\mathcal{U},\mathcal{V})^{\perp});$$

$$(\binom{\mathcal{U}}{\mathcal{V}},\ \binom{\mathcal{U}}{\mathcal{V}}^{\perp}) = ({}^{\perp}\nabla(\mathcal{X},\mathcal{Y}),\ \nabla(\mathcal{X},\mathcal{Y})).$$

An important example

We claim that generally $({}^{\perp}\left(\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right),\ \left(\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right)) \neq (\Delta(\mathcal{U},\mathcal{V}),\ \Delta(\mathcal{U},\mathcal{V})^{\perp})$; $({}^{\perp}\nabla(\mathcal{X},\mathcal{Y}),\ \nabla(\mathcal{X},\mathcal{Y})) \neq (\left(\begin{smallmatrix}\mathcal{U}\\ \mathcal{V}\end{smallmatrix}\right),\ \left(\begin{smallmatrix}\mathcal{U}\\ \mathcal{V}\end{smallmatrix}\right)^{\perp})$.

Example. Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B\end{smallmatrix}\right) = \left(\begin{smallmatrix} A & M \\ M & A\end{smallmatrix}\right)$, $A = B = k(1 \longrightarrow 2)$, where $\mathrm{char}\ k \neq 2$. Thus $e_1Ae_2 = 0$ and $e_2Ae_1 \cong k$. Take $M = N = Ae_2 \otimes_k e_1A$. Then $M \otimes_A N = 0 = N \otimes_A M$. The AR quiver of A is



(1) Take $(\mathcal{U},\mathcal{X}) = (A\text{-Mod},\ {}_A\mathcal{I}) = (\mathcal{V},\mathcal{Y})$, $L = \begin{pmatrix} Ae_1 \\ Ae_1 \end{pmatrix}_{\sigma,\sigma}$. Then

$$L \in \mathrm{Mon}(\Lambda) = \Delta(A\text{-Mod}, A\text{-Mod}) = \Delta(\mathcal{U},\mathcal{V}), \qquad L \in \begin{pmatrix} \mathcal{I} \\ \mathcal{I} \end{pmatrix} = \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix}.$$

The following exact sequence of Λ-modules does not split:

$$0 \to \begin{pmatrix} Ae_1 \\ Ae_1 \end{pmatrix}_{\sigma,\sigma} \xrightarrow{\begin{pmatrix} \binom{1}{0} \\ \binom{1}{0} \end{pmatrix}} \begin{pmatrix} Ae_1 \oplus Ae_1 \\ Ae_1 \oplus Ae_1 \end{pmatrix}_{\left(\begin{smallmatrix} \sigma & \sigma \\ 0 & \sigma \end{smallmatrix}\right),\left(\begin{smallmatrix} \sigma & \sigma \\ 0 & \sigma \end{smallmatrix}\right)} \xrightarrow{\begin{pmatrix} (0,1) \\ (0,1) \end{pmatrix}} \begin{pmatrix} Ae_1 \\ Ae_1 \end{pmatrix}_{\sigma,\sigma} \to 0$$

i.e., $\mathrm{Ext}^1_\Lambda(L,L) \neq 0$. So $\mathrm{Mon}(\Lambda) \not\subseteq {}^{\perp}\begin{pmatrix} \mathcal{I} \\ \mathcal{I} \end{pmatrix}$, and hence $(\mathrm{Mon}(\Lambda),\ \mathrm{Mon}(\Lambda)^{\perp}) \neq ({}^{\perp}\begin{pmatrix} \mathcal{I} \\ \mathcal{I} \end{pmatrix},\ \begin{pmatrix} \mathcal{I} \\ \mathcal{I} \end{pmatrix})$.

(2) Take $(\mathcal{U},\mathcal{X}) = (\mathcal{P}, A\text{-Mod}) = (\mathcal{V},\mathcal{Y})$, $L = \begin{pmatrix} Ae_1 \\ Ae_1 \end{pmatrix}_{\sigma,\sigma} \in \begin{pmatrix} \mathcal{P} \\ \mathcal{P} \end{pmatrix} = \begin{pmatrix} \mathcal{U} \\ \mathcal{V} \end{pmatrix}$.
Then $L \in \mathrm{Eip}(\Lambda) = \nabla(A\text{-Mod}, A\text{-Mod}) = \nabla(\mathcal{X},\mathcal{Y})$. By (1), $\mathrm{Ext}^1_\Lambda(L,L) \neq 0$, which shows $\mathrm{Epi}(\Lambda) \not\subseteq \begin{pmatrix} \mathcal{P} \\ \mathcal{P} \end{pmatrix}^{\perp}$, and hence

$$({}^{\perp}\mathrm{Epi}(\Lambda),\ \mathrm{Epi}(\Lambda)) \neq (\begin{pmatrix} \mathcal{P} \\ \mathcal{P} \end{pmatrix},\ \begin{pmatrix} \mathcal{P} \\ \mathcal{P} \end{pmatrix}^{\perp}).$$

