On F-polynomials for generalized quantum cluster algebras and Gupta's formula

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Outline

Generalized quantum cluster algebra

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- A *compatible pair* $(\tilde{B}, \Lambda)$ consists of an integer $m \times n$-matrix $\tilde{B}$ and a skew-symmetric integer $m \times m$-matrix Λ such that

$$\tilde{B}^T \Lambda = [D\ 0],$$

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- The *quantum torus* $\mathcal{T}_\Lambda$ associated with Λ is the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra generated by the distinguished $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis $\{\mathbf{X}(\alpha) | \alpha \in \mathbb{Z}^m\}$ with multiplication given by

$$\mathbf{X}(\alpha)\mathbf{X}(\beta) = q^{\frac{1}{2}\alpha^T \Lambda \beta}\mathbf{X}(\alpha + \beta)$$

  for any $\alpha, \beta \in \mathbb{Z}^m$. Let $\mathcal{F}_\Lambda$ be skew field of fractions of $\mathcal{T}_\Lambda$.

- The *mutation data* $(R, \mathbf{h})$, where $R = diag\{r_1, \cdots, r_n\}$ is a diagonal n $\times$ n matrix whose diagonal coefficients are positive integers and $\mathbf{h} = (\mathbf{h}_1; \cdots; \mathbf{h}_n)$, $\mathbf{h}_k := \{h_{k,0}(q^{\frac{1}{2}}), h_{k,1}(q^{\frac{1}{2}}), ..., h_{k,r_k}(q^{\frac{1}{2}})\}$, where $h_{k,i}(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ satisfying $h_{k,i}(q^{\frac{1}{2}}) = h_{k,r_{k-i}}(q^{\frac{1}{2}})$ and $h_{k,0}(q^{\frac{1}{2}}) = h_{k,r_k}(q^{\frac{1}{2}}) = 1$.

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- A $(R, \mathbf{h})$-*quantum seed* is a triple $\Sigma = (\mathrm{X}, \tilde{B}, \Lambda)$, where $(\tilde{B}, \Lambda)$ is a compatible pair and $\mathrm{X} = (X_1, \ldots, X_m)$ is an m-tuple of elements of $\mathcal{F}_\Lambda$ such that
    - $X_1, \ldots, X_m$ generated $\mathcal{F}_\Lambda$ over $\mathbb{Q}(q^{\frac{1}{2}})$;
    - $X_iX_j = q^{\lambda_{ij}}X_jX_i$, where $\Lambda = (\lambda_{ij})$

We define $X(\alpha) := q^{\frac{1}{2}\sum_{i<j}\lambda_{ij}}X_1^{a_1}\cdots X_m^{a_m}$, where $\alpha = (a_1, \ldots, a_m)^T \in \mathbb{Z}^m$.

- Let $k \in [1,n] := \{1,2,\cdots,n\}$.

$$E_{k,\varepsilon}^{\tilde{B}R} := \begin{bmatrix} 1 & 0 & \cdots & [-\varepsilon b_{1k}r_k]_+ & \cdots & 0 \\ 0 & 1 & \cdots & [-\varepsilon b_{2k}r_k]_+ & \cdots & 0 \\ & & & \vdots & & \\ 0 & 0 & \cdots & -1 & \cdots & 0 \\ & & & \vdots & & \\ 0 & 0 & \cdots & [-\varepsilon b_{m-1k}r_k]_+ & \cdots & 0 \\ 0 & 0 & \cdots & [-\varepsilon b_{mk}r_k]_+ & \cdots & 1 \end{bmatrix},$$

$$F_{k,\varepsilon}^{R\tilde{B}} := \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ & & & \vdots & & \\ [\varepsilon r_k b_{k1}]_+ & [\varepsilon r_k b_{k2}]_+ & \cdots & -1 & \cdots & [\varepsilon r_k b_{kn}]_+ \\ & & & \vdots & & \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

where $\varepsilon \in \{1,-1\}$ and $\tilde{B} = (b_{ij})$.

