Quiver loci, Kazhdan-Lusztig varieties, and Zelevinsky map

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August 9, 2024

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Quiver loci

- Quiver
$$Q = (Q_0, Q_1, s, t),$$
and a fixed dimension vector
$$\mathbf{d} = (d_x)_{x \in Q_0}.$$
- Quiver representation $V = (V_\alpha)_{\alpha \in Q_1}$, where
$$V_\alpha : k^{d_{s(\alpha)}} \to k^{d_{t(\alpha)}}.$$
- The representation space with fixed dimension vector
$$\mathrm{rep}_Q(\mathbf{d}) \cong \prod_{\alpha \in Q_1} \mathbb{A}^{d_{t(\alpha)} \times d_{s(\alpha)}}$$
is a product of matrix spaces.

Quiver loci

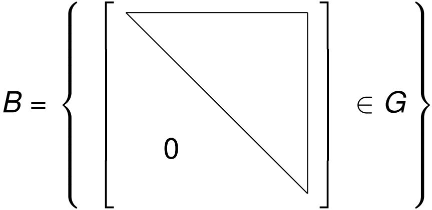
- $\mathrm{GL}(\mathbf{d}) = \prod_{x \in Q_0} \mathrm{GL}_{d_x}$ acts on $\mathrm{rep}_Q(\mathbf{d})$ as base change group:

$$\forall g = (g_x)_{x \in Q_0} \in \mathrm{GL}(\mathbf{d}), V = (V_\alpha)_{\alpha \in Q_1} \in \mathrm{rep}_Q(\mathbf{d}),$$
$$g.V = (g_{t(\alpha)} V_\alpha g_{s(\alpha)}^{-1})_{\alpha \in Q_1}.$$

- Given $V \in \mathrm{rep}_Q(\mathbf{d})$:
  the orbit $\mathcal{O}_V$ and its Zariski closure $\overline{\mathcal{O}_V}$ in $\mathrm{rep}_Q(\mathbf{d})$.
- The orbit closures $\overline{\mathcal{O}_V}$ are called **quiver loci**. The quiver loci are called of type A (D, or E), if the quiver Q is of Dynkin type A (D, or E).

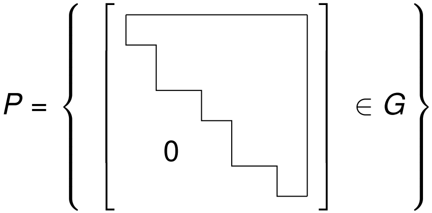
Kazhdan-Lusztig varieties

- $G = \mathrm{GL}_N$ (over an algebraically closed field k).
- T maximal torus, diagonal matrices in G.
- B fixed Borel subgroup, upper triangular matrices in G.



Kazhdan-Lusztig varieties

- $G = \mathrm{GL}_N$ (over an algebraically closed field k).
- T maximal torus, diagonal matrices in G.
- B fixed Borel subgroup, upper triangular matrices in G.
- P parabolic subgroup, $P \supset B$, block upper triangular matrices in G.



Kazhdan-Lusztig varieties

- $G = \mathrm{GL}_N$ (over an algebraically closed field k).
- T maximal torus, diagonal matrices in G.
- B fixed Borel subgroup, upper triangular matrices in G.
- P parabolic subgroup, $P \supset B$, block upper triangular matrices in G.
- G/P the partial or complete ($P = B$) flag variety.
- $W(G), W(P)$ the Weyl groups with respect to T.

Theorem (Bruhat Decomposition)

$$G = \coprod_{w \in W(G)/W(P)} BwP,$$

$$G/P = \coprod_{w \in W(G)/W(P)} BwP/P.$$

Kazhdan-Lusztig varieties

For $w, v \in W(G)/W(P)$

- $X_w^\circ = BwP/P$ the Schubert cell.
- $X_w = \overline{X_w^\circ}$ the Schubert variety.
- B^- the opposite Borel subgroup, lower triangular matrices in G.
- $X_\circ^v = B^- wP/P$ the opposite Schubert cell, isomorphic to an affine matrix space.
- $X_w \cap X_\circ^v$ the Kazhdan-Lusztig variety.

