Some endotrivial module of the symmetric group

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Notations

- Let p be a prime number and $\mathbb{F}$ be an algebraically closed field of characteristic p.
- Let G be a finite group with $p \mid |G|$.
- The $\mathbb{F}G$-modules are all assumed to be finitely generated modules.
- For $\mathbb{F}G$-modules M, N, $M \otimes N = M \otimes_{\mathbb{F}} N$.

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Definition

Let M be an $\mathbb{F}G$-module. Then we say M is endotrivial if

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Example

The trivial module $\mathbb{F}$ is endotrivial since $\mathbb{F}^* \otimes \mathbb{F} \cong \mathbb{F}$.

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Consider the Green ring of the stable module category of $\mathbb{F}G$, M is invertible if and only if $M \otimes N \cong \mathbb{F} \oplus P$ for some module N and some projective module P. In particular, endotrivial modules are invertible.

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For $\mathbb{F}G$-modules M, N, $\mathbb{F} | M \otimes N$, if and only if $N \cong M^*$ and $\dim M$ is coprime to p.

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- The endotrivial modules are exactly all invertible elements in the Green ring of the stable module category.
- They induce stable equivalences on the stable module category of $\mathbb{F}G$.

Properties

Some known properties of endotrivial modules:

Proposition

Let M, N be $\mathbb{F}G$-modules. We have

- *if there exist a projective $\mathbb{F}G$-module P and a short exact sequence*

$$0 \to M \to P \to N \to 0,$$

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- *if M is endotrivial, then $M \cong M_0 \oplus$ (proj) for some indecomposable endotrivial module M_0;*
- *if M, N are endotrivial, then so are $M \otimes N$ and M^*.*

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The endotrivial modules are closed under taking tensor product but not direct sum!

Definition

Two endotrivial $\mathbb{F}G$-modules M N are equivalent if $M \oplus P \cong N \oplus Q$ for some projective $\mathbb{F}G$-modules P, Q.
The group of endotrivial modules of G, denoted by $T(G)$, is the set of equivalence classes $[M]$ of endotrivial $\mathbb{F}G$-module M together with

$$[M] + [N] = [M \otimes N].$$

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$$T(G) = TT(G) \oplus TF(G)$$

where $TT(G)$ is the torsion subgroup and $TF(G)$ is the torsion-free subgroup of finite rank.

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Theorem (Alperin)

Let n_G denote the number of conjugacy classes of maximal elementary abelian p-subgroups of G of order p^2. The rank of $TF(G)$ is n_G if G has p-rank at most 2, and is equal to $n_G + 1$ otherwise.

Some known classification on $T(G)$:

- G is abelian p-group (Dade);
- G is finite groups of Lie type in the defining characteristic (Carlson–Mazza–Nakano);
- G with a normal Sylow p-subgroup (Mazza);
- G with a cyclic Sylow p-subgroup (Mazza–Thévenaz);
- G is a symmetric or alternating group (Carlson–Mazza–Nakano,Carlson-Hemmer–Mazza–Nakano);

...

Let $\mathfrak{S}_n$ be the symmetric group of degree n.

Theorem

If $p = 2$, then $TT(\mathfrak{S}_n) = 0$ and

$$TF(\mathfrak{S}_n) = \begin{cases} \{0\} & \text{if } n \leq 3, \\ \mathbb{Z}^2 & \text{if } n = 4, 5, \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

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In the case when $n \geq 4$, the syzygies of $\mathbb{F}$, i.e. $\langle \Omega(\mathbb{F}) \rangle \cong \mathbb{Z}$ is always a direct summand of $TT(\mathfrak{S}_n)$.

Theorem

If $p \geq 3$, then

$$T(\mathfrak{S}_n) = \begin{cases} TT(\mathfrak{S}_n) & \text{if } 1 \leq n < 2p, \\ \mathbb{Z} \oplus (Z/2\mathbb{Z})^2 & \text{if } 2p \leq n < 3p, \\ \mathbb{Z} \oplus Z/2\mathbb{Z} & \text{if } 3p \leq n < p^2 \text{ or } p^2 + p \leq n, \\ (\mathbb{Z})^2 \oplus Z/2\mathbb{Z} & \text{if } p^2 \leq n < p^2 + p. \end{cases}$$

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When $p \geq 3$ and $2p \leq n$, the sign representation always generates a direct summand isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and $\langle \Omega(\mathbb{F}) \rangle \cong \mathbb{Z}$ is a direct summand of $TT(\mathfrak{S}_n)$.

