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Auslander–Reiten sequences in minimal A_∞-structures of the module category of a representation directed algebra

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Representation finite algebra

Λ finite dimensional algebra over a field k

Theorem [Kerner–Skowronski 1991, Bongartz, Auslander]

Λ is representation finite $\iff$ $\mathrm{rad}^{\infty}(\Lambda\text{-}\mathrm{mod}) = 0$

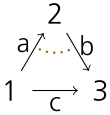
$\Rightarrow$ All morphisms can be constructed from the irreducible ones

Theorem [Butler 1981, Auslander 1984]

Λ is representation finite $\iff$ the relations in $K_0(\Lambda\text{-}\mathrm{mod})$ are generated by the AR-sequences

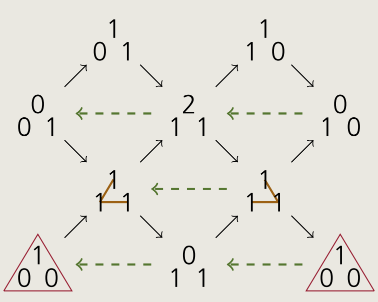
$\Rightarrow$ Can the other short exact sequences be constructed from the AR-sequences?

Approximating short exact sequences: An example

$Q =$ 

and $\Lambda = kQ/(ba)$

$$0 \longrightarrow \begin{smallmatrix} & 1 & \\ 0 & & 0 \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & & 0 \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 0 & \\ 1 & & 0 \end{smallmatrix} \longrightarrow 0$$



Auslander–Reiten quiver of Λ

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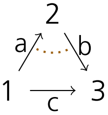
$Q =$ 

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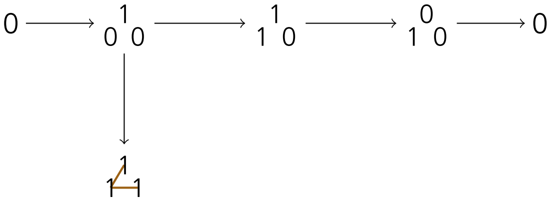

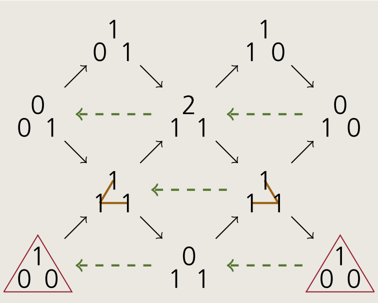

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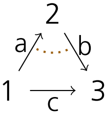
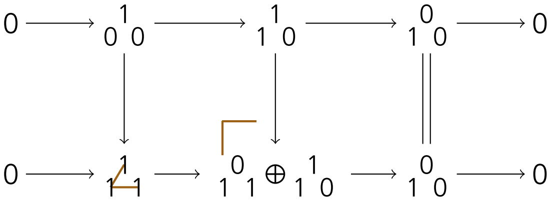
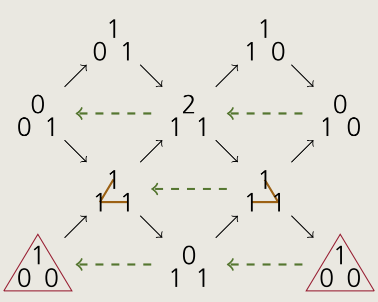
$Q =$ 

and $\Lambda = kQ/(ba)$





Auslander–Reiten quiver of Λ

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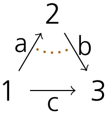
$Q =$ 

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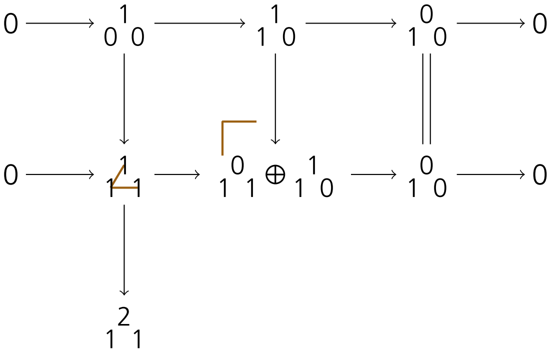
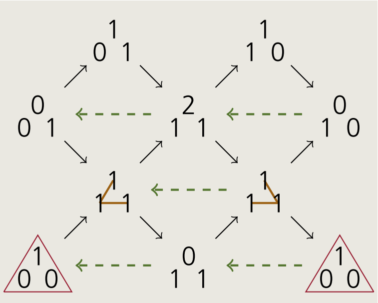
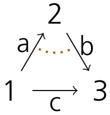