Type A Zelevinsky map

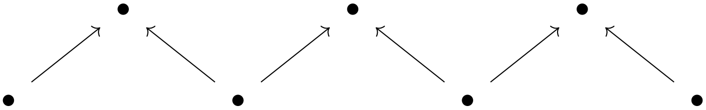
$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

- **Equioriented** Zelevinsky map:
  1985, A. Zelevinsky; 1998, V. Lakshmibai and P. Magyar

$$\overline{\mathcal{O}_V} \cong X_w \cap X_\circ^v.$$

- The same idea also has been applied in many studies on type A quiver loci:
  - Quiver polynomials of equioriented type A quivers (A. Knutson, E. Miller, and M. Shimozono).
  - Type of singularities of type A quiver loci (G. Bobiński and G. Zwara).
  - Maximal singular points of Buchsbaum-Eisenbud varieties, the varieties of complexes (N. Gonciulea).

Type A Zelevinsky map



- **Bipartite** Zelevinsky map:
  2015, R. Kinser and J. Rajchgot.
- For arbitrary orientation, up to a smooth factor Y,

$$\overline{\mathcal{O}_V} \times Y$$

  is isomorphic to an open subvariety of some Kazhdan-Lusztig variety.

Type A Zelevinsky map

Is there a direct isomorphism from type A quiver locus with arbitrary orientation to a Kazhdan-Lusztig variety?

Contents

Rank parameter of type A quiver locus

The rank parameter **r** of a representation V:
(S. Abeasis, A. Der Fra and H. Kraft, for type A)

- An **array of ranks** of 'interval' matrices $M_{[a,b]}(V)$:

$$\mathbf{r} = (\operatorname{rank} M_{[a,b]}(V))_{1 \le a < b \le n},$$

- $W \in \overline{\mathcal{O}_V}$ if and only if

$$\operatorname{rank} M_{[a,b]}(W) \le \operatorname{rank} M_{[a,b]}(V), \forall 1 \le a < b \le n.$$

- We also denote $\mathcal{O}_V = \mathcal{O}_{\mathbf{r}}$.

Rank parameter of type A quiver locus

Now consider a type A quiver Q with

- Vertices $1, 2, \cdots, n$ (1 the leftmost, n the rightmost).
- Left arrows α_i, right arrows β_i.
- For $V \in \text{rep}_Q(\mathbf{d})$, denote $V_{\alpha_i} = A_i$, $V_{\beta_i} = B_i$.

Example

$$Q = 1 \xrightarrow[\beta_1]{} 2 \xrightarrow[\beta_2]{} 3 \xleftarrow[\alpha_1]{} 4 \xleftarrow[\alpha_2]{} 5 \xrightarrow[\beta_3]{} 6 \xleftarrow[\alpha_3]{} 7$$

$$V = V_1 \xrightarrow[\beta_1]{B_1} V_2 \xrightarrow[\beta_2]{B_2} V_3 \xleftarrow[\alpha_1]{A_1} V_4 \xleftarrow[\alpha_2]{A_2} V_5 \xrightarrow[\beta_3]{B_3} V_6 \xleftarrow[\alpha_3]{A_3} V_7$$

Rank parameter of type A quiver locus

- The rank parameter of V is an array of the ranks of interval matrices $M_{[a,b]}(V)$, $1 \leq a < b \leq n$.

Example

$$V = V_1 \xrightarrow[\beta_1]{B_1} V_2 \xrightarrow[\beta_2]{B_2} V_3 \xleftarrow[\alpha_1]{A_1} V_4 \xleftarrow[\alpha_2]{A_2} V_5 \xrightarrow[\beta_3]{B_3} V_6 \xleftarrow[\alpha_3]{A_3} V_7$$

$$M_{[1,2]}(V) = B_1, M_{[2,4]}(V) = [A_1, B_2],$$

$$M_{[1,7]}(V) = \begin{bmatrix} A_3 & B_3 & 0 \\ 0 & A_1A_2 & B_2B_1 \end{bmatrix}.$$

