The Deligne – Simpson Problem

via

weighted projective lines

and

deformed preprojective algebras

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Overview

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The Deligne – Simpson Problem

Connections on $\mathbb{P}^1$

Today: purely algebraic approach.

Atiyah defined a functorial short exact sequence

$$\alpha(E): \quad 0 \longrightarrow E(-2) \longrightarrow A(E) \longrightarrow E \longrightarrow 0$$

Explicitly, given $E = (E^{\pm}; \theta) \in \operatorname{coh} \mathbb{P}^1$, the sheaf $A(E)$ has charts $E^{\pm} \ltimes E^{\pm}$, where

$$s^{\pm} \quad \text{acts via} \quad \begin{pmatrix} s^{\pm} & 0 \\ \pm 1 & s^{\pm} \end{pmatrix} \qquad \text{and glue is} \quad \begin{pmatrix} \theta & 0 \\ 0 & s^{-2}\theta \end{pmatrix}$$

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Now ∇ is a **connection** on E if and only if $\binom{1}{\nabla}$ is a section of $\alpha(E)$.

A connection is given by k-linear endomorphisms $\nabla^{\pm}$ of $E^{\pm}$ such that

$$\nabla^{\pm}(s^{\pm}x) = s^{\pm}\nabla^{\pm}(x) \pm x \quad \text{and} \quad \nabla^{+}(\theta(x)) = s^{-2}\theta(\nabla^{-}(x))$$

Generalised Atiyah classes

Functoriality gives for $f\colon E \to F$

$$A(E \oplus F) = A(E) \oplus A(F) \quad \text{and} \quad A(f)\colon A(E) \to A(f)$$

but construction does not commute with shift

$$A(E(1)) \ncong A(E)(1).$$

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Led to 2-parameter deformation. For $\mu, \nu \in k$ define functorial

$$\alpha_{\mu,\nu}(E)\colon \qquad 0 \longrightarrow E(-2) \longrightarrow A_{\mu,\nu}(E) \longrightarrow E \longrightarrow 0$$

For $E = (E^{\pm}; \theta)$ the sheaf $A_{\mu,\nu}(E)$ has charts $E^{\pm} \ltimes E^{\pm}$, where

$$s^{\pm} \quad \text{acts via} \quad \begin{pmatrix} s^{\pm} & 0 \\ \pm\nu & s^{\pm} \end{pmatrix} \qquad \text{and glue is} \quad \begin{pmatrix} \theta & 0 \\ \mu s^{-}\theta & s^{-2}\theta \end{pmatrix}$$

Generalised Atiyah classes

Proposition

Shift $\alpha_{\mu,\nu}(E)(1) = \alpha_{\mu+\nu,\nu}(E(1))$

Baer sum $\alpha_{\mu,\nu} + t\alpha_{\mu',\nu'} = \alpha_{\mu+t\mu',\nu+t\nu'}$

Serre pairing $\langle \mathrm{id}, \alpha_{\mu,\nu}(E)\rangle = \mu \operatorname{rank} E - \nu \deg E$

Thus have family of functorial short exact sequences indexed by linear functionals $K_0(\operatorname{coh} \mathbb{P}^1) \to k$.

Atiyah sequence is for $(\mu, \nu) = (0, 1)$.

Connections on $\mathbb{P}^1$

By analogy call ∇ a (μ,ν)-**connection** on E if $\binom{1}{\nabla}$ is a section of $\alpha_{\mu,\nu}(E)$.

Proposition

Sheaf E admits (μ,ν)-connection $\iff \mu \operatorname{rank} E' = \nu \deg E'$ for all indecomposable summands E' of E.

Special cases (in characteristic zero)

E admits $(m,1)$-connection $\iff E \in \operatorname{add}\mathcal{O}(m)$.

E admits $(1,0)$-connection $\iff E \in \operatorname{tor}\mathbb{P}^1$.

In characteristic p every indecomposable torsion sheaf of length divisible by p admits an $(m,1)$- connection

Connections on $\mathbb{X}$

Extend this idea to construct family of functorial short exact sequences in $\operatorname{coh}\mathbb{X}$ indexed by linear functionals $K_0(\operatorname{coh}\mathbb{X}) \to k$.