Auslander–Reiten quiver of Λ

Approximating short exact sequences: An example

$Q =$ 

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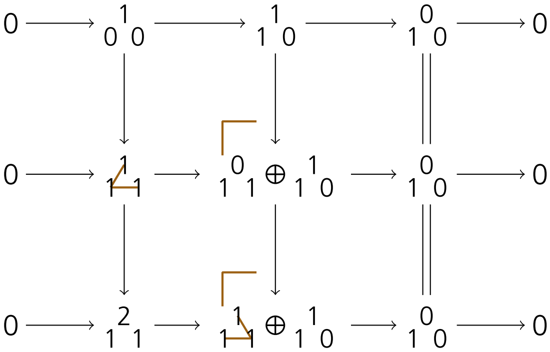

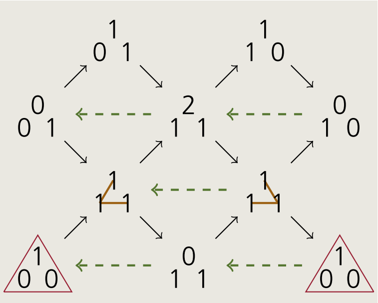

Auslander–Reiten quiver of Λ

A problem: The Yoneda product and AR-sequences

Let M be a basic additive generator for Λ-mod

- $\Rightarrow$ $\operatorname{Ext}^*_\Lambda(M,M)$ encodes all exact sequences in Λ-mod
- $\Rightarrow$ Products of s.e.s. with morphisms are computed via pushout/pullback

$$
\begin{array}{lccccccccc}
\eta: & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & \tau^- X & \longrightarrow & 0 \\
 & & & \Big\downarrow {\scriptstyle \operatorname{rad}(X,X') \ni f} & & \Big\downarrow & & \Big\| & & \\
f\eta: & 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & \tau^- X & \longrightarrow & 0
\end{array}
$$

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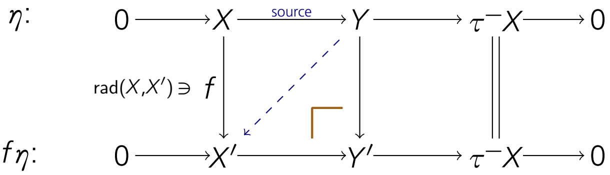

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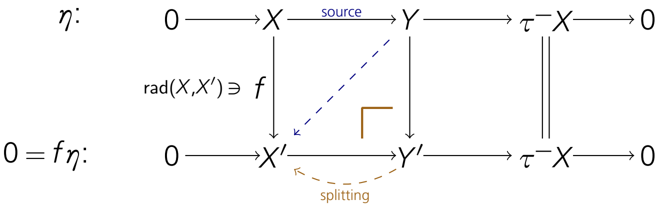


The AR sequences together with the irreducible morphisms do **not** generate $\mathrm{Ext}^*_A(M,M)$ as graded algebra in general

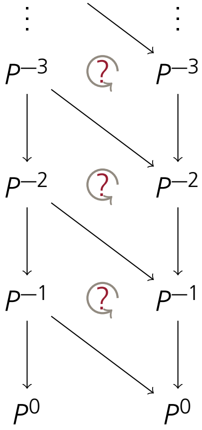
Change of perspective

Let $P^\bullet$ be a projective resolution of M

- dg endomorphism algebra $\mathscr{H}\mathrm{om}^*_\Lambda(P^\bullet,P^\bullet)$
  - in degree $r \in \mathbb{Z}$ have $(f^n \colon P^n \to P^{n+r})_{n\in\mathbb{Z}}$, no comparability to the differential
  - differential of degree r morphisms f is given by

$$\partial(f) := d_P \circ f - (-1)^r f \circ d_P$$

- $\mathrm{Ext}^*_\Lambda(M,M) = H^*(\mathscr{H}\mathrm{om}^*_\Lambda(P^\bullet,P^\bullet))$
- Not necessarily quasi isomorphic as dg algebras