Rank parameter of type A quiver locus

Complete list of the interval matrices $M_{[a,b]}(V), 1 \leq a < b \leq n$:

$$B_1, B_2, A_1, A_2, B_3, A_3, B_2B_1, [A_1, B_2], A_1A_2, \begin{bmatrix} B_3 \\ A_2 \end{bmatrix}, [A_3, B_3],$$

$$[A_1, B_2B_1], [A_1A_2, B_2], \begin{bmatrix} B_3 \\ A_1A_2 \end{bmatrix}, \begin{bmatrix} A_3 & B_3 \\ 0 & A_2 \end{bmatrix},$$

$$[A_1A_2, B_2B_1], \begin{bmatrix} B_3 & 0 \\ A_1A_2 & B_2 \end{bmatrix}, \begin{bmatrix} A_3 & B_3 \\ 0 & A_1A_2 \end{bmatrix}, \begin{bmatrix} B_3 & 0 \\ A_1A_2 & B_2B_1 \end{bmatrix},$$

$$\begin{bmatrix} A_3 & B_3 & 0 \\ 0 & A_1A_2 & B_2 \end{bmatrix}, \begin{bmatrix} A_3 & B_3 & 0 \\ 0 & A_1A_2 & B_2B_1 \end{bmatrix}$$

The (radical) defining ideal $I_{\mathbf{r}}$ of $\overline{\mathcal{O}_{\mathbf{r}}}$ is generated by the minors of $M_{[a,b]}$ (C. Riedtmann and G. Zwara).

The opposite Schubert cell

Depending on Q and $\mathbf{d}$, we consider a Kazhdan-Lusztig variety:

- Let $G = \mathrm{GL}_N$, with $N = \sum_{x \in Q_0} d_x$
- View the matrices in G as block matrices.
- Label the block rows/columns with vertices of Q (in specially designed orders).
- The block row/column has height/width d_x if it is labeled by vertex $x \in Q_0$.

The opposite Schubert cell

- v the index of the opposite Schubert cell.

Example

$$Q = 1 \xrightarrow[\beta_1]{} 2 \xrightarrow[\beta_2]{} 3 \xleftarrow[\alpha_1]{} 4 \xleftarrow[\alpha_2]{} 5 \xrightarrow[\beta_3]{} 6 \xleftarrow[\alpha_3]{} 7$$

$$v = \begin{array}{c} 1 \\ 2 \\ 5 \\ 7 \\ 6 \\ 4 \\ 3 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix} \\ \begin{matrix} 7 & 5 & 4 & 1 & 2 & 3 & 6 \end{matrix} \end{array}.$$

The parabolic P has block sizes as the same as the sizes of the block columns of v.

The opposite Schubert cell

The opposite Schubert cell $X_\circ^v = B^- vP/P$ has generic matrix like

$$Z = \begin{matrix} 1 \\ 2 \\ 5 \\ 7 \\ 6 \\ 4 \\ 3 \\ \end{matrix} \begin{matrix} \begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & * & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & I \\ * & * & I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & I & 0 \end{bmatrix} \\ \begin{matrix} 7 & 5 & 4 & 1 & 2 & 3 & 6 \end{matrix} \end{matrix}.$$

Defining ideal of Kazhdan-Lusztig variety

In this affine space, the (radical) defining ideal I_w of $X_w \cap X_\circ^v$ is generated by the minors of **southwest** submatrices of Z.

$$Z = \begin{array}{c} 1 \\ 2 \\ 5 \\ 7 \\ 6 \\ 4 \\ 3 \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & * & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & I \\ * & * & I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & I & 0 \end{bmatrix} \\ \begin{matrix} 7 & 5 & 4 & 1 & 2 & 3 & 6 \end{matrix} \end{array}.$$

Fulton's essential set:
we only need to consider those southwest submatrices whose boundary coincides with the boundary of the blocks,
if w is in a certain special form (called Z-type, as a pre-def of Zelevinsky permutation).