Equality

Theorem

Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$, and $(\mathcal{U},\ \mathcal{X})$, $(\mathcal{V},\ \mathcal{Y})$ cotorsion pairs in $A\text{-Mod}$ and in $B\text{-Mod}$, respectively.
(1) Assume that $\mathrm{Tor}_1^A(M,\ \mathcal{U}) = 0 = \mathrm{Tor}_1^B(N,\ \mathcal{V})$. If $M \otimes_A \mathcal{U} \subseteq \mathcal{Y}$ or $N \otimes_B \mathcal{V} \subseteq \mathcal{X}$, then

$$(\Delta(\mathcal{U},\mathcal{V}),\ \Delta(\mathcal{U},\mathcal{V})^{\perp}) = ({}^{\perp}\left(\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right),\ \left(\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right)).$$

Thus $(\Delta(\mathcal{U},\mathcal{V}),\ \left(\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right))$ is a cotorsion pair.
(2) Assume that $\mathrm{Ext}_B^1(M,\mathcal{Y}) = 0 = \mathrm{Ext}_A^1(N,\mathcal{X})$. If $\mathrm{Hom}_B(M,\mathcal{Y}) \subseteq \mathcal{U}$ or $\mathrm{Hom}_A(N,\mathcal{X}) \subseteq \mathcal{V}$, then

$$({}^{\perp}\nabla(\mathcal{X},\mathcal{Y}),\ \nabla(\mathcal{X},\mathcal{Y})) = (\left(\begin{smallmatrix} \mathcal{U} \\ \mathcal{V} \end{smallmatrix}\right),\ \left(\begin{smallmatrix} \mathcal{U} \\ \mathcal{V} \end{smallmatrix}\right)^{\perp}).$$

Thus $(\left(\begin{smallmatrix} \mathcal{U} \\ \mathcal{V} \end{smallmatrix}\right),\ \nabla(\mathcal{X},\mathcal{Y}))$ is a cotorsion pair.

Theorem

Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ be a noetherian Morita ring with $M \otimes_A N = 0 = N \otimes_B M$. Assume that A and B are quasi-Frobenius rings, ${}_AN$ and ${}_BM$ are projective, and that M_A and N_B are flat. Then

(1) Λ is a Gorenstein ring with $\mathrm{inj.dim}_\Lambda \Lambda \leq 1$.

(2) $({}^\perp\binom{\mathcal{I}}{\mathcal{I}}, \ \binom{\mathcal{I}}{\mathcal{I}}) = (\mathrm{Mon}(\Lambda), \ \mathrm{Mon}(\Lambda)^\perp)$; and it is the Gorenstein-proj. cotorsion pair $(\mathrm{GP}(\Lambda), \ {}_\Lambda\mathcal{P}^{\leq 1})$. So it is complete and hereditary, and

$$\mathrm{GP}(\Lambda) = \mathrm{Mon}(\Lambda) = {}^\perp\ {}_\Lambda\mathcal{P}, \qquad \mathrm{Mon}(\Lambda)^\perp = \ {}_\Lambda\mathcal{P}^{\leq 1}.$$

(2)$'$ $(\binom{\mathcal{P}}{\mathcal{P}}, \ \binom{\mathcal{P}}{\mathcal{P}}^\perp) = ({}^\perp\mathrm{Epi}(\Lambda), \ \mathrm{Epi}(\Lambda))$; and it is the Gorenstein-inj. cotorsion pair $({}_\Lambda\mathcal{P}^{\leq 1}, \ \mathrm{GI}(\Lambda))$. So it is complete and hereditary, and

$$\mathrm{GI}(\Lambda) = \mathrm{Epi}(\Lambda) = \ {}_\Lambda\mathcal{I}^\perp, \qquad {}^\perp\mathrm{Epi}(\Lambda) = \ {}_\Lambda\mathcal{P}^{\leq 1}.$$

Note that $\mathrm{GP}(\Lambda) = \mathrm{Mon}(\Lambda)$ is a new result.