- The *mutation* μ_k *in direction* k transforms the compatible pair $(\tilde{B}, \Lambda)$ into $\mu_k(\tilde{B},\ \Lambda) := (\tilde{B}', \Lambda')$, where

$$\tilde{B}' = E_{k,\varepsilon}^{\tilde{B}R} \tilde{B} F_{k,\varepsilon}^{R\tilde{B}},\ \Lambda' = (E_{k,\varepsilon}^{\tilde{B}R})^T \Lambda E_{k,\varepsilon}^{\tilde{B}R}$$

transforms the quantum cluster $\mathrm{X} = (X_1, \cdots, X_m)$ into $\mu_k(X) = \mathrm{X}' = (X_1', \ldots, X_m')$ is given by

$$X_i' := \mathrm{X}'(e_i) = \begin{cases} \mathrm{X}(e_i) & \text{if } i \neq k; \\ \sum_{s=0}^{r_k} h_{k,s}(q^{\frac{1}{2}}) \mathrm{X}(s[\varepsilon \mathrm{b}_k]_+ + (r_k - s)[-\varepsilon \mathrm{b}_k]_+ - e_i) & \text{if } i = k, \end{cases}$$

where b_k is the k-th column vector of $\tilde{B}$ and $\varepsilon \in \{\pm 1\}$, $e_1, \cdots, e_m$ is the standard basis of $\mathbb{Z}^m$. In fact, μ_k is an isomorphism $\mathcal{F}_\Lambda \to \mathcal{F}_{\Lambda'}$. Moreover the mutation μ_k is an involution. And $(\mathrm{X}', \tilde{B}', \Lambda')$ is also a A $(R, \mathbf{h})$-*quantum seed* in $\mathcal{F}_{\Lambda'}$.

- Let $\mathbb{T}_n$ be a n-regular tree. Fix a root vertex $t_0 \in \mathbb{T}_n$, $\Sigma_{t_0} = (\mathrm{X}, \tilde{B}, \Lambda)$ the initial $(R, \mathbf{h})$-quantum seed. We assign each vertex $t \in \mathbb{T}_n$ an $(R, \mathbf{h})$-quantum seed Σ_t which can be obtained from Σ_{t_0} by iterated mutations such that if $t \overset{k}{\longrightarrow} t'$, then $\Sigma_{t'} = \mu_k(\Sigma_t)$. We call such an assignment $t \to \Sigma_t$ an $(R, \mathbf{h})$-quantum seed pattern.

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- For each vertex t, we refer to $\mathrm{X}_t = (X_{1;t}, \cdots, X_{m;t})$ a quantum cluster, $X_{i;t}(1 \leq i \leq n)$ quantum cluster variables and $X_{n+i;t}(1 \leq i \leq m-n)$ coefficients.

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- Let $\mathcal{X}_q := \{X_{i;t|1 \leq i \leq n}\}$, the $(R, \mathbf{h})$-quantum cluster algebra $\mathcal{A}_q(\Sigma_{t_0})$ is the $\mathbb{Z}[q^{\pm\frac{1}{2}}][X_{n+1}^{\pm 1}, ..., X_m^{\pm 1}]$ subalgebra of $\mathcal{F}_q := \mathcal{F}_{\Lambda_{t_0}}$ generated by elements of $\mathcal{X}_q$.

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$$X_{i,t} \in \mathbb{Z}[q^{\pm\frac{1}{2}}][X_1^{\pm 1}, \cdots, X_m^{\pm 1}],$$

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- We aim to prove the existence of F-polynomial and establish the separation formula for generalized quantum cluster algebras. Moreover we will give a computing method for F-polynomial using c-vectors and g-vectors.

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Notation:

- For the edge $t \overset{k}{\longrightarrow} t'$ in $\mathbb{T}_n$. The mutation μ_k in direction *k* yields a unique $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra isomorphism $\mu_{k;t} : \mathcal{F}_{\Lambda_{t'}} \to \mathcal{F}_{\Lambda_t}$ such that

$$\mu_{k;t}(\mathrm{X}_{t'}(e_i)) = \begin{cases} \mathrm{X}_t(e_i) & \text{if } i \neq k; \\ \sum_{s=0}^{r_k} h_{k,s}(q^{\frac{1}{2}})\mathrm{X}_t(s[\varepsilon \mathrm{b}_k]_+ + (r_k - s)[-\varepsilon \mathrm{b}_k]_+ - e_i) & \text{if } i = k. \end{cases}$$

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$$(\sum_{s=0}^{r_k} h_{k,s}(q^{\frac{1}{2}})(q^{\frac{b}{2}}z)^s)^{\{a\}} := \begin{cases} \prod_{i=1}^{a}(\sum_{s=0}^{r_k} h_{k,s}(q^{\frac{1}{2}})(q^{\frac{b(2i-1)}{2}}z)^s) & \text{if } a > 0; \\ 1 & \text{if } a = 0; \\ \prod_{i=a}^{-1}(\sum_{s=0}^{r_k} h_{k,s}(q^{\frac{1}{2}})(q^{\frac{b(2i+1)}{2}}z)^s)^{-1} & \text{if } a < 0. \end{cases}$$