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$$\begin{array}{ccccccc} 7 & 5 & 4 & 1 & 2 & 3 & 6 \end{array}$$

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$$
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\begin{bmatrix}
0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & * & I & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & * & * & 0 & I \\
* & * & I & 0 & 0 & 0 & 0 \\
* & * & * & * & * & I & 0
\end{bmatrix}.
\quad \begin{matrix} 7 & 5 & 4 & 1 & 2 & 3 & 6 \end{matrix}
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$$\begin{bmatrix} I & 0 & 0 & 0 \\ * & * & 0 & * \\ * & * & I & 0 \\ * & * & * & * \end{bmatrix}$$

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we only need to consider those southwest submatrices whose boundary coincides with the boundary of the blocks,
if w is in a certain special form (called Z-type, as a pre-def of Zelevinsky permutation).

Construction of the map

Ranks	vs.	Ranks
of interval matrices for type A quiver loci		of southwest matrices for Kazhdan-Lusztig varieties

Construction of the map

$$\zeta_Q : \mathrm{rep}_Q(\mathbf{d}) \to X^v_\circ$$

Example

$$V = V_1 \underset{\beta_1}{\xrightarrow{B_1}} V_2 \underset{\beta_2}{\xrightarrow{B_2}} V_3 \underset{\alpha_1}{\xleftarrow{A_1}} V_4 \underset{\alpha_2}{\xleftarrow{A_2}} V_5 \underset{\beta_3}{\xrightarrow{B_3}} V_6 \underset{\alpha_3}{\xleftarrow{A_3}} V_7$$

$$\zeta_Q(V) = \begin{array}{c} 1 \\ 2 \\ 5 \\ 7 \\ 6 \\ 4 \\ 3 \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ A_3 & B_3 & 0 & 0 & 0 & 0 & I \\ 0 & A_2 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_1 & B_2B_1 & B_2 & I & 0 \end{bmatrix} \\ \begin{matrix} 7 & 5 & 4 & 1 & 2 & 3 & 6 \end{matrix} \end{array}$$

Construction of the map

$$\begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ A_3 & B_3 & 0 & 0 & 0 & 0 & I \\ 0 & A_2 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_1 & B_2 B_1 & B_2 & I & 0 \end{bmatrix}$$

$$M_{[1,2]}(V) = B_1, \; M_{[2,4]}(V) = [A_1, B_2],$$

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Construction of the map

$$\begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ A_3 & B_3 & 0 & 0 & 0 & 0 & I \\ 0 & A_2 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_1 & B_2 B_1 & B_2 & I & 0 \end{bmatrix}$$

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Construction of the map

$$\begin{bmatrix} 0 & 0 & 0 & B_1 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ A_3 & B_3 & 0 & 0 \\ 0 & A_2 & I & 0 \\ 0 & 0 & A_1 & B_2 B_1 \end{bmatrix} \xrightarrow[\textit{transformation}]{\textit{elementary}} \begin{bmatrix} 0 & 0 & 0 & B_1 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{[1,2]}(V) = B_1, M_{[2,4]}(V) = [A_1, B_2],$$

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Construction of the map

$$\begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ A_3 & B_3 & 0 & 0 & 0 & 0 & I \\ 0 & A_2 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_1 & B_2 B_1 & B_2 & I & 0 \end{bmatrix}$$

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Construction of the map

$$\begin{bmatrix} 0 & 0 & A_1 & B_2 B_1 & B_2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & A_1 & 0 & B_2 \end{bmatrix}$$

$$M_{[1,2]}(V) = B_1, M_{[2,4]}(V) = [A_1, B_2],$$

$$M_{[1,7]}(V) = \begin{bmatrix} A_3 & B_3 & 0 \\ 0 & A_1 A_2 & B_2 B_1 \end{bmatrix}.$$

Construction of the map

$$\begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ A_3 & B_3 & 0 & 0 & 0 & 0 & I \\ 0 & A_2 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & A_1 & B_2 B_1 & B_2 & I & 0 \end{bmatrix}$$

$$M_{[1,2]}(V) = B_1, M_{[2,4]}(V) = [A_1, B_2],$$

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Construction of the map

$$
\left[\begin{array}{cccc|ccc}
0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & B_1 & I & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
A_3 & B_3 & 0 & 0 & 0 & 0 & I \\
0 & A_2 & I & 0 & 0 & 0 & 0 \\
0 & 0 & A_1 & B_2 B_1 & B_2 & I & 0
\end{array}\right]
$$

