Specialization map for quiver Grassmannians

G. Cerulli Irelli[1] F. Esposito[2] X. Fang[3] G. Fourier[3]

[1]Sapienza-Università di Roma (ITALY).

[2]University of Padova (ITALY).

[3]RWTH Aachen University (Germany).

ICRA XXI, Shanghai, 6th Aug. 2024

Motivation

- $Q = (Q_0, Q_1, s, t)$: an acyclic quiver.
- $\mathbf{e}, \mathbf{d} \in \mathbb{Z}^{Q_0}$: two dimension vectors.
- $R_{\mathbf{d}}(Q) = \prod_{\alpha \in Q_1} \mathrm{Hom}(\mathbb{C}^{\mathbf{d}_{s(\alpha)}}, \mathbb{C}^{\mathbf{d}_{t(\alpha)}})$
- $M \in R_{\mathbf{d}}(Q)$: a Q-representation of dimension vector $\mathbf{d}$
- $\mathrm{Gr}_{\mathbf{e}}(M) = \{N \subseteq M | \mathbf{dim}\, N = \mathbf{e}\}$: a quiver Grassmannian
- $\iota_M : \mathrm{Gr}_{\mathbf{e}}(M) \hookrightarrow \prod_{i \in Q_0} \mathrm{Gr}_{\mathbf{e}_i}(\mathbb{C}^{\mathbf{d}_i}) = \mathrm{Gr}_{\mathbf{e}}(0_{\mathbf{d}})$
- $\iota_M^* : \mathrm{H}^\bullet(\prod_{i \in Q_0} \mathrm{Gr}_{\mathbf{e}_i}(M_i)) \to \mathrm{H}^\bullet(\mathrm{Gr}_{\mathbf{e}}(M))$

Theorem (CI- Esposito-Franzen-Reineke, 2021)

$$M \textit{ rigid} \implies \iota_M^* \textit{ surjective}$$

Aim

Given $x, y \in R_{\mathbf{d}}(Q)$ such that $y \in \overline{Gx}$ we want to define a map

$$c_{y,x} : H^\bullet(\mathrm{Gr}_{\mathbf{e}}(y)) \to H^\bullet(\mathrm{Gr}_{\mathbf{e}}(x))$$

with favourable properties, the most important being that if $y = 0_{\mathbf{d}}$ then

$$c_{0_{\mathbf{d}},x} = \iota_M^* : H^\bullet(\mathrm{Gr}_{\mathbf{e}}(0_{\mathbf{d}})) \to H^\bullet(\mathrm{Gr}_{\mathbf{e}}(x))$$

is induced by the inclusion

$$\iota_M : \mathrm{Gr}_{\mathbf{e}}(x) \hookrightarrow \prod \mathrm{Gr}_{\mathbf{e}_i}(\mathbb{C}^{\mathbf{d}_i}) = \mathrm{Gr}_{\mathbf{e}}(0_{\mathbf{d}}) \, .$$

SAPIENZA
UNIVERSITÀ DI ROMA

Setting

- $Q = (Q_0, Q_1, s, t)$: a *Dynkin* quiver
- $\mathbf{e}, \mathbf{d} \in \mathbb{Z}^{Q_0}$: two dimension vector
- $Y = \prod_{\alpha \in Q_1} \mathbb{C}^{\mathbf{d}_{t(\alpha)} \times \mathbf{d}_{s(\alpha)}}$: a representation variety.
- $G = \prod_{i \in Q_0} \mathrm{GL}(\mathbf{d}_i, \mathbb{C})$: the structure group of Y.
- $G \times Y \to Y$: change of basis action of G on Y.
- $X = \{(U = (U_i)_{i \in Q_0}, y = (y_\alpha)_{\alpha \in Q_1}) \in \prod \mathrm{Gr}_{\mathbf{e}_i}(\mathbb{C}^{\mathbf{d}_i}) \times Y | \, y_\alpha(U_{s(\alpha)}) \subset U_{t(\alpha)}\}$
- $\pi : X \to Y : (U, y) \mapsto y$
- $\pi^{-1}(y) = \mathrm{Gr}_{\mathbf{e}}(y)$: quiver Grassmannian.
- G acts on X and Y and π is G-equivariant.