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- $\hat{Y}_t(\alpha) := \mathrm{X}_t(\tilde{B}_t\alpha), \alpha \in \mathbb{Z}^n$. We also denote $\hat{Y}_{k;t} := \hat{Y}_t(f_k)$, $k \in [1, n]$, where $f_1, \cdots, f_n$ is the standard basis of $\mathbb{Z}^n$.

For edge $t \overset{k}{\longrightarrow} t'$ in $\mathbb{T}_n$ and $k \in [1, n]$, we have two types of $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebras isomorphisms.:

- $$\psi_{k;t}(\widehat{\mathrm{Y}}_t^{\alpha}) : \mathcal{F}_{\Lambda_t} \to \mathcal{F}_{\Lambda_t}$$
  $$\mathrm{X}_t(\beta) \mapsto \mathrm{X}_t(\beta)\Big(\sum_{s=0}^{r_k} h_{k,s}(q^{\frac{1}{2}})(q^{\frac{1}{2d_k}}\widehat{\mathrm{Y}}_t^{\alpha})^s\Big)^{-\{d_k(\bar{\beta},\alpha)_D\}},$$

  where $\bar{\beta}$ is the first n entries of $\beta \in \mathbb{Z}^m$, $\alpha \in \mathbb{Z}^n$.

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  where $\bar{\beta}$ is the first n entries of $\beta \in \mathbb{Z}^m$, $\alpha \in \mathbb{Z}^n$.
- $$\phi_{k;t;\varepsilon} : \mathcal{F}_{\Lambda_{t'}} \to \mathcal{F}_{\Lambda_t}$$
  $$\mathrm{X}_{t'}(\alpha) \mapsto \mathrm{X}_t(E_{k,\varepsilon}^{\tilde{B}_t R}\alpha)$$

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Proposition 2.1

For an edge $t \overset{k}{\text{———}} t'$ in $\mathbb{T}_n$ and $\varepsilon \in \{\pm 1\}$, we have

$$\mu_{k;t} = \psi_{k;t}(\widehat{\mathrm{Y}}_{k;t}^{\varepsilon})^{\varepsilon} \circ \phi_{k;t;\varepsilon}.$$

- C-matrices $C_t = (c_{ij;t})$:
  - $C_{t_0} = I_n$;
  - If $t \overset{k}{\longrightarrow} t' \in \mathbb{T}_n$, then

$$c_{ij:t'} = \begin{cases} -c_{ij;t} & \text{if } j = k; \\ c_{ij;t} + r_k(c_{ik;t}[\varepsilon b_{kj;t}]_+ + [-\varepsilon c_{ik;t}]_+ b_{kj;t}) & \text{if } j \neq k; \end{cases}$$

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- G-matrices $\tilde{G}_t = (\tilde{\mathbf{g}}_{1;t}, \cdots, \tilde{\mathbf{g}}_{m;t})$:
  - $G_{t_0} = I_m$;
  - If $t \overset{k}{\text{———}} t' \in \mathbb{T}_n$, then

$$\tilde{\mathbf{g}}_{i;t'} = \begin{cases} \tilde{\mathbf{g}}_{i;t} & \text{if } i \neq k; \\ -\tilde{\mathbf{g}}_{k;t} + r_k(\sum_{j=1}^{m}[-b_{jk;t}]_+ \tilde{\mathbf{g}}_{j;t} - \sum_{j=1}^{n}[-c_{jk;t}]_+ \mathbf{b}_{j;t_0}) & \text{if } i = k. \end{cases}$$

- The following identities hold for G- and C-matrices:

$$\mathbf{c}_{i;t}^T[D \quad 0]\tilde{\mathbf{g}}_{j;t} = (\mathbf{c}_{i;t}, \mathbf{g}_{j;t})_D = d_i^{-1}\delta_{ij}, \text{ for } i, j \in [1, n],\ t \in \mathbb{T}_n,$$
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- For each $t \in \mathbb{T}_n$, we have a

  path in $\mathbb{T}_n$ $\mathbf{i}: \ t_0 \overset{i_1}{\longrightarrow} t_1 \overset{i_2}{\longrightarrow} t_2 \overset{i_3}{\longrightarrow} \cdots \overset{i_k}{\longrightarrow} t_k = t$,

  subpath in $\mathbb{T}_n$ $\mathbf{i}_j: \ t_0 \overset{i_1}{\longrightarrow} t_1 \overset{i_2}{\longrightarrow} t_2 \overset{i_3}{\longrightarrow} \cdots \overset{i_j}{\longrightarrow} t_j$, for $j \in [1, k]$.

