The Deligne – Simpson Problem

via

weighted projective lines

and

deformed preprojective algebras

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Overview

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Sheaves

Talk 2

Connections

Talk 3

The Deligne – Simpson Problem

Sheaves on $\mathbb{P}^1$

Fix field k. Projective line $\mathbb{P}^1 = \operatorname{Proj} k[u, v]$ has open affine cover

$$U^+ = \mathbb{P}^1 - \{\infty\} = \operatorname{Spec} k[s]$$
$$U^- = \mathbb{P}^1 - \{0\} = \operatorname{Spec} k[s^-]$$

where $s = u/v$.

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where $s = u/v$.

A **coherent sheaf** $E = (E^+, E^-; \phi)$ consists of

1. a finitely generated $k[s]$-module E^+
2. a finitely generated $k[s^-]$-module E^-
3. a $k[s, s^-]$-isomorphism $\theta \colon k[s, s^-] \otimes E^- \xrightarrow{\sim} k[s, s^-] \otimes E^+$

We call $E^\pm$ the **charts** and θ the **glue**.

Morphisms

A morphism $f = (f^+, f^-)\colon E \to F$ consists of

1. a $k[s]$-linear map $f^+\colon E^+ \to F^+$
2. a $k[s^-]$-linear map $f^-\colon E^- \to F^-$
3. fitting into a commutative square

$$\begin{array}{ccc}
k[s,s^-] \otimes E^- & \xrightarrow{1\otimes f^-} & k[s,s^-] \otimes F^- \\
\wr\downarrow \theta & & \wr\downarrow \phi \\
k[s,s^-] \otimes E^+ & \xrightarrow{1\otimes f^+} & k[s,s^-] \otimes F^+
\end{array}$$

Category coh $\mathbb{P}^1$

We obtain a k-linear category coh $\mathbb{P}^1$. This is furthermore

1. **abelian**, where

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

   is exact provided it is exact on both charts

$$0 \longrightarrow E^{\pm} \longrightarrow F^{\pm} \longrightarrow G^{\pm} \longrightarrow 0$$

2. with finite dimensional homomorphism and extension spaces
3. and is **hereditary**, so $\mathrm{Ext}^2(-,-) \equiv 0$
4. and **noetherian**, so ascending chains stabilise.

Examples

1. $\mathcal{O}(m) = (k[s], k[s^-]; s^m)$. Indecomposable, endomorphism ring k.
2. In general, for $E = (E^\pm; \theta)$, have shift $E(m) = (E^\pm; s^m\theta)$.
   Note, $(E(m))(n) = E(m+n)$ for all $m, n \in \mathbb{Z}$. $\mathbb{Z} \to \mathrm{Aut}(\mathrm{coh}\,\mathbb{P}^1)$
   Can identify $\mathrm{Hom}(E, F) = \mathrm{Hom}(E(m), F(m))$.

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   Can identify $\mathrm{Hom}(E, F) = \mathrm{Hom}(E(m), F(m))$.
3. Let $\sigma \in k[u, v]$ be homogeneous of degree d. Define

$$\sigma^+ = \sigma(s, 1) \in k[s], \quad \sigma^- = \sigma(1, s^-) \in k[s^-]$$

   Multiplication by $\sigma^\pm$ on $E^\pm$ gives morphism $\sigma\colon E \to E(d)$.
   Get natural transformation $\sigma\colon \mathrm{id} \to (d)$. $k[u, v] \to Z_{\mathrm{gr}}(\mathrm{coh}\,\mathbb{P}^1, (1))$

Examples

4. Every morphism $\mathcal{O} \to \mathcal{O}(d)$ is uniquely of the form σ. So $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(d))$ has dimension $d+1$.

5. Let $\sigma \in k[u,v]$ be homogeneous of degree d. Get short exact sequence

$$0 \longrightarrow \mathcal{O}(-d) \xrightarrow{\sigma} \mathcal{O} \longrightarrow S_\sigma \longrightarrow 0$$

where $S_\sigma = (k[s^\pm]/(\sigma^\pm); \mathrm{id})$. Note

$S_\sigma(m) \cong S_\sigma$ for all m, and

S_σ indecomposable $\iff \sigma = \tau^m$ with τ irreducible.

Split torsion pair

1. Sheaf E is **locally free** if $E^{\pm}$ both free.
   $\operatorname{loc}\mathbb{P}^1$ is an exact subcategory.
   Every locally free sheaf is uniquely a direct sum of $\mathcal{O}(m)$.

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2. Sheaf E is **torsion** if $E^{\pm}$ both torsion.
   $\operatorname{tor}\mathbb{P}^1$ is a Serre subcategory. closed under subquots, exts
   Every torsion sheaf is uniquely $\bigoplus S_{\sigma^m}$, σ irreducible.

