# Point varieties of noncommutative conics

Hú Hǎi gāng Wǔ Wén chāo
胡 海 刚 & 伍 文 超
(中国科学技术大学 USTC)

2024-8-6

**ICRA21 2024@ 上海交通大学 SJTU**

# Notations

- $k$ is an algebraically closed field of characteristic 0. All algebras are over $k$.
- $V$ is a vector space over $k$, and $T(V)$ is the tensor algebra.
- A quadratic algebra is a graded algebra $A = T(V)/(R)$, where $R \subset V \otimes V$.
- $k[x_1, \ldots, x_n]$ is the polynomial algebra.

# Introduction

In projective geometry, the projective scheme

$$X = \operatorname{Proj} k[x_1, \dots, x_n]/(f),$$

where $0 \neq f \in k[x_1, \dots, x_n]_2$, is called a quadric hypersurface. If $n = 3$, $X$ is called a conic.

# Introduction

In projective geometry, the projective scheme

$$X = \operatorname{Proj} k[x_1, \ldots, x_n]/(f),$$

where $0 \neq f \in k[x_1, \ldots, x_n]_2$, is called a quadric hypersurface. If $n = 3$, $X$ is called a conic.

In noncommutative algebraic geometry, the noncommutatve projective scheme

$$X_{nc} = \operatorname{Proj}_{nc} S/(f)$$

where $S$ is an $n$-dim quantum polynomial algebra and $0 \neq f \in S_2$ a regular central element, is called a noncommutative quadric hypersurface. If $n = 3$, $X_{nc}$ is called a noncommutative conic.
(For convenience, we will simply say $S/(f)$ is a noncommutative quadric hypersurface.)

**Remark**

Classifications of quantum polynomial algebras and noncommutative quadric hypersurfaces are very important problems in noncommutative algebraic geometry!

## Remark

Classifications of quantum polynomial algebras and noncommutative quadric hypersurfaces are very important problems in noncommutative algebraic geometry!

- Point varieties are important invariants in the classification of 3-dim quantum polynomial algebras ([Artin-Tate-Van den Bergh, 91]).
- NOT every quantum polynomial algebra or every noncommutative quadric hypersurface has the point variety.

### Remark

Classifications of quantum polynomial algebras and noncommutative quadric hypersurfaces are very important problems in noncommutative algebraic geometry!

- Point varieties are important invariants in the classification of 3-dim quantum polynomial algebras ([Artin-Tate-Van den Bergh, 91]).
- NOT every quantum polynomial algebra or every noncommutative quadric hypersurface has the point variety.

### Goal

Prove the existence of point varieties of noncommutative conics (main result 1), and give some classification results of point varieties (main result 2).

# Preliminaries

**Definition (Artin-Schelter, 87)**

A graded algebra $S = k \oplus S_1 \oplus S_2 \oplus \cdots$ generated in degree 1 is called an $n$-dimensional quantum polynomial algebra (qpa for short) if

1. $\operatorname{gldim} S = n$;
2. $\underline{\operatorname{Ext}}^i_S(k, S) = \begin{cases} k & \text{if } i = n, \\ 0 & \text{otherwise;} \end{cases}$
3. $H_S(t) := \sum_{i=0}^{\infty} (\dim_k S_i) t^i = 1/(1-t)^n$.

# Preliminaries

**Definition (Artin-Schelter, 87)**

A graded algebra $S = k \oplus S_1 \oplus S_2 \oplus \cdots$ generated in degree 1 is called an *$n$-dimensional quantum polynomial algebra* (qpa for short) if

1. $\operatorname{gldim} S = n$;
2. $\underline{\operatorname{Ext}}^i_S(k, S) = \begin{cases} k & \text{if } i = n, \\ 0 & \text{otherwise;} \end{cases}$
3. $H_S(t) := \sum_{i=0}^{\infty} (\dim_k S_i) t^i = 1/(1-t)^n$.

In noncommutative algebraic geometry, we regard an $n$-dim qpa as a noncommutative version of $k[x_1, \dots, x_n]$.

- $k[x_1, \dots, x_n]$ is an $n$-dim qpa.
- $k\langle x, y, z\rangle/(xy + yx, xz + zx, yz + zy)$ is a 3-dim qpa.

**Fact (Artin-Tate-Van den Bergh, 91)**

A 3-dim quantum polynomial algebra is a domain.

**Fact (Artin-Tate-Van den Bergh, 91)**

A 3-dim quantum polynomial algebra is a domain.

**Fact (Artin-Schelter, 87; Artin-Tate-Van den Bergh, 90, 91; Itaba-Matsuno, 21, 22; Matsuno, 21)**

3-dim quantum polynomial algebras are completely classified.

**Fact (Artin-Tate-Van den Bergh, 91)**

A 3-dim quantum polynomial algebra is a domain.