  $\varepsilon_j :=$ the common sign of components of $\mathbf{c}_{i_j;t_{j-1}}$ and $\mathbf{c}_j^+ := \varepsilon_j\mathbf{c}_{i_j;t_{j-1}}$ for $j \in [1, k]$.

Now define

$$\mu_{t_k}^{t_0} := \mu_{i_1;t_0} \circ \mu_{i_2;t_1} \circ \cdots \mu_{i_k;t_{k-1}} : \mathcal{F}_{\Lambda_{t_k}} \to \mathcal{F}_{\Lambda_{t_0}}.$$

For each $j \in [1, k]$, we also set

$$\begin{aligned}\psi(\mathbf{i}_j) :=& \psi_{i_1;t_0}(\widehat{\mathrm{Y}}_{t_0}^{\mathbf{c}_1^+})^{\varepsilon_1} \circ \psi_{i_2;t_0}(\widehat{\mathrm{Y}}_{t_0}^{\mathbf{c}_2^+})^{\varepsilon_2} \circ \cdots \circ \psi_{i_j;t_0}(\widehat{\mathrm{Y}}_{t_0}^{\mathbf{c}_j^+})^{\varepsilon_j} : \mathcal{F}_{\Lambda_{t_0}} \to \mathcal{F}_{\Lambda_{t_0}} \\ \phi_{t_j}^{t_0} :=& \phi_{i_1;t_0;\varepsilon_1} \circ \phi_{i_2;t_1;\varepsilon_2} \circ \cdots \circ \phi_{i_j;t_{j-1};\varepsilon_j} : \mathcal{F}_{\Lambda_{t_j}} \to \mathcal{F}_{\Lambda_{t_0}}.\end{aligned}$$

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Keep the notation as above, we have

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Key: $\phi_{t_j}^{t_0} \circ \psi_{i_{j+1};t_j}(\widehat{\mathrm{Y}}_{i_{j+1};t_j}^{\varepsilon_{j+1}})^{\varepsilon_{j+1}} = \psi_{i_{j+1};t_0}(\widehat{\mathrm{Y}}_{t_0}^{\mathbf{c}_{j+1}^+})^{\varepsilon_{j+1}} \circ \phi_{t_j}^{t_0}$.

Outline

Notation

For simplicity of notation, we also denote by $d_{(j)} = d_{i_j}$, $r_{(j)} = r_{i_j}$, $\mathbf{c}_j = \mathbf{c}_{i_j;t_{j-1}}$, $\mathbf{c}_j^+ = \varepsilon_j \mathbf{c}_j$, $\hat{\mathbf{c}}_j^+ = B\mathbf{c}_j^+$, $\tilde{\mathbf{g}}_j = \tilde{\mathbf{g}}_{i_j;t_j}$, $\mathbf{g}_j = \overline{\tilde{\mathbf{g}}_j}$.

We first define a set of elements $\{L_{i,j} \mid i, j \in [1,k]\}$ of $\mathcal{F}_{\Lambda_{t_0}}$ by the initial condition

$$L_{1,i} := \widehat{Y}_{t_0}^{\mathbf{c}_i^+} \Big(\sum_{s=0}^{r_{(1)}} h_{i_1,s}(q^{\frac{1}{2}})(q^{\frac{1}{2d_{(1)}}} \widehat{Y}_{t_0}^{\mathbf{c}_1^+})^s\Big)^{-\varepsilon_1\{(d_{(1)}\mathbf{c}_1^+, \hat{\mathbf{c}}_i^+)_D\}} \text{ for } i \in [1,k]$$

with recurrence relations: for $j \in [1,n]$,

$$L_{j+1,i} = L_{j,i}\Big(\sum_{s=0}^{r_{(j+1)}} h_{i_{j+1},s}(q^{\frac{1}{2}})(q^{\frac{1}{2d_{(j+1)}}} L_{j,j+1})^s\Big)^{-\varepsilon_{j+1}\{(d_{(j+1)}\mathbf{c}_{j+1}^+, \hat{\mathbf{c}}_i^+)_D\}}.$$