Theorem

Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ be a Morita ring with $\phi = 0 = \psi$, and $(\mathcal{V}, \mathcal{Y})$ a complete ctp in B-Mod. Suppose that N_B is flat and ${}_BM$ is projective.

(1) If $M \otimes_A \mathcal{P} \subseteq \mathcal{Y}$, then $\left({}^{\perp}\binom{A\text{-Mod}}{\mathcal{Y}}, \binom{A\text{-Mod}}{\mathcal{Y}}\right)$ is a complete cotorsion pair in Λ-Mod.

If $M \otimes_A N = 0 = N \otimes_B M$, then one has the complete cotorsion pair:

$$(\mathrm{T}_A({}_A\mathcal{P}) \oplus \mathrm{T}_B(\mathcal{V}), \ \binom{A\text{-Mod}}{\mathcal{Y}}).$$

(2) If $\mathrm{Hom}_A(N, \ {}_A\mathcal{I}) \subseteq \mathcal{V}$, then $\left(\binom{A\text{-Mod}}{\mathcal{V}}, \binom{A\text{-Mod}}{\mathcal{V}}^{\perp}\right)$ is a complete cotorsion pair in Λ-Mod.

If $M \otimes_A N = 0 = N \otimes_B M$, then one has the complete cotorsion pair:

$$(\binom{A\text{-Mod}}{\mathcal{V}}, \ \mathrm{H}_A({}_A\mathcal{I}) \oplus \mathrm{H}_B(\mathcal{Y})).$$

- If B is left noetherian and ${}_BM$ is inj., then $M \otimes_A \mathcal{P} \subseteq \mathcal{Y}$ always holds.
- If B is quasi-Frobenius and N_B is flat, then $\mathrm{Hom}_A(N, {}_A\mathcal{I}) \subseteq \mathcal{V}$ always holds.

Theorem

Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ be a Morita ring with $\phi = 0 = \psi$, and $(\mathcal{U}, \mathcal{X})$ a complete ctp in A-Mod. Suppose that M_A is flat and $_AN$ is proj..

(1) If $N \otimes_B \mathcal{P} \subseteq \mathcal{X}$, then $\left({}^{\perp}\binom{\mathcal{X}}{B\text{-Mod}}, \binom{\mathcal{X}}{B\text{-Mod}}\right)$ is a complete ctp in Λ-Mod.

If $M \otimes_A N = 0 = N \otimes_B M$, then one has complete ctp

$$(\mathrm{T}_A(\mathcal{U}) \oplus \mathrm{T}_B({}_B\mathcal{P}), \ \binom{\mathcal{X}}{B\text{-Mod}}).$$

(2) If $\mathrm{Hom}_B(M, \ {}_B\mathcal{I}) \subseteq \mathcal{U}$, then $\left(\binom{\mathcal{U}}{B\text{-Mod}}, \binom{\mathcal{U}}{B\text{-Mod}}^{\perp}\right)$ is a complete ctp in Λ-Mod.

If $M \otimes_A N = 0 = N \otimes_B M$, then one has complete ctp

$$(\binom{\mathcal{U}}{B\text{-Mod}}, \ \mathrm{H}_A(\mathcal{X}) \oplus \mathrm{H}_B({}_B\mathcal{I})).$$

- If A is left noetherian and $_AN$ is inj., then $N \otimes_B \mathcal{P} \subseteq \mathcal{X}$ always holds.
- If A is quasi-Frobenius and M_A is flat, then $\mathrm{Hom}_B(M, \ {}_B\mathcal{I}) \subseteq \mathcal{U}$ always holds.

Model structures on Morita rings

Theorem

Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$, and $(\mathcal{V}', \ \mathcal{Y}, \ \mathcal{W})$ a Hovey triple in B-Mod. Suppose that N_B is flat and $_BM$ is projective.