Recall: Standard presentation of $\mathcal{E} \in \operatorname{coh}\mathbb{X}$ ended

$$\pi_! E_0 \oplus \bigoplus_{i,p} \pi_! E_{px_i}(px_i) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{E}$$

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For $\omega = -2c + \bar{w} \in \mathbb{L}$ this yields surjection

$$\Psi\colon \operatorname{Ext}_{\mathbb{P}}(E_0, E_0(-2)) \oplus \bigoplus_{i,p} \operatorname{Ext}_{\mathbb{P}}(E_{px_i}, E_{px_i}(-2)) \twoheadrightarrow \operatorname{Ext}_{\mathbb{X}}(\mathcal{E}, \mathcal{E}(\omega))$$

Serre pairing satisfies

$$\langle f, \Psi(\eta_0, \eta_{px_i}) \rangle_{\mathbb{X}} = \langle f_0, \eta_0 \rangle_{\mathbb{P}} + \sum_{i,p} \langle f_{px_i}, \eta_{px_i} \rangle_{\mathbb{P}}$$

Connections on $\mathbb{X}$

Theorem

Take $\chi\colon K_0(\mathbb{X}) \to k$, *say*

$$\chi[\mathcal{E}] = \mu \operatorname{rank} \mathcal{E} - \nu_0 \deg E_0 - \sum_{i,p} \nu_{ip} \deg E_{px_i}$$

Then

$$\beta_\chi(\mathcal{E}) = \Psi(\alpha_{\mu,\nu_0}(E_0), \alpha_{0,\nu_{ip}}(E_{px_i}))$$

gives functorial short exact sequence

$$\beta_\chi(\mathcal{E})\colon \qquad 0 \longrightarrow \mathcal{E}(\omega) \longrightarrow B_\chi(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0$$

such that

$$\langle \mathrm{id}, \beta_\chi(\mathcal{E}) \rangle_{\mathbb{X}} = \chi[\mathcal{E}]$$

Category of connections

By analogy call ∇ a χ-connection on $\mathcal{E}$ if $\binom{1}{\nabla}$ is a section of $\beta_\chi(\mathcal{E})$.

As B_χ functorial, get category $\operatorname{conn}_\chi \mathbb{X}$ having

1. objects $(\mathcal{E}, \nabla)$ for sheaf $\mathcal{E} \in \operatorname{coh} \mathbb{X}$ with connection ∇
2. morphism $f\colon (\mathcal{E}, \nabla) \to (\mathcal{E}', \nabla')$ is $f\colon \mathcal{E} \to \mathcal{E}'$ fitting into commutative square

$$\begin{array}{ccc} B_\chi(\mathcal{E}) & \xleftarrow{\binom{1}{\nabla}} & \mathcal{E} \\ {\scriptstyle B_\chi(f)}\downarrow & & \downarrow{\scriptstyle f} \\ B_\chi(\mathcal{E}') & \xleftarrow[\binom{1}{\nabla'}]{} & \mathcal{E}' \end{array}$$

Category of connections

Theorem

The category $\mathrm{conn}_\chi \mathbb{X}$ *is* k*-linear, abelian and noetherian.*

If $\mathrm{char}\, k = 0$ *and* $\nu = \nu_0 + \sum_{ip} \nu_{ip}$ *nonzero, then length category.*

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When $\mathrm{conn}_\chi \mathbb{X}$ is a length category want to describe the simple objects.

First: Describe classes $[\mathcal{E}] \in K_0(\mathrm{coh}\, \mathbb{X})$ for simples $(\mathcal{E}, \nabla)$.

This is essentially the Deligne – Simpson Problem !

Proof: abelian

Easy: $\mathrm{conn}_\chi \mathbb{X}$ is k-linear and additive.

Show abelian. Take morphism $f\colon (\mathcal{E}, \nabla) \to (\mathcal{E}', \nabla')$.