$$
M_{[1,2]}(V) = B_1, \; M_{[2,4]}(V) = [A_1, B_2],
$$

$$
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$$

Construction of the map

$$\begin{bmatrix} A_3 & B_3 & 0 & 0 \\ 0 & A_2 & I & 0 \\ 0 & 0 & A_1 & B_2 B_1 \end{bmatrix} \longrightarrow \begin{bmatrix} A_3 & B_3 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & A_1 A_2 & 0 & B_2 B_1 \end{bmatrix}$$

$$M_{[1,2]}(V) = B_1, M_{[2,4]}(V) = [A_1, B_2],$$

$$M_{[1,7]}(V) = \begin{bmatrix} A_3 & B_3 & 0 \\ 0 & A_1 A_2 & B_2 B_1 \end{bmatrix}.$$

Construction of the map

Any rank parameter $\mathbf{r}$ uniquely determines a (Z-type) permutation $w(\mathbf{r})$ (called the Zelevinsky permutation). For the surjective homomorphism of k-algebras induced by ζ_Q:

$$\zeta_Q^* : k[X_\circ^v] \to k[\mathrm{rep}_Q(\mathbf{d})],$$

we can prove that

Theorem

$(\zeta_Q^*)^{-1}(I_{\mathbf{r}}) = I_{w(\mathbf{r})}$. *The restriction of ζ_Q provides an isomorphism*

$$\overline{\mathcal{O}_{\mathbf{r}}} \to X_{w(\mathbf{r})} \cap X_\circ^v.$$

Contents

Applications: immediate corollaries

- Provide new and direct interpretations for some existing results.
- Improve some important proofs.

Corollary (G. Bobiński and G. Zwara)

Type A quiver loci are normal, Cohen-Macaulay, and have rational singularities (when char $k = 0$*).*

Corollary (G. Lusztig)

The intersection cohomology of type A quiver loci vanishes in odd degrees.
(Improve a step in Luszitg's proof of existence of type A canonical bases, in geometric approach)

Applications: geometric vertex decomposability and glicci property

Kazhdan-Lusztig varieties have a nice family of defining equations: the minors of the southwest submatrices form a Gröbner basis of the ideal under a term order. This ideal Gröbner degenerates to a Stanley-Reisner ideal of subword complex (A. Woo and A. Yong) .

Theorem

*The defining ideals of type A quiver loci are geometrically vertex decomposable for arbitrary orientation. Moreover, type A quiver loci are glicci (in the **G**orenstein **li**aison **c**lass of a **c**omplete **i**ntersection) in* $\mathrm{rep}_Q(\mathbf{d})$.

Applications: Krull-Schmidt decomposition

- I_{pq} the indecomposable representations of Q, $1 \leq p \leq q \leq n$.

Example

$V = \bigoplus m_{pq} I_{pq}$, then the numbers of non-zero entries in the blocks of $w(\mathbf{r})$ are

$$\begin{array}{c} 1 \\ 2 \\ 5 \\ 7 \\ 6 \\ 4 \\ 3 \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & m_{11} & n_{12} & 0 & 0 \\ 0 & 0 & 0 & m_{12} & m_{22} & n_{23} & 0 \\ 0 & m_{55} & m_{45} & m_{15} & m_{25} & m_{35} & n_{56} \\ m_{77} & m_{57} & m_{47} & m_{17} & m_{27} & m_{37} & m_{67} \\ n_{67} & m_{56} & m_{46} & m_{16} & m_{26} & m_{36} & m_{66} \\ 0 & n_{45} & m_{44} & m_{14} & m_{24} & m_{34} & 0 \\ 0 & 0 & n_{34} & m_{13} & m_{23} & m_{33} & 0 \end{bmatrix} \\ \begin{matrix} 7 & 5 & 4 & 1 & 2 & 3 & 6 \end{matrix} \end{array}$$

where $n_{xy} = \sum_{p \leq x \leq y \leq q} m_{pq}$.

Applications: the maximal singular points

Working on this 'multiplicity matrix', it allows us to compute the maximal singular points (i.e., the irreducible components of the singular loci) of any type A quiver loci.

- Find the 'bad' patterns of the positive entries in the multiplicity matrix.
- 'Bad' patterns: (3412) and (4231) quadruples with some reduced conditions.