Homotopy Transfer Theorem [Kadeishvili 1982]

The cohomology of a dg algebra carries an induced A_∞-algebra structure A_∞-quasi isomorphic to the original dg algebra

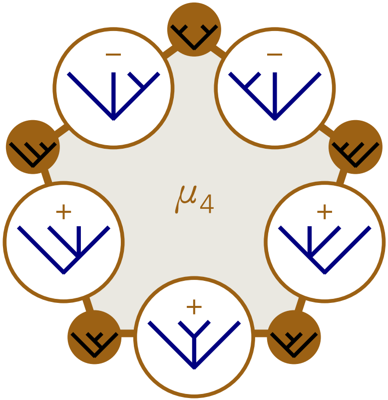
Higher multiplications in A_∞-algebras

An A_∞-algebra is a dg vector space $E = \coprod_{i\in\mathbb{Z}} E^i$ with differential $d^i : E^i \to E^{i+1}$ equipped with

$$\mu_n : E^{\otimes n} \to E, \quad 2 \leq n \in \mathbb{N}$$

with $\partial(\mu_n)$ a coherence of lower multiplications.

- μ_n of degree $2-n$
- $\partial(\mu_2) = 0$, i.e. μ_2 satisfies graded Leibniz rule
- $\partial(\mu_3) = \mu_2(1 \otimes \mu_2 - \mu_2 \otimes 1)$ (non-)associativity
- $\partial(\mu_4)$ is "boundary of the associahedron K_4"
- ...



Towards a solution

Consider the following pushout diagram in Λ-mod.

$$
\begin{array}{llllllllll}
t: & 0 & \longrightarrow & X & \xrightarrow{j} & Y & \xrightarrow{p} & Z & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \| & & \\
s: & 0 & \longrightarrow & X' & \xrightarrow[j']{} & Y' & \xrightarrow[p']{} & Z & \longrightarrow & 0
\end{array}
$$

Furthermore, let $r \in \mathrm{Ext}^1_\Lambda(Y', X)$ represent $0 \to X \to X' \oplus Y \to Y' \to 0$.

Proposition [T.]

In any minimal model of $\mathscr{H}\mathrm{om}^*_\Lambda(P^\bullet, P^\bullet)$ there exist $f \in \mathrm{Hom}_\Lambda(Z, Y')$ and $g \in \mathrm{Hom}_\Lambda(X', X)$ s.t.

$$\mu_3(r, j', s) - \mu_2(g, s) - \mu_2(r, f)$$

is the class of the original short exact sequence t.

Proof idea: A_∞-triangles as per [Kontsevich]

Main result

Let Λ be a representation finite f.d. algebra with M a basic additive generator of Λ-mod which has projective resolution $P^\bullet$.

Theorem [T.]

Let E be a minimal model of $\mathscr{H}\mathrm{om}^*_\Lambda(P^\bullet, P^\bullet)$, i.e. $E \underset{\text{gr vsp}}{\cong} H^*(\mathscr{H}\mathrm{om}^*_\Lambda(P^\bullet, P^\bullet)) = \mathrm{Ext}^*_\Lambda(M, M)$.