Let $\iota\colon \mathcal{K} \to \mathcal{E}$ be kernel for f in $\mathrm{coh}\,\mathbb{X}$. Then $B_\chi(\iota)$ is kernel of $B_\chi(f)$.

Composite

$$\mathcal{K} \xrightarrow{\quad\iota\quad} \mathcal{E} \xrightarrow{\binom{1}{\nabla}} B_\chi(\mathcal{E}) \xrightarrow{B_\chi(f)} B_\chi(\mathcal{E}')$$

equals $\binom{1}{\nabla'} f\iota = 0$, so $\binom{1}{\nabla}\iota$ factors uniquely through $B_\chi(\iota)$.

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equals $\binom{1}{\nabla'} f \iota = 0$, so $\binom{1}{\nabla}\iota$ factors uniquely through $B_\chi(\iota)$.

Get section $\mathcal{K} \to B_\chi(\mathcal{K})$. Similarly for cokernels.

Finally, map from coimage of f to image of f is an isomorphism in $\operatorname{coh} \mathbb{X}$, so is an isomorphism in $\operatorname{conn}_\chi \mathbb{X}$.

Proof: noetherian

Take ascending chain inside $(\mathcal{E}, \nabla)$

$$(\mathcal{E}_1, \nabla_1) \subseteq (\mathcal{E}_2, \nabla_2) \subseteq \cdots$$

Get ascending chain of sheaves inside $\mathcal{E}$. Stabilises since $\operatorname{coh}\mathbb{X}$ noetherian.

Get inclusions $(\mathcal{E}', \nabla'_1) \subseteq (\mathcal{E}', \nabla'_2) \subseteq \cdots$

Identity on $\mathcal{E}'$ so must have $\nabla'_i = \nabla'_{i+1}$ for all i.

Proof: length category

Assume: only finitely many indecomposable torsion sheaves $\mathcal{G}_1, \ldots, \mathcal{G}_m$ admit χ-connection.

To show: $\operatorname{conn}_\chi \mathbb{X}$ is artinian.

Take descending chain inside $(\mathcal{E}, \nabla)$

$$(\mathcal{E}, \nabla) \supseteq (\mathcal{E}_1, \nabla_1) \supseteq (\mathcal{E}_2, \nabla_2) \supseteq \cdots$$

Then $\operatorname{rank} \mathcal{E}_i$ form decreasing sequence, so stabilises.

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Then $\operatorname{rank} \mathcal{E}_i$ form decreasing sequence, so stabilises.

May assume ranks constant, so $\mathcal{T}_i = \mathcal{E}/\mathcal{E}_i$ torsion.

Enough to show induced sequence stabilises

$$(\mathcal{E}, \nabla) \twoheadrightarrow \cdots \twoheadrightarrow (\mathcal{T}_2, \nabla'_2) \twoheadrightarrow (\mathcal{T}_1, \nabla'_1)$$

Proof: length category

Have sequence

$$(\mathcal{E}, \nabla) \twoheadrightarrow \cdots \twoheadrightarrow (\mathcal{T}_2, \nabla'_2) \twoheadrightarrow (\mathcal{T}_1, \nabla'_1)$$

Assumption gives: each $\mathcal{T}_i$ lies in $\mathrm{add}(\mathcal{G}_1, \ldots, \mathcal{G}_m)$.

Proof: length category

Have sequence

$$(\mathcal{E}, \nabla) \twoheadrightarrow \cdots \twoheadrightarrow (\mathcal{T}_2, \nabla_2') \twoheadrightarrow (\mathcal{T}_1, \nabla_1')$$

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Surjection $\mathcal{E} \twoheadrightarrow \mathcal{T}_i$, so multiplicity of $\mathcal{G}_j$ in $\mathcal{T}_i$ bounded by

$$\dim \operatorname{Hom}(\mathcal{E}, \mathcal{G}_j)$$

Only finitely many choices for $\mathcal{T}_i$. One $\mathcal{T}$ occurs infinitely often.

For surjection $(\mathcal{T}, \nabla') \twoheadrightarrow (\mathcal{T}, \nabla'')$ must have $\nabla' = \nabla''$.