**Fact (Artin-Schelter, 87; Artin-Tate-Van den Bergh, 90, 91; Itaba-Matsuno, 21, 22; Matsuno, 21)**

3-dim quantum polynomial algebras are completely classified.

**Fact (Zhang, 96; Shelton-Tingey, 01)**

Quantum polynomial algebras and noncommutative quadric hypersurfaces are Koszul (and thus quadratic).

## Definition (Artin-Tate-Van den Bergh, 90; Mori, 06)

We say a quadratic algebra $A = T(V)/(R)$ satisfies condition (G1) if $\exists$ a projective variety $E_A \subset \mathbb{P}(V^*)$ and $\sigma_A \in \operatorname{Aut} E$ s.t.

$$\begin{aligned}\mathcal{V}(R) :=& \{(p,q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid f(p,q) = 0,\ \forall f \in R\} \\ =& \{(p, \sigma_A(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E_A\}.\end{aligned}$$

If $A$ satisfies (G1), we call $(E_A, \sigma_A)$ the geometric pair of $A$, and $E_A$ the point variety of $A$.

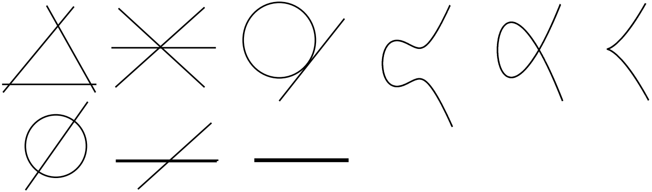
**Theorem (Artin-Tate-Van den Bergh, 90)**

Every 3-dimensional quantum polynomial algebra satisfies condition (G1).

## Theorem (Artin-Tate-Van den Bergh, 90)

Every 3-dimensional quantum polynomial algebra satisfies condition (G1).

The point varieties of 3-dim qpa:



We say a 3-dim qpa $S$ is of type EC if the point variety of $S$ is an elliptic curve. Type EC is the "generic" case of 3-dim qpas.

**Theorem (Mori, 06)**

For two quantum polynomial algebras (or noncommutative quadric hypersurfaces) $A, A'$ satisfy condition (G1). We have

$$\operatorname{Proj}_{nc} A \cong \operatorname{Proj}_{nc} A' \Rightarrow E_A \cong E_{A'}.$$

**Theorem (Mori, 06)**

For two quantum polynomial algebras (or noncommutative quadric hypersurfaces) $A, A'$ satisfy condition (G1). We have

$$\mathrm{Proj}_{nc}\, A \cong \mathrm{Proj}_{nc}\, A' \Rightarrow E_A \cong E_{A'}.$$

$$\begin{array}{ccc}
A \cong A' & \Longrightarrow & \mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A' \\
\text{defn.}\Downarrow & & \Downarrow\text{[Zhang]} \\
E_A \cong E_{A'} & \underset{\text{[Mori]}}{\Longleftarrow} & \mathrm{Proj}_{nc}\, A \cong \mathrm{Proj}_{nc}\, A'.
\end{array}$$

# Main result 1

**Theorem (H-W)**

Every noncommutative conic $A = S/(f)$ satisfies condition (G1).

# Main result 1

**Theorem (H-W)**

Every noncommutative conic $A = S/(f)$ satisfies condition (G1).

Sketch of proof:

1. Consider $A = T(V)/(R)$. For fixed basises of $V$ and $R$, construct the linear resolutions of $k_A$ and ${}_Ak$ explicitly:

$$\cdots \longrightarrow A^{\oplus 4} \xrightarrow[N]{} A^{\oplus 4} \xrightarrow[M]{} A^{\oplus 4} \xrightarrow[J]{} A^{\oplus 3} \longrightarrow A \to k_A \to 0,$$

and

$$\cdots \longrightarrow A^{\oplus 4} \xrightarrow[N]{} A^{\oplus 4} \xrightarrow[M]{} A^{\oplus 4} \xrightarrow[L]{} A^{\oplus 3} \longrightarrow A \to {}_Ak \to 0,$$

where $M, N \in V^{4\times 4}, J \in V^{3\times 4}, L \in V^{4\times 3}$ are matrices.

2. Show that

$$\begin{aligned} X_L :=& \mathcal{Z}(\{\text{3-minors of } L\}) \\ =& \mathcal{Z}(\{\text{3-minors of } N\}) \\ =& \mathcal{Z}(\{\text{3-minors of } J\}) =: X_J. \end{aligned}$$

(Where $\mathcal{Z}(\{?\})$ is the projective variety defined by the zero set of $\{?\}$.)

**Remark**

It is HARD to compare $X_L$ and $X_J$ directly.

3. Use the fact

$$A \text{ satisfies (G1)} \Leftrightarrow X_L = X_J.$$

# Classification of type EC case

Let $A = S/(f)$ be a noncommutative conic. We say $A$ is of type EC if $S$ is of type EC.

There are 2 steps to classify point varieties of noncommutative conics $A = S/(f)$ of type EC.