Then set

$$L_1 = \sum_{s=0}^{r_{(1)}} h_{i_1,s}(q^{\frac{1}{2}})(q^{\frac{1}{2d_{(1)}}} \widehat{Y}_{t_0}^{\mathbf{c}_1^+})^s,$$

$$L_{j+1} = \sum_{s=0}^{r_{(j+1)}} h_{i_{j+1},s}(q^{\frac{1}{2}})(q^{\frac{1}{2d_{(j+1)}}} L_{j,j+1})^s, j \in [1, k-1].$$

Calculating $\mathrm{X}_{t_0}(-\tilde{\mathbf{g}}_k)\mu_{t_k}^{t_0}(\mathrm{X}_{i_k;t_k})$, we have that

$$\begin{aligned}\mathrm{X}_{t_0}(-\tilde{\mathbf{g}}_k)\mu_{t_k}^{t_0}(\mathrm{X}_{i_k;t_k}) =&\mathrm{X}_{t_0}(-\tilde{\mathbf{g}}_k)\psi(\mathbf{i}_1)(\mathrm{X}_{t_0}(\tilde{\mathbf{g}}_k))\psi(\mathbf{i}_1)(\mathrm{X}_{t_0}\tilde{(}-\mathbf{g}_k))\psi(\mathbf{i}_2)(\mathrm{X}_{t_0}(\tilde{\mathbf{g}}_k))\\ &\cdots\psi(\mathbf{i}_{k-1})(\mathrm{X}_{t_0}(-\tilde{\mathbf{g}}_k))\psi(\mathbf{i}_k)(\mathrm{X}_{t_0}(\tilde{\mathbf{g}}_k))\\ =&L_1^{-\varepsilon_1\{(\mathbf{g}_k,d_{(1)}\mathbf{c}_1^+)_D\}}L_2^{-\varepsilon_2\{(\mathbf{g}_k,d_{(2)}\mathbf{c}_2^+)_D\}}\cdots L_k^{-\varepsilon_k\{(\mathbf{g}_k,d_{(k)}\mathbf{c}_k^+)_D\}}.\end{aligned}$$

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$$\begin{aligned}\mathrm{X}_{t_0}(-\tilde{\mathbf{g}}_k)\mu_{t_k}^{t_0}(\mathrm{X}_{i_k;t_k}) =& \mathrm{X}_{t_0}(-\tilde{\mathbf{g}}_k)\psi(\mathbf{i}_1)(\mathrm{X}_{t_0}(\tilde{\mathbf{g}}_k))\psi(\mathbf{i}_1)(\mathrm{X}_{t_0}\tilde{(}-\mathbf{g}_k))\psi(\mathbf{i}_2)(\mathrm{X}_{t_0}(\tilde{\mathbf{g}}_k)) \\ & \cdots\psi(\mathbf{i}_{k-1})(\mathrm{X}_{t_0}(-\tilde{\mathbf{g}}_k))\psi(\mathbf{i}_k)(\mathrm{X}_{t_0}(\tilde{\mathbf{g}}_k)) \\ =& L_1^{-\varepsilon_1\{(\mathbf{g}_k,d_{(1)}\mathbf{c}_1^+)_D\}}L_2^{-\varepsilon_2\{(\mathbf{g}_k,d_{(2)}\mathbf{c}_2^+)_D\}}\cdots L_k^{-\varepsilon_k\{(\mathbf{g}_k,d_{(k)}\mathbf{c}_k^+)_D\}}.\end{aligned}$$

Lemma 3.1

Keep the notation as above. We have

$$\mathrm{X}_{t_0}(-\tilde{\mathbf{g}}_k)\mu_{t_k}^{t_0}(\mathrm{X}_{i_k;t_k}) = \overrightarrow{\prod_{j\in[1,k]}} L_j^{-\varepsilon_j\{d_{(j)}(\mathbf{c}_j^+,\mathbf{g}_k)_D\}}.$$

Remark 3.2

The above Product as a rational polynomial in $\hat{Y}_{t_0}$ *only depends on the principal part of the exchange matrix* $\tilde{B}$.