SAPIENZA
UNIVERSITÀ DI ROMA

Geometric setting

- G: a reductive group.
- X, Y: G-varieties.
- In Y there are a finite number of G-orbits.
- Every orbit closures $\overline{\mathcal{O}}$ in Y is unibranch.
- The G-stabilizer of each point of Y is connected.
- $\pi : X \to Y$ is a proper and G-equivariant.

Definition

The quadruple (G, X, Y, π) is a *geometric setting* if it satisfied the properties above.

Main Theorem part 1

Theorem

In a geometric setting (G, X, Y, π) *for every G-orbit* $\mathcal{O} \subset Y$ *there is a canonical graded algebra* $H^\bullet(X_{[\mathcal{O}]})$ *such that for every* $y \in \mathcal{O}$ *there exists a canonical isomorphism* $H^\bullet(\pi^{-1}(y)) \overset{\simeq}{\to} H^\bullet(X_{[\mathcal{O}]})$

Proof: Define

$$H^\bullet(X_{[\mathcal{O}]}) = \varprojlim_{y_1, y_2 \in \mathcal{O}} \left(\varphi_{y_1, y_2} : H^\bullet(\pi^{-1}(y_1)) \to H^\bullet(\pi^{-1}(y_2)) \right).$$

SAPIENZA
UNIVERSITÀ DI ROMA

Nice little neighborhoods

Definition

An open neighborhood U of a point $y \in Y$ is called a *nice little neighborhood* of y if it satisfies the following two conditions:

(i) U is contractible;

(ii) The map in cohomology $\psi : H^\bullet(\pi^{-1}(U)) \to H^\bullet(\pi^{-1}(y))$ induced by the inclusion $\{y\} \subset U$ is an isomorphism.

SAPIENZA UNIVERSITÀ DI ROMA

Main Theorem part 2

Theorem

Suppose that the datum (G, X, Y, π) is a geometric setting. Then for every two G-orbits $\mathcal{O}_1 \subset \overline{\mathcal{O}_2} \subset Y$ there is a canonical specialization map $c_{[\mathcal{O}_1],[\mathcal{O}_2]} : H^\bullet(X_{[\mathcal{O}_1]}) \to H^\bullet(X_{[\mathcal{O}_2]})$.

Proof: Choose $y_1 \in \mathcal{O}_1$, then a nice little neighborhood U of y_1 and then $y_2 \in U \cap \mathcal{O}_2$. Define c_{y_1,U,y_2} as the composite

$$\begin{array}{ccccc} H^\bullet(X_{[\mathcal{O}_1]}) & \xrightarrow{\simeq} & H^\bullet(\pi^{-1}(y_1)) & \xrightarrow{\simeq} & H^\bullet(\pi^{-1}(U)) \\ \downarrow{\scriptstyle c_{y_1,U,y_2}} & & & & \downarrow \\ H^\bullet(X_{[\mathcal{O}_2]}) & \xleftarrow{\simeq} & H^\bullet(\pi^{-1}(y_2)) & = & H^\bullet(\pi^{-1}(y_2)) \end{array}$$

Need to show that c_{y_1,U,y_2} depends only on $\mathcal{O}_1$ and $\mathcal{O}_2$.

Specialization map for quiver Grassmannians

Theorem

Let Q be a Dynkin quiver and let $[M]$ and $[N]$ be two isomorphism classes of Q-representations of the same dimension vector $\mathbf{d}$ such that $M \leq_{deg} N$.

(i) *There are well-defined cohomology algebras $H^\bullet(\mathrm{Gr}_{\mathbf{e}}([M]))$ and $H^\bullet(\mathrm{Gr}_{\mathbf{e}}([N]))$.*

(ii) *There is a well-defined map of graded algebras $c_{[N],[M]} : H^\bullet(\mathrm{Gr}_{\mathbf{e}}([N])) \to H^\bullet(\mathrm{Gr}_{\mathbf{e}}([M]))$ such that $c_{[0_{\mathbf{d}}],[M]} = \iota_M^*$.*

(iii) *If Q is of type A then $c_{[N],[M]}$ is surjective.*

SAPIENZA UNIVERSITÀ DI ROMA

arXiv: 2206.10281