1. Find all noncommutative conics of type EC.
2. Calculate point varieties.

# Find all noncommutative conics of type EC

There are infinitely many isomorphism classes of 3-dim qpas $S$ of type EC. The hard thing is to determine when the center $Z(S)_2$ is nontrivial.

# Find all noncommutative conics of type EC

There are infinitely many isomorphism classes of 3-dim qpas $S$ of type EC. The hard thing is to determine when the center $Z(S)_2$ is nontrivial.

**Theorem (H-Matsuno-Mori, 23)**

Let $S$ be a 3-dim qpa, and $(E, \sigma)$ the geometric pair of $S$. If $Z(S)_2 \neq \{0\}$, then

(∗) there is an irreducible component $E' \subset E$ such that $(\sigma|_{E'})^2 = \mathrm{id}$.

## Theorem (Matsuno, 21)

A 3-dim qpa $S$ of type EC satisfies $(*)$ if and only if $S$ is isomorphic to one of the following:

A. $k\langle x, y, z\rangle/(yz + zy + \lambda x^2, zx + xz + \lambda z^2, xy + yx + \lambda z^2)$.

B. $k\langle x, y, z\rangle/(x^2 + y^2 + \lambda z^2, xz + zy + \lambda yx, zx + yz + \lambda xy)$.

Where $0 \neq \lambda \in k$, and $\lambda^3 \neq 1, -8$.

## Theorem (Matsuno, 21)

A 3-dim qpa $S$ of type EC satisfies $(*)$ if and only if $S$ is isomorphic to one of the following:

A. $k\langle x, y, z\rangle/(yz + zy + \lambda x^2, zx + xz + \lambda z^2, xy + yx + \lambda z^2)$.

B. $k\langle x, y, z\rangle/(x^2 + y^2 + \lambda z^2, xz + zy + \lambda yx, zx + yz + \lambda xy)$.

Where $0 \neq \lambda \in k$, and $\lambda^3 \neq 1, -8$.

## Lemma

A. $Z(S)_2 = \{ax^2 + by^2 + cz^2 \mid a, b, c \in k\}$.

B. $Z(S)_2 = \{a(xy + yx) + bz^2 \mid a, b \in k\}$.

# Calculate point varieties

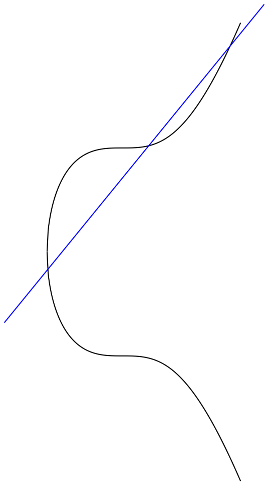
Let $S$ be a 3-dim qpa, and $(E, \sigma)$ the geometric pair of $S$.

**Theorem (H-Matsuno-Mori, 23; H-W)**

Let $A = S/(f)$ be a noncommutative conic of type EC. Then $f$ is reducible. Moreover

A. $f = g^2$ where $g \in S_1$, and $E_A = (E \cap \mathcal{Z}(g)) \cup \sigma(E \cap \mathcal{Z}(g))$.

B. $f = gg'$ where $g, g' \in S_1$, and $E_A = (E \cap \mathcal{Z}(g)) \cup \tau\sigma(E \cap \mathcal{Z}(g))$, where $\tau \in \operatorname{Aut} \mathbb{P}^2$ defined by $\tau(a : b : c) = (b : a : c)$.



There are 3 intersection points $p_1, p_2, p_3$ in $E \cap \mathcal{Z}(g)$ above. Then

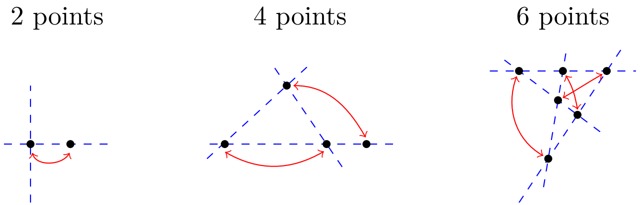
$$E_A = \begin{cases} \{p_1, p_2, p_3, \sigma(p_1), \sigma(p_2), \sigma(p_3)\} & \text{in case A,} \\ \{p_1, p_2, p_3, \tau\sigma(p_1), \tau\sigma(p_2), \tau\sigma(p_3)\} & \text{in case B.} \end{cases}$$

# Main result 2

**Theorem (H-W)**

There are exactly 3 isomorphism classes of point varieties of noncommutaive conics of type EC.

They are:



# Cohen-Macaulay representations

**Theorem (H-Matsuno-Mori, 23; H-W)**

For two noncommutative conics $A, A'$ of type EC. Then

$$E_A \cong E_{A'} \Leftrightarrow \underline{\mathrm{CM}}^{\mathbb{Z}} A \cong \underline{\mathrm{CM}}^{\mathbb{Z}} A'.$$

Thank you for listening!