Thus sequence stabilises.

Proof: finitely many indec torsion with connection

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Have finitely many classes $x_i \in K_0(\operatorname{coh} \mathbb{X})$ such that every indecomposable torsion $\mathcal{T}$ has class

$$[\mathcal{T}] = x_m + [\pi_! S] \quad \text{some } i, \text{ some } S \in \operatorname{tor} \mathbb{P}^1 \text{ with } \pi_! S \subseteq \mathcal{T}.$$

Take $x_0 = 0$ and classes of indecomposables in $\operatorname{tor}_{a_i} \mathbb{X}$ of length $< w_i$.

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Sheaf $\pi_! S \in \operatorname{tor}\mathbb{X}$ has rank 0 and $\deg(\pi_! S)_d = b$ constant.

Thus $\chi[\pi_! S] = -b(\nu_0 + \sum_{ip} \nu_{ip}) = -b\nu$.

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Thus $\chi[\pi_! S] = -b(\nu_0 + \sum_{ip} \nu_{ip}) = -b\nu$.

If char $k = 0$ and $\nu \neq 0$, then this is never zero for $S \neq 0$.

Thus $\chi[\mathcal{T}] = 0$ implies $x_i \neq 0$, so $\mathcal{T}$ supported at some weighted point a_i.

Then unique choice for b, and unique choice for S with $\pi_! S \subseteq \mathcal{T}$.

Existence

Theorem

Suppose k has characteristic zero, or is algebraically closed.
A sheaf $\mathcal{E} \in \operatorname{coh} \mathbb{X}$ *admits* χ*-connection if and only if*

$$\chi[\mathcal{E}'] = 0 \quad \forall \text{ direct summands } \mathcal{E}' \text{ of } \mathcal{E}$$

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Have $\beta_\chi(\mathcal{E}) = 0$ if and only if $\beta_\chi(\mathcal{E}') = 0$ for all summands $\mathcal{E}'$.

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Suppose $\mathcal{E}$ indecomposable. Have $\Lambda = \operatorname{End}(\mathcal{E})$ local and map

$$\Lambda \to \operatorname{End}_k(\Lambda/J(\Lambda)), \quad f \mapsto \bar{f}$$

Then Serre pairing $\langle f, \beta_\chi(\mathcal{E})\rangle$ is proportional to $\operatorname{tr}(\bar{f})\chi[\mathcal{E}]$.

Easy first property

Want to understand simple objects $(\mathcal{E}, \nabla) \in \mathrm{conn}_\chi \mathbb{X}$.

Lemma

Take $(\mathcal{E}, \nabla) \in \mathrm{conn}_\chi \mathbb{X}$. If $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$ with $\mathrm{Ext}(\mathcal{E}'', \mathcal{E}') = 0$, then have subobject $(\mathcal{E}', \nabla') \subseteq (\mathcal{E}, \nabla)$.

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Proof

Consider composite $\mathcal{E}' \xrightarrow{\binom{1}{\nabla}\iota} B_\chi(\mathcal{E}) \longrightarrow B_\chi(\mathcal{E}'')$

Vanishes when composed with $B_\chi(\mathcal{E}'') \to \mathcal{E}''$,
so factors uniquely through kernel $\mathcal{E}''(\omega)$
but $\operatorname{Hom}(\mathcal{E}', \mathcal{E}''(\omega)) \cong D\operatorname{Ext}(\mathcal{E}'', \mathcal{E}') = 0$.

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Thus $\binom{1}{\nabla}\iota \colon \mathcal{E}' \to B_\chi(\mathcal{E})$ factors uniquely through $B_\chi(\mathcal{E}')$.

Domestic case

Let Q_* be Dynkin. Then $\operatorname{coh}\mathbb{X}$ is derived equivalent to $\operatorname{mod}\Lambda$ for tame hereditary algebra Λ.

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Have $K_0(\operatorname{coh}\mathbb{X}) = \mathbb{Z}\partial \oplus K_0(Q_*)$ and $K_0(\Lambda) = \mathbb{Z}\delta \oplus K_0(Q_*)$.
Identify via $\partial \mapsto \delta$.