Then the smallest A_∞-subalgebra $\tilde{E}$ of E containing

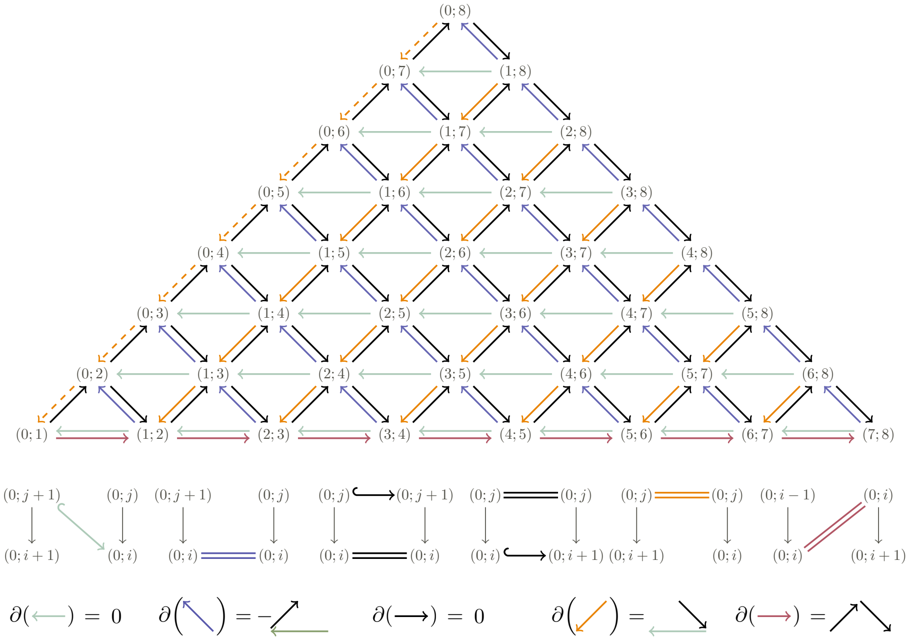
- the automorphisms of each indecomposable direct summand of M,
- the irreducible morphisms between indecomposable direct summands of M and
- representatives of each Auslander–Reiten sequence in Λ-mod

will already contain $\mathrm{Ext}^1_\Lambda(M, M)$. In particular, $\tilde{E} = E$.

Another example: linearly oriented A_N

$$1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow N-1 \leftarrow N$$

$\Rightarrow$ Have to compute $\mathscr{H}\mathrm{om}^*_{A_N}(P_\bullet, P_\bullet)$



(0; 8)
(0; 7) (1; 8)
(0; 6) (1; 7) (2; 8)
(0; 5) (1; 6) (2; 7) (3; 8)
(0; 4) (1; 5) (2; 6) (3; 7) (4; 8)
(0; 3) (1; 4) (2; 5) (3; 6) (4; 7) (5; 8)
(0; 2) (1; 3) (2; 4) (3; 5) (4; 6) (5; 7) (6; 8)
(0; 1) (1; 2) (2; 3) (3; 4) (4; 5) (5; 6) (6; 7) (7; 8)
(0; j + 1) (0; j) (0; i + 1) (0; i)
(0; j + 1) (0; j) (0; i) (0; i)
(0; j) (0; j + 1) (0; i) (0; i)
(0; j) (0; j) (0; i) (0; i + 1)
(0; j) (0; j) (0; i + 1) (0; i)
(0; i − 1) (0; i) (0; i) (0; i + 1)
∂( ) = 0
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∂( ) = 0
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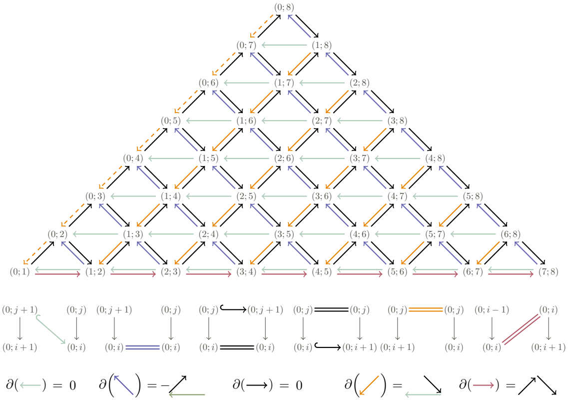
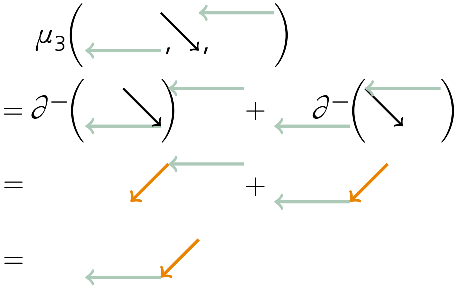
Another example: linearly oriented A_N