(1) If $M \otimes_A \mathcal{P} \subseteq \mathcal{Y} \cap \mathcal{W}$, then

$$(\mathrm{T}_A({}_A\mathcal{P}) \oplus \mathrm{T}_B(\mathcal{V}'), \ \left(\begin{smallmatrix} A\text{-Mod} \\ \mathcal{Y} \end{smallmatrix}\right), \ \left(\begin{smallmatrix} A\text{-Mod} \\ \mathcal{W} \end{smallmatrix}\right))$$

is a Hovey triple in Λ-Mod; and it is hereditary with $\mathrm{Ho}(\Lambda) \cong \mathrm{Ho}(B)$, provided that $(\mathcal{V}', \mathcal{Y}, \mathcal{W})$ is hereditary.

(2) If $\mathrm{Hom}_A(N, \ {}_A\mathcal{I}) \subseteq \mathcal{V}' \cap \mathcal{W}$, then

$$(\left(\begin{smallmatrix} A\text{-Mod} \\ \mathcal{V}' \end{smallmatrix}\right), \ \mathrm{H}_A({}_A\mathcal{I}) \oplus \mathrm{H}_B(\mathcal{Y}), \ \left(\begin{smallmatrix} A\text{-Mod} \\ \mathcal{W} \end{smallmatrix}\right))$$

is a Hovey triple in Λ-Mod; and it is hereditary with $\mathrm{Ho}(\Lambda) \cong \mathrm{Ho}(B)$, provided that $(\mathcal{V}', \mathcal{Y}, \mathcal{W})$ is hereditary.

Theorem

Let $\Lambda = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$, and $(\mathcal{U}', \ \mathcal{X}, \ \mathcal{W})$ a Hovey triple in $A\text{-Mod}$. Suppose that M_A is flat and $_A N$ is projective.

(1) If $N \otimes_B \mathcal{P} \subseteq \mathcal{X} \cap \mathcal{W}$, then

$$(\mathrm{T}_A(\mathcal{U}') \oplus \mathrm{T}_B({}_B\mathcal{P}), \ \left(\begin{smallmatrix} \mathcal{X} \\ B\text{-Mod} \end{smallmatrix}\right), \ \left(\begin{smallmatrix} \mathcal{W} \\ B\text{-Mod} \end{smallmatrix}\right))$$

is a Hovey triple; and it is hereditary with $\mathrm{Ho}(\Lambda) \cong \mathrm{Ho}(A)$, provided that $(\mathcal{U}', \mathcal{X}, \mathcal{W})$ is hereditary.

(2) If $\mathrm{Hom}_B(M, \ {}_B\mathcal{I}) \subseteq \mathcal{U}' \cap \mathcal{W}$, then

$$(\left(\begin{smallmatrix} \mathcal{U}' \\ B\text{-Mod} \end{smallmatrix}\right), \ \mathrm{H}_A(\mathcal{X}) \oplus \mathrm{H}_B({}_B\mathcal{I}), \ \left(\begin{smallmatrix} \mathcal{W} \\ B\text{-Mod} \end{smallmatrix}\right))$$

is a Hovey triple; and it is hereditary with $\mathrm{Ho}(\Lambda) \cong \mathrm{Ho}(A)$, provided that $(\mathcal{U}', \mathcal{X}, \mathcal{W})$ is hereditary.