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Let $\Phi \subset K_0(Q_*)$ be root system. Roots for Λ contained in $\mathbb{Z}\delta \times (\Phi \cup \{0\})$.

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Classes of indecomposable Λ-modules correspond to positive roots.

Theorem

Classes of indecomposable locally free sheaves correspond to

$$\textit{roots} \quad a\partial + x \quad \textit{with} \quad x_* > 0$$

Classes of indecomposable torsion sheaves correspond to

$$\textit{roots} \quad a\partial \quad \textit{with} \quad a > 0 \qquad \textit{or} \qquad \textit{roots} \quad a\partial + x \quad \textit{with} \quad a \geq 0,\ x_* = 0$$

Domestic case

For a root $x \in \Phi$ there is at most one $a \in \mathbb{Z}$ with $\chi(a\partial + x) = 0$.

As $\chi(\partial) = \nu \neq 0$ only finitely many indecomposable sheaves admit χ-connection.

We may asume they all lie in $\operatorname{Gen} \mathcal{T}$ for some tilting $\mathcal{T} \in \operatorname{loc} \mathbb{X}$ with $\operatorname{End}(\mathcal{T}) \cong \Lambda$.

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Theorem

The fully faithful functor

$$\operatorname{Hom}(\mathcal{T}, -)\colon \mathit{Gen}\, \mathcal{T} \to \operatorname{mod} \Lambda$$

lifts to an equivalence

$$\operatorname{conn}_\chi \mathbb{X} \cong \operatorname{mod} \Pi^\chi$$

where Π^χ is a deformed preprojective algebra.

Domestic case

Have an equivalence $\operatorname{conn}_\chi \mathbb{X} \cong \operatorname{mod} \Pi^\chi$.

The dimension vectors of the simple Π^χ-modules were described by Crawley-Boevey and Holland.

Let $R_\chi = \{\text{roots } a\delta + x \mid \chi(a\delta + x) = 0\}$. This is finite since $\chi(\delta) \neq 0$.

Let $\Sigma_\chi \subseteq R_\chi$ be the minimal positive elements.

Theorem

There exists a simple Π^χ-module of dimension vector $a\delta + x$ if and only if $a\delta + x \in \Sigma_\chi$.

Logarithmic connections

Following Mihai, consider pushout of Atiyah sequence along product $\sigma = \sigma_1 \cdots \sigma_n$, having degree $h = \sum_i h_i$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(-2) & \longrightarrow & A(E) & \longrightarrow & E & \longrightarrow & 0 \\ & & \big\downarrow \sigma & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & E(h-2) & \longrightarrow & M(E) & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

Explicitly, the sheaf $M(E)$ has charts $E^{\pm} \ltimes E^{\pm}$, where

$$s^{\pm} \quad \text{acts via} \quad \begin{pmatrix} s^{\pm} & 0 \\ \pm\sigma^{\pm} & s^{\pm} \end{pmatrix} \qquad \text{and glue is} \quad \begin{pmatrix} \theta & 0 \\ 0 & s^{h-2}\theta \end{pmatrix}$$

Logarithmic connections

Call ∇ a **log connection** if $\binom{1}{\nabla}$ is a section of the pushout sequence

$$0 \longrightarrow E(h-2) \longrightarrow M(E) \longrightarrow E \longrightarrow 0$$

Thus log connection given by k-linear $\nabla^{\pm}$ such that

$$\nabla^{\pm}(s^{\pm}x) = s^{\pm}\nabla^{\pm}(x) \pm \sigma^{\pm}x \quad \text{and} \quad \nabla^{+}(\theta(x)) = s^{h-2}\theta(\nabla^{-}(x))$$

Logarithmic connections have at most simple poles at the points a_i

Again, get abelian category $\log\mathrm{conn}\,\mathbb{P}^1$ having objects (E,∇) consisting of a sheaf E equipped with a log connection ∇.

Example

Take $\{a_1, a_2\} = \{0, \infty\}$, so $\sigma = uv$.