Homotopy Transfer Theorem

Higher multiplications are given by "inverting differential after multiplying"



μ_3 encodes pasting of bicartesian squares for rep. directed algebras

A_∞-triangles

$$\left\{\text{s.e.s. in } \Lambda\text{-mod}\right\} \overset{1:1}{\longleftrightarrow} \left\{\text{exact functors } S\colon \mathrm{rep}(A_2) \to \Lambda\text{-mod}\right\}$$

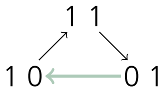
Following this idea, we can write accoding to [Kontsevich]

$$\left\{ A_\infty\text{-triangles in } \left(\Lambda\text{-mod}\right)_{A_\infty} \right\} \overset{1:1}{\longleftrightarrow} \left\{ A_\infty\text{-functors } T\colon \left(\mathrm{rep}(A_2)\right)_{A_\infty} \to \left(\Lambda\text{-mod}\right)_{A_\infty} \right\}$$

Proposition

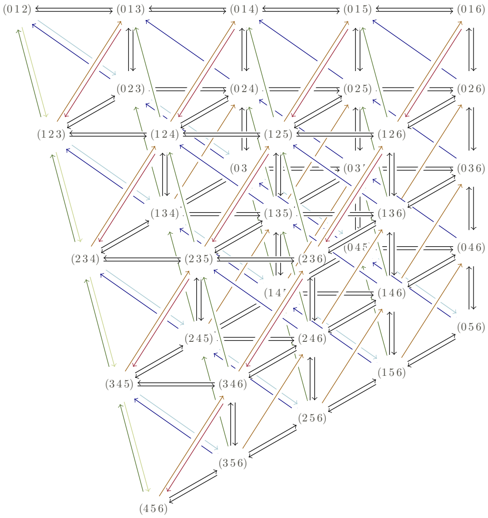
Any A_∞-triangle gives rise to long exact sequences via $\mathrm{Hom}_\Lambda(X,—)$ and $\mathrm{Hom}_\Lambda(—,Y)$. Furthermore, lifts in these sequences can be computed via an explicit formula.

Key ingredient: The s.e.s. in $\mathrm{rep}(A_2)$ give a "3-way isomorphism" in $\left(\mathrm{rep}(A_2)\right)_{A_\infty}$



Future research

- Is this a **characterisation** of finite representation type?
- **Generalize** to algebraic extriangulated categories with $\mathrm{rad}^{\infty} = 0$
- Use generative properties of the AR-sequences to better **describe** the A_∞-structure of $\mathrm{Ext}^*_\Lambda(M,M)$
- **Generalize** to higher Auslander–Reiten theory with A_∞-d-angles



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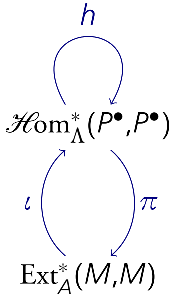
Details on the Homotopy Transfer Theorem



$\pi\iota = \mathrm{id}$

$\partial(h) = \iota\pi - \mathrm{id}$

The Ext-algebra $\mathrm{Ext}^*_A(M,M)$ of any A-module M carries an A_∞-algebra structure induced by the dg algebra $\mathscr{H}om^*_\Lambda(P^\bullet, P^\bullet)$, $P^\bullet$ a projective resolution of M. Recursively define

$$\mu'_n := \sum_{k=1}^{n-1} (-1)^{k-1} m \circ (h\mu'_k \otimes h\mu'_{n-k})$$

for $2 \leq n \in \mathbb{N}$ with $h\mu'_1 := \mathrm{id}$. Then, the higher multiplications are given by

$$\mu_n = \pi \circ \mu'_n \circ \iota^{\otimes n}.$$