	Hereditary cotorsion pairs in Series I $\varphi = 0 = \psi$		Cotorsion pairs in Series II $M \otimes_A N = 0 = N \otimes_B M$	
$({}_A\mathcal{U},\ {}_A\mathcal{X})$ $({}_B\mathcal{V},\ {}_B\mathcal{Y})$	$\mathrm{Tor}_1(M, \mathcal{U}) = 0$ $\mathrm{Tor}_1(N, \mathcal{V}) = 0$: $({}^{\perp}\binom{\mathcal{X}}{\mathcal{Y}}, \binom{\mathcal{X}}{\mathcal{Y}})$	$\mathrm{Ext}^1(N, \mathcal{X}) = 0$ $\mathrm{Ext}^1(M, \mathcal{Y}) = 0$: $(\binom{\mathcal{U}}{\mathcal{V}}, \binom{\mathcal{U}}{\mathcal{V}}^{\perp})$	$(\Delta(\mathcal{U}, \mathcal{V}),\ \Delta(\mathcal{U}, \mathcal{V})^{\perp})$	$({}^{\perp}\nabla(\mathcal{X}, \mathcal{Y}),\ \nabla(\mathcal{X}, \mathcal{Y}))$
$(\mathcal{P}, \mathcal{A})$ $(\mathcal{P}, \mathcal{B})$	$({}_{\Lambda}\mathcal{P}, \Lambda\text{-Mod})$	${}_AN,\ {}_BM$ proj.: $(\binom{\mathcal{P}}{\mathcal{P}},\ \binom{\mathcal{P}}{\mathcal{P}}^{\perp})$	$({}_{\Lambda}\mathcal{P}, \Lambda\text{-Mod})$	$({}^{\perp}\mathrm{Epi}(\Lambda),\ \mathrm{Epi}(\Lambda))$. Even if ${}_AN,\ {}_BM$ proj., $({}^{\perp}\mathcal{E}, \mathcal{E}) \neq (\binom{\mathcal{P}}{\mathcal{P}},\ \binom{\mathcal{P}}{\mathcal{P}}^{\perp})$ in general.
$(\mathcal{P}, \mathcal{A})$ $(\mathcal{B}, \mathcal{I})$	N_B flat: $({}^{\perp}\binom{\mathcal{A}}{\mathcal{I}},\ \binom{\mathcal{A}}{\mathcal{I}})$	${}_AN$ proj.: $(\binom{\mathcal{P}}{\mathcal{B}},\ \binom{\mathcal{P}}{\mathcal{B}}^{\perp})$	$(\Delta(\mathcal{P}, \mathcal{B}), \Delta(\mathcal{P}, \mathcal{B})^{\perp})$. If N_B flat then it is $({}^{\perp}\binom{\mathcal{A}}{\mathcal{I}},\ \binom{\mathcal{A}}{\mathcal{I}})$ thus it is $(\Delta(\mathcal{P}, \mathcal{B}),\ \binom{\mathcal{A}}{\mathcal{I}})$	$({}^{\perp}\nabla(\mathcal{A}, \mathcal{I}), \nabla(\mathcal{A}, \mathcal{I}))$. If ${}_AN$ proj. then it is $(\binom{\mathcal{P}}{\mathcal{B}},\ \binom{\mathcal{P}}{\mathcal{B}}^{\perp})$ thus it is $(\binom{\mathcal{P}}{\mathcal{B}},\ \nabla(\mathcal{A}, \mathcal{I}))$
$(\mathcal{A}, \mathcal{I})$ $(\mathcal{P}, \mathcal{B})$	M_A flat: $({}^{\perp}\binom{\mathcal{I}}{\mathcal{B}},\ \binom{\mathcal{I}}{\mathcal{B}})$	${}_BM$ proj.: $(\binom{\mathcal{A}}{\mathcal{P}},\ \binom{\mathcal{A}}{\mathcal{P}}^{\perp})$	$(\Delta(\mathcal{A}, \mathcal{P}), \Delta(\mathcal{A}, \mathcal{P})^{\perp})$. If M_A flat then it is $({}^{\perp}\binom{\mathcal{I}}{\mathcal{B}}, \binom{\mathcal{I}}{\mathcal{B}})$ thus it is $(\Delta(\mathcal{A}, \mathcal{P}),\ \binom{\mathcal{I}}{\mathcal{B}})$	$({}^{\perp}\nabla(\mathcal{I}, \mathcal{B}), \nabla(\mathcal{I}, \mathcal{B}))$. If ${}_BM$ proj. then it is $(\binom{\mathcal{A}}{\mathcal{P}},\ \binom{\mathcal{A}}{\mathcal{P}}^{\perp})$ thus it is $(\binom{\mathcal{A}}{\mathcal{P}},\ \nabla(\mathcal{I}, \mathcal{B}))$
$(\mathcal{A}, \mathcal{I})$ $(\mathcal{B}, \mathcal{I})$	$M_A,\ N_B$ flat: $({}^{\perp}\binom{\mathcal{I}}{\mathcal{I}},\ \binom{\mathcal{I}}{\mathcal{I}})$	$(\Lambda\text{-Mod},\ {}_{\Lambda}\mathcal{I})$	$(\mathrm{Mon}(\Lambda),\ \mathrm{Mon}(\Lambda)^{\perp})$. Even if $M_A,\ N_B$ flat, $(\mathcal{M},\ \mathcal{M}^{\perp}) \neq ({}^{\perp}\binom{\mathcal{I}}{\mathcal{I}},\ \binom{\mathcal{I}}{\mathcal{I}})$ in general	$(\Lambda\text{-Mod},\ {}_{\Lambda}\mathcal{I})$