For all $m \in \mathbb{Z}$ and $\alpha \in k$ have log connection ∇_α on $\mathcal{O}(m)$ given by

$$\nabla^+_\alpha(s^r) = (\alpha + r)s^r \quad \text{and} \quad \nabla^-_\alpha(s^{-r}) = (\alpha + m - r)s^{-r}$$

Morphism $u^n \colon \mathcal{O}(m) \to \mathcal{O}(m+n)$ yields morphism

$$u^n \colon (\mathcal{O}(m), \nabla_\alpha) \to (\mathcal{O}(m+n), \nabla_{\alpha-n}).$$

Similarly for morphism v^n.

Note

Taking their cokernels shows that torsion sheaves supported at a_i admit log connections.

Residues

If $E \in \operatorname{loc} \mathbb{P}^1$, then $\sigma\colon E(-h) \to E$ is injective with cokernel

$$E[\sigma] = \bigoplus_i E[a_i]$$

the direct sum of the fibres.

Thus $\sigma\colon E(-2) \to E(h-2)$, and also $A(E) \to M(E)$, has cokernel $E[\sigma](h-2)$.

Need explicit isomorphism to $E[\sigma]$.

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Need explicit isomorphism to $E[\sigma]$. Set

$$\rho^+ = d(\sigma^+)/d(s^+) \quad \text{and} \quad \rho^- = -d(\sigma^-)/d(s^-).$$

Then $\rho^+ = s^{h-2}\rho^- + hs^-\sigma^+$. As σ^+ acts as zero on all $\mathcal{T} \in \operatorname{add}(S_\sigma)$, get isomorphism

$$\rho\colon \mathcal{T} \xrightarrow{\simeq} \mathcal{T}(h-2) \quad \forall\, \mathcal{T} \in \operatorname{add}(S_\sigma).$$

Residues

Let ∇ be a log connection on locally free sheaf E.

Have exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(-h) & \xrightarrow{\sigma} & E & \longrightarrow & E[\sigma] & \longrightarrow & 0 \\ & & \left(\begin{smallmatrix}\sigma \\ \rho+\nabla\end{smallmatrix}\right)\Big\downarrow & & \Big\downarrow\left(\begin{smallmatrix}1 \\ \nabla\end{smallmatrix}\right) & & \exists\Big\downarrow \rho\,\mathrm{Res}\,\nabla & & \\ 0 & \longrightarrow & A(E) & \longrightarrow & M(E) & \longrightarrow & E[\sigma](h-2) & \longrightarrow & 0 \end{array}$$

Call induced map $\mathrm{Res}\,\nabla \in \mathrm{End}(E[\sigma])$ the **residue** of ∇.

Decomposes to give residue maps $\mathrm{Res}_i\,\nabla \in \mathrm{End}(E[a_i])$ on fibres.

Example

Take $\{a_1, a_2\} = \{0, \infty\}$, so $\sigma = uv$. Take $\alpha \in k$. Have log connection ∇ on $\mathcal{O}(m)$

$$\nabla^+(s^r) = (\alpha + r)s^r, \quad \nabla^-(s^{-r}) = (\alpha + m - r)s^{-r}$$

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Take $\{a_1, a_2\} = \{0, \infty\}$, so $\sigma = uv$. Take $\alpha \in k$. Have log connection ∇ on $\mathcal{O}(m)$

$$\nabla^+(s^r) = (\alpha + r)s^r, \quad \nabla^-(s^{-r}) = (\alpha + m - r)s^{-r}$$

Have $\rho^{\pm} = \pm 1$. To compute $\mathrm{Res}_0 \nabla$ use charts on U^+

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k[s] & \xrightarrow{\;s\;} & k[s] & \longrightarrow & k & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \begin{pmatrix} s \\ 1+\nabla^+ \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1 \\ \nabla^+ \end{pmatrix}} & & \Big\downarrow {\scriptstyle \mathrm{Res}_0 \nabla} & & \\
0 & \longrightarrow & k[s] \ltimes k[s] & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}} & k[s] \ltimes k[s] & \longrightarrow & k & \longrightarrow & 0
\end{array}$$

Get $\mathrm{Res}_0 \nabla = \alpha$. Using charts on U^- get $\mathrm{Res}_\infty \nabla = -(\alpha + m)$.