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Definition 3.3

The element $F_{i_k,t_k} := \overrightarrow{\prod_{j\in[1,k]}} L_j^{-\varepsilon_j\{d_{(j)}(\boldsymbol{c}_j^+,\boldsymbol{g}_k)_D\}}$ is called the F-polynomial of $\mathrm{X}_{t_k}(e_{i_k})$ whenever $F_{i_k;t_k}$ is a polynomial.

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Theorem 3.4

(1) *The element $F_{i_k;t_k}$ is a Laurent polynomial in $\hat{Y}_{1;t_0},\ldots,\hat{Y}_{n;t_0}$.*

(2) *Suppose that $h_{i,s}(1) > 0$ for each $i \in [1,n]$ and $s \in [1, r_i - 1]$, then $F_{i_k;t_k}$ is a polynomial in $\hat{Y}_{1;t_0},\ldots,\hat{Y}_{n;t_0}$.*

We give a brief proof for Theorem 3.4: Observing the definition of $L_i, i \in [1,k]$, there are two polynomials $A(\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0}), P(\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0})$ with coefficients in $\mathbb{N}[q^{\pm\frac{1}{2}}]$ for $h_{i,s}(q^{\frac{1}{2}}), 1 \le i \le n, 1 \le s \le r_i$ (as variables) and $\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0}$ satisfying the following equation

$$F_{i_k;t_k}(\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0}) = A(\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0})P(\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0})^{-1}.$$

Then using Newton polytope for Laurent polynomials in $\mathrm{X}_{1;t_0},\cdots,\mathrm{X}_{2n;t_0}$ of this equation. We will get that $F_{i_k;t_k}(\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0})$ also is a Laurent polynomial in $\widehat{\mathrm{Y}}_{1;t_0},\cdots,\widehat{\mathrm{Y}}_{n;t_0}$. Moreover in a mild condition $h_{i,s}(1) > 0$ for $i \in [1,n]$ and $s \in [1, r_i - 1]$. Setting $q^{\frac{1}{2}} = 1$ does not shrink New$(F_{i_k;t_k}(\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0}))$.

$$\begin{aligned}\text{New}(F_{i_k;t_k}(\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0})) =& \text{New}(F_{i_k;t_k}(\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0})|_{q^{\frac{1}{2}}=1}) \\ =& \text{New}(F_{i_k;t_k}(\widehat{\mathrm{y}},\mathrm{z})|_{z_{i,s}=h_{i,s}(1), i\in[1,n], s\in[1,r_i-1]}),\end{aligned}$$

where $F_{i_k;t_k}(\mathrm{y},\mathrm{z})$ is the F-polynomial of the cluster variable $x_{i_k;t_k}$ of the corresponding generalized cluster algebra with principal coefficients. Thus New$(F_{i_k;t_k}(\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0}))$ does not contain any points with negative coordinates. It follows that $F_{i_k;t_k}(\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0})$ is a polynomial in $\widehat{\mathrm{Y}}_{1;t_0},\dots,\widehat{\mathrm{Y}}_{n;t_0}$.

Theorem 3.5 (Separation formula)

Suppose that $h_{i,s}(1) > 0$ *for each* $i \in [1, n]$ *and* $s \in [1, r_i - 1]$. *For each* $i \in [1, n]$ *and* $t \in \mathbb{T}_n$, *let* $F_{i;t}[Z_1, \ldots, Z_n]$ *be the associated* F*-polynomial of* $\mathrm{X}_{i;t}$ *and* $\tilde{\boldsymbol{g}}_{i;t}$ *the* g*-vector of* $\mathrm{X}_{i;t}$. *We have*

$$\mathrm{X}_{i;t} = \mathrm{X}_{t_0}(\tilde{\boldsymbol{g}}_{i;t}) F_{i;t}(\hat{Y}_{1;t_0}, \ldots, \hat{Y}_{n;t_0}).$$

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$$\mathrm{X}_{i;t} = \mathrm{X}_{t_0}(\tilde{\mathbf{g}}_{i;t}) F_{i;t}(\hat{Y}_{1;t_0}, \ldots, \hat{Y}_{n;t_0}).$$

Remark 3.6

We call the equation $F_{i_k,t_k} := \overrightarrow{\prod_{j \in [1,k]}} L_j^{-\varepsilon_j \{d_{(j)}(\boldsymbol{c}_j^+, \mathbf{g}_k)_D\}}$ Gupta' s formula for F-polynomials of generalized quantum cluster algebras. When $R = I_n$, it specializes to Gupta' s formula for quantum cluster algebras.

Thanks for your attention!