Example

$$Q = 1 \xleftarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 3 \xrightarrow{\beta_1} 4 \xleftarrow{\alpha_3} 5 \ .$$

Let $V = 3I_{14} \oplus 2I_{15} \oplus I_{25} \oplus I_{55} \in \mathrm{rep}_Q(\mathbf{d})$ *with rank parameter* $\mathbf{r}$.

Applications: the maximal singular points

The multiplicity matrix of $w(\mathbf{r})$ is

$$\begin{matrix} 3 \\ 5 \\ 4 \\ 2 \\ 1 \\ \end{matrix} \begin{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 6 \\ 1 & 0 & 1 & 2 & 0 \\ 3 & 0 & 0 & 3 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \end{bmatrix} \\ \begin{matrix} 5 & 3 & 2 & 1 & 4 \end{matrix} \end{matrix}$$

Applications: the maximal singular points

The (4231) quadruple is

$$\begin{array}{c} 3 \\ 5 \\ 4 \\ 2 \\ 1 \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 & + \\ \boxed{+} & 0 & + & \boxed{+} & 0 \\ \boxed{+} & 0 & 0 & \boxed{+} & 0 \\ 0 & + & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 \end{bmatrix} \\ \begin{matrix} 5 & 3 & 2 & 1 & 4 \end{matrix} \end{array}$$

Applications: the maximal singular points

The (3412) quadruples are

$$\begin{matrix} 3 \\ 5 \\ 4 \\ 2 \\ 1 \\ \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & + \\ + & 0 & \boxed{+} & + & 0 \\ \boxed{+} & 0 & 0 & \boxed{+} & 0 \\ 0 & \boxed{+} & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 \end{bmatrix} \text{ and } \begin{matrix} 3 \\ 5 \\ 4 \\ 2 \\ 1 \\ \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & + \\ + & 0 & \boxed{+} & + & 0 \\ \boxed{+} & 0 & 0 & \boxed{+} & 0 \\ 0 & + & 0 & 0 & 0 \\ 0 & 0 & \boxed{+} & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} 5 & 3 & 2 & 1 & 4 \end{matrix} \qquad \begin{matrix} 5 & 3 & 2 & 1 & 4 \end{matrix}$$

Applications: the maximal singular points

All the irreducible components of the singular locus of $\overline{\mathcal{O}_V}$ are $\overline{\mathcal{O}_{W'}}, \overline{\mathcal{O}_{W''}}, \overline{\mathcal{O}_{W'''}}$, where

$$W' = 4I_{14} \oplus I_{15} \oplus I_{25} \oplus 2I_{55},$$
$$W'' = I_{12} \oplus 2I_{14} \oplus 2I_{15} \oplus I_{24} \oplus I_{34} \oplus 2I_{55},$$
$$W''' = I_{11} \oplus 2I_{14} \oplus 2I_{15} \oplus 2I_{24} \oplus 2I_{55}.$$

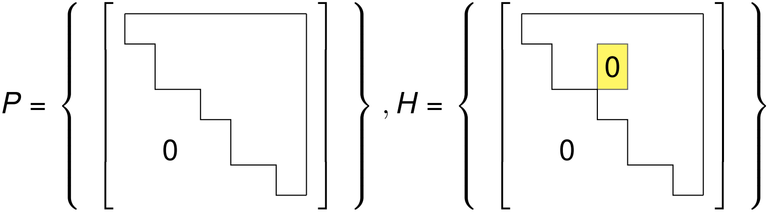
Contents

The generalization to type D

Motivation:
Achieving this concept for two types in almost the same ways!

The generalization to type D

- $G = \mathrm{GL}_N$.
- P the parabolic subgroup, block upper triangular matrices in G.
- H the subgroup of P with an additional condition: a certain block in the block superdiagonal is always zero.