Connections on parabolic sheaves

If (E, V) is a parabolic sheaf, and ∇ a log connection on E, then can ask how the residues

$$\mathrm{Res}_i \nabla \colon E[a_i] \to E[a_i]$$

interact with the flags

$$0 = V_{i,w_i} \subseteq \cdots V_{i,1} \subseteq V_{i,0} = E[a_i]$$

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Given scalars $\zeta_{ip} \in k$, call ∇ a ζ-**connection** on (E, V) provided

$$(\mathrm{Res}_i \nabla - \zeta_{ip})(V_{i,p-1}) \subseteq V_{i,p} \quad \forall\ i, p$$

Get category $\mathrm{conn}_\zeta \operatorname{par} \mathbb{P}^1$.

Connections on parabolic sheaves

Recall: Equivalence of categories $\operatorname{par}\mathbb{P}^1 \cong \operatorname{loc}\mathbb{X}$.

Define subcategory $\operatorname{conn}_\chi \operatorname{loc}\mathbb{X}$ by taking those $(\mathcal{E}, \nabla) \in \operatorname{conn}_\chi \mathbb{X}$ with $\mathcal{E} \in \operatorname{loc}\mathbb{X}$.

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Theorem

Fix scalars ζ_{ip}. Define $\chi\colon K_0(\operatorname{coh}\mathbb{X}) \to k$ using

$$\mu = -\sum_i h_i \zeta_{iw_i}, \quad \nu_0 = 1 + \sum_i (\zeta_{i1} - \zeta_{iw_i}), \quad \nu_{ip} = \zeta_{ip+1} - \zeta_{ip}.$$

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Then the equivalence $\operatorname{par}\mathbb{P}^1 \cong \operatorname{loc}\mathbb{X}$ *extends to an equivalence*

$$\operatorname{conn}_\zeta \operatorname{par}\mathbb{P}^1 \cong \operatorname{conn}_\chi \operatorname{loc}\mathbb{X}$$

of categories of sheaves equipped with a connection.

Connections on parabolic sheaves

Idea of proof.

Equivalence $\operatorname{par}\mathbb{P}^1 \cong \operatorname{loc}\mathbb{X}$ follows from pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_! E_0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{V} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_! E_0 & \longrightarrow & \pi_* E_0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0
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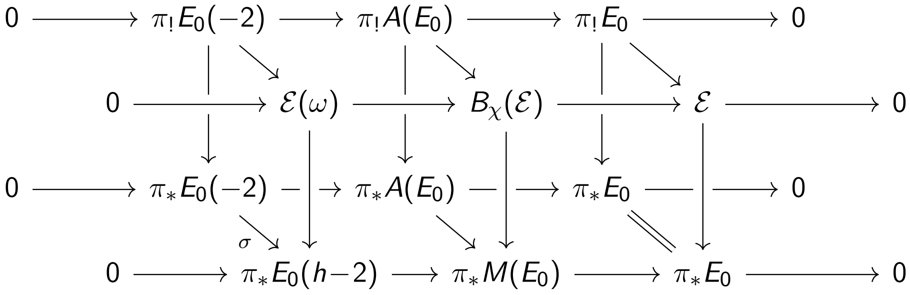
and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E(-2) & \longrightarrow & A(E) & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \| & & \\
0 & \longrightarrow & E(h-2) & \longrightarrow & M(E) & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

Connections on parabolic sheaves

Now combine to get diagram



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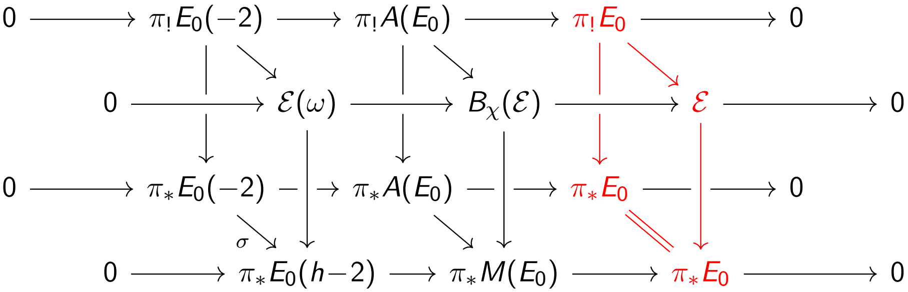