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For $p = 2$, $n = 4, 5$ or $p \geq 3$, $p^2 \leq n < p^2 + p$, there will be another module that generates a direct summand $\mathbb{Z}$ of $TT(\mathfrak{S}_n)$.

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Theorem (Auslander-Carlson)

Suppose $p = 2$ and the Sylow 2-subgroups of G are dihedral. Let $P(\mathbb{F})$ denote the projective cover of $\mathbb{F}$. Then we have

$$\mathbf{Rad}(P(\mathbb{F}))/\mathbf{Soc}(P(\mathbb{F})) \cong M \oplus M^*,$$

for some endotrivial module M. Moreover, we have

$$TF(G) \cong \langle \Omega(\mathbb{F}), M \rangle \cong \mathbb{Z}^2$$

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In particular, the above is true when $G = \mathfrak{S}_4$ or $\mathfrak{S}_5$.

Let $p = 2$ and consider Specht module for $\mathbb{F}\mathfrak{S}_4$, $S^{(3,1)}$.

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There is a nonsplit short exact sequence:

$$0 \to \mathbb{F} \to S^{(3,1)} \to D \to 0,$$

where D is a simple self-dual $\mathbb{F}\mathfrak{S}_4$-module of dimension 2.

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Then

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We have $TT(\mathfrak{S}_4) \cong \langle \Omega(\mathbb{F}), S^{3,1)} \rangle$.

Theorem (Symonds, Karagueuzian-Symonds)

The number of nonprojective summands appearing in the symmetric algebra of an $\mathbb{F}G$-module M, i.e. $\bigoplus_{n\geq 0} \mathbf{Sym}^n M$, is finite (up to isomorphism).

Let M be the natural module for $\mathbb{F}\mathfrak{S}_4$ given by $\begin{matrix} \mathbb{F} \\ D \\ \mathbb{F} \end{matrix}$.

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Theorem (Erdmann-W.)

All the indecomposable nonprojective direct summands appear in $\mathbf{Sym}^n S^{(3,1)}$ *for* $n \geq 0$ *are given by*

$$\mathbb{F}, D, Q, \Omega(D), \Omega(Q), M.$$

For $k \geq 0$, we have (in the stable module category):

$$
\begin{aligned}
n = 4k: &\quad \mathbf{Sym}^n S^{(3,1)} \cong \mathbb{F} \oplus a_n D \oplus b_n Q \oplus c_n M \\
n = 4k+1: &\quad \mathbf{Sym}^n S^{(3,1)} \cong S^{(3,1)} \oplus a_n \Omega(D) \oplus b_n \Omega(Q) \oplus c_n M \\
n = 4k+2: &\quad \mathbf{Sym}^n S^{(3,1)} \cong a_n D \oplus b_n Q \oplus c_n M \\
n = 4k+3: &\quad \mathbf{Sym}^n S^{(3,1)} \cong a_n \Omega(D) \oplus b_n \Omega(Q) \oplus c_n M
\end{aligned}
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In fact, there is an embedding of $\mathfrak{S}_4$ into $SL_3(2)$ such the Sylow 2-subgroup of $\mathfrak{S}_4$ is also a Sylow 2-subgroup of $SL_3(2)$.
Let V denote the natural module for $SL_3(2)$.

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We have

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$$\mathbb{F}, V, \mathbf{Sym}^2 V, \mathbf{Sym}^3 V, R, \Omega(R),$$

for some indecomposable $\mathbb{F}SL_3(2)$*-module* R.

For $k \geq 0$, we have (in the stable module category):

$$n = 4k : \qquad \mathbf{Sym}^n V \cong \mathbb{F} \oplus a_n \mathbf{Sym}^2 V \oplus b_n R$$

$$n = 4k+1 : \qquad \mathbf{Sym}^n V \cong V \oplus a_n \mathbf{Sym}^3 V \oplus b_n \Omega(R)$$

$$n = 4k+2 : \qquad \mathbf{Sym}^n V \cong a_n \mathbf{Sym}^2 V \oplus b_n R$$

$$n = 4k+3 : \qquad \mathbf{Sym}^n V \cong a_n \mathbf{Sym}^3 V \oplus b_n \Omega(R)$$

Thank you!