The generalization to type D

- $G = \mathrm{GL}_N$.
- P the parabolic subgroup, block upper triangular matrices in G.
- H the subgroup of P with an additional condition: a certain block in the block superdiagonal is always zero.
- G/H is a spherical variety, the B-orbit closures are multiplicity-free.

$$G/P = \{0 \subset V_{(1)} \subset V_{(2)} \subset V_{(3)} \subset V_{(4)} \subset V_{(5)} = k^N | \ldots\}$$

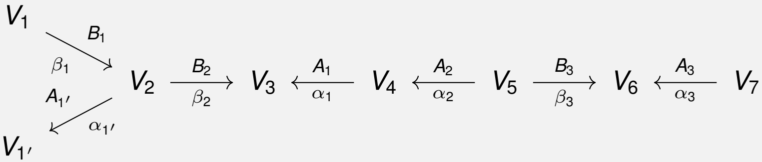
$$G/H = \{0 \subset V_{(1)} \begin{array}{c} \subset \\ \subset \end{array} \begin{array}{c} V_{(2)} \\ V_{(3)} \end{array} \begin{array}{c} \subset \\ \subset \end{array} V_{(4)} \subset V_{(5)} = k^N | \\ (V_{(2)}/V_{(1)}) \cap (V_{(3)}/V_{(1)}) = 0, \ldots\}$$

The generalization to type D

- $G = \mathrm{GL}_N$.
- P the parabolic subgroup, block upper triangular matrices in G.
- H the subgroup of P with an additional condition: a certain block in the block superdiagonal is always zero.
- G/H is a spherical variety, the B-orbit closures are multiplicity-free.
- These B-orbit closures are normal, Cohen-Macaulay, and have rational singularities (when $chark = 0$). The same holds for the generalization of the Kazhdan-Lusztig varieties we are considering.

The generalization to type D

$$\zeta_Q : \mathrm{rep}_Q(\mathbf{d}) \to B^- vH/H.$$



$$\zeta_Q(V) = \begin{array}{c} 1 \\ 2 \\ 5 \\ 7 \\ 6 \\ 4 \\ 3 \\ 1' \\ \end{array} \begin{array}{c} \left[\begin{array}{cccccccc} 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_3 & B_3 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & A_2 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_1 & B_2B_1 & B_2 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & A_{1'} & I & 0 & 0 \end{array}\right] \\ \begin{array}{cccccccc} 7 & 5 & 4 & 1 & 2 & 1' & 3 & 6 \end{array} \end{array}.$$

The generalization to type D

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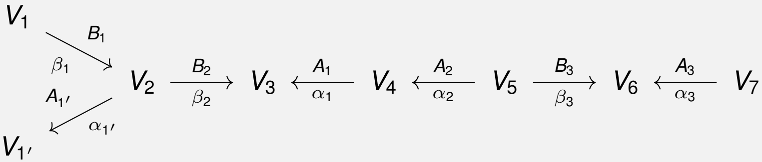

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The generalization to type D

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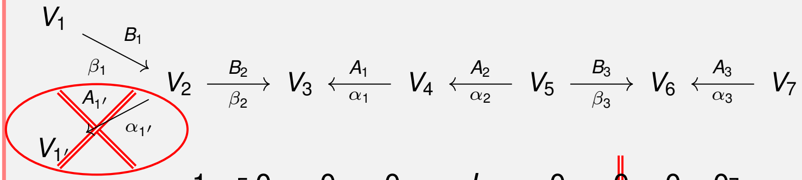

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The generalization to type D

And type D rank parameter uniquely determines a (specially formed) index matrix $u(\mathbf{r})$.

- $(\zeta_Q^*)^{-1}(I_{\mathbf{r}}) = I_{u(\mathbf{r})}$.
- $(\zeta_Q^*)^{-1}(\sqrt{I_{\mathbf{r}}}) = \sqrt{I_{u(\mathbf{r})}}$.
- $\overline{\mathcal{O}_{\mathbf{r}}} \cong \overline{Bu(\mathbf{r})H/H} \cap B^- vH/H$.

Further problems

- $I_{\mathbf{r}} = \sqrt{I_{\mathbf{r}}}$ (for type D)?
- Is there such an isomorphism for types E_6, E_7, E_8? Which target variety should be chosen?
- Is there such an isomorphism for extended Dynkin types and affine flag varieties? (G. Lusztig: cyclic type A)
- Could it be achieved by other classical groups G?
- Better interpretation in higher levels ...

That's all

Thanks !