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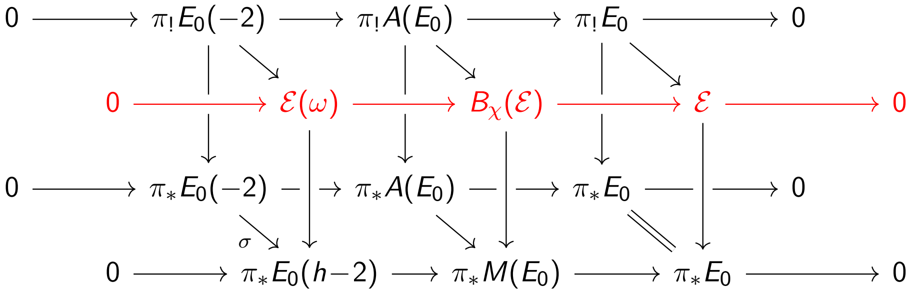

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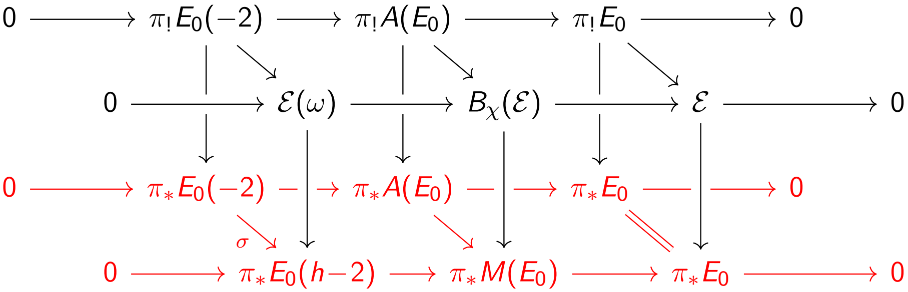

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Have defined category $\operatorname{conn}_\chi \mathbb{X}$ of sheaves together with a χ-connection. It is k-linear, abelian, noetherian. In nice cases also a length category.

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Given ζ there exists χ and an equivalence $\operatorname{conn}_\chi \operatorname{loc} \mathbb{X} \cong \operatorname{conn}_\zeta \operatorname{par} \mathbb{P}^1$. Thus locally free sheaves on $\mathbb{X}$ with a connection are the same as parabolic sheaves on $\mathbb{P}^1$ with an appropriate log connection.

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Given ζ there exists χ and an equivalence $\operatorname{conn}_\chi \operatorname{loc} \mathbb{X} \cong \operatorname{conn}_\zeta \operatorname{par} \mathbb{P}^1$.
Thus locally free sheaves on $\mathbb{X}$ with a connection are the same as parabolic sheaves on $\mathbb{P}^1$ with an appropriate log connection.

Basic aim is to describe classes $[\mathcal{E}] \in K_0(\operatorname{coh} \mathbb{X})$ for simple objects $(\mathcal{E}, \nabla) \in \operatorname{conn}_\chi \mathbb{X}$.
Can do this easily in domestic type by passing to deformed preprojective algebras.

Next time

Introduce the Deligne – Simpson Problem.

Relate to parabolic sheaves with connection via functor

$$\operatorname{conn}_\zeta \operatorname{par} \mathbb{P}^1 \to \log \operatorname{conn} \mathbb{P}^1, \quad (E, V, \nabla) \mapsto (E, \nabla)$$

forgetting the parabolic structure.

Reinterpret in terms of $\operatorname{conn}_\chi \mathbb{X}$.

Give local description of $\operatorname{conn}_\chi \mathbb{X}$ via deformed preprojective algebras.

Thank You !