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2. Sheaf E is **torsion** if $E^{\pm}$ both torsion.
   $\operatorname{tor}\mathbb{P}^1$ is a Serre subcategory.
   Every torsion sheaf is uniquely $\bigoplus S_{\sigma^m}$, σ irreducible.
3. $\operatorname{Hom}(\operatorname{tor}\mathbb{P}^1, \operatorname{loc}\mathbb{P}^1) = 0 = \operatorname{Ext}(\operatorname{loc}\mathbb{P}^1, \operatorname{tor}\mathbb{P}^1)$.
4. Have functorial short exact sequence

$$0 \longrightarrow E_{\mathrm{tor}} \longrightarrow E \longrightarrow E_{\mathrm{loc}} \longrightarrow 0$$

Torsion sheaves

Take $a = (\sigma) \in \operatorname{Proj} k[u, v]$. Have Serre subcategory

$$\operatorname{tor}_a \mathbb{P}^1 = \operatorname{add}(S_{\sigma^m}, m \geq 1)$$

It is a uniserial length category with unique simple S_σ.

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Have decomposition

$$\operatorname{tor} \mathbb{P}^1 = \bigvee_{a \in \mathbb{P}} \operatorname{tor}_a \mathbb{P}^1$$

so no homomorphisms or extensions between sheaves supported at distinct points.

Grothendieck group

The **Grothendieck group** $K_0(\operatorname{coh}\mathbb{P}^1)$ is $\mathbb{Z}^2$, where

$$[\mathcal{O}(m)] = (1, m) \quad \text{and} \quad [S_\sigma] = (0, \deg\sigma).$$

In general write $[E] = (\operatorname{rank} E, \deg E)$. Note $\operatorname{rank} E = \operatorname{rank} E^{\pm}$.

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The **Euler form**

$$\{E, F\} = \dim \operatorname{Hom}(E, F) - \dim \operatorname{Ext}^1(E, F)$$

descends to bilinear form on $K_0(\operatorname{coh}\mathbb{P}^1)$

$$\{(r, d), (r', d')\} = rr' + rd' - dr'.$$

Extensions

Take $E \in \operatorname{coh} \mathbb{P}^1$ and $F \in \operatorname{loc} \mathbb{P}^1$. A short exact sequence

$$0 \longrightarrow E \longrightarrow M \longrightarrow F \longrightarrow 0$$

is split on charts, so $M^\pm = F^\pm \oplus E^\pm$.

The glue is then of the form

$$\begin{pmatrix} \phi & 0 \\ \gamma\phi & \theta \end{pmatrix}$$

for some

$$\gamma \colon k[s, s^-] \otimes F^+ \to k[s, s^-] \otimes E^+.$$

Write $\eta_\gamma \in \operatorname{Ext}(F, E)$ for the extension.

Examples

Up to equivalence, every short exact sequence

$$0 \longrightarrow \mathcal{O}(-d) \longrightarrow M \longrightarrow \mathcal{O} \longrightarrow 0$$

is uniquely of the form $M^{\pm} = k[s^{\pm}]^2$ with glue

$$\begin{pmatrix} 1 & 0 \\ \gamma & s^{-d} \end{pmatrix}, \quad \gamma \in \operatorname{span}\{s^{-}, s^{-2}, \ldots, s^{1-d}\}.$$

The extension

$$0 \longrightarrow \mathcal{O}(-2m) \xrightarrow{(v^m, u^m)^t} \mathcal{O}(-m)^2 \xrightarrow{(u^m, -v^m)} \mathcal{O} \longrightarrow 0$$

corresponds to $\gamma = s^{-m}$.

Serre duality

Take $E \in \operatorname{coh} \mathbb{P}^1$, $F \in \operatorname{loc} \mathbb{P}^1$, and short exact sequence

$$0 \longrightarrow E(-2) \longrightarrow M \longrightarrow F \longrightarrow 0$$

Let M have charts $M^\pm = F^\pm \oplus E^\pm$ and glue

$$\begin{pmatrix} \phi & 0 \\ \gamma\phi & s^{-2}\theta \end{pmatrix}, \quad \gamma \colon k[s, s^-] \otimes F^+ \to k[s, s^-] \otimes E^+.$$

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$$\begin{pmatrix} \phi & 0 \\ \gamma\phi & s^{-2}\theta \end{pmatrix}, \quad \gamma\colon k[s, s^{-}] \otimes F^{+} \to k[s, s^{-}] \otimes E^{+}.$$

Given $f\colon E \to F$, have $f^{+}\colon k[s, s^{-}] \otimes E^{+} \to k[s, s^{-}] \otimes F^{+}$

so $f^{+}\gamma \in \operatorname{End}(k[s, s^{-}] \otimes F^{+}) \cong M_r(k[s, s^{-}])$, where $r = \operatorname{rank} F$.

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Let M have charts $M^\pm = F^\pm \oplus E^\pm$ and glue

$$\begin{pmatrix} \phi & 0 \\ \gamma\phi & s^{-2}\theta \end{pmatrix}, \quad \gamma\colon k[s, s^-] \otimes F^+ \to k[s, s^-] \otimes E^+.$$

Given $f\colon E \to F$, have $f^+\colon k[s, s^-] \otimes E^+ \to k[s, s^-] \otimes F^+$

so $f^+\gamma \in \operatorname{End}(k[s, s^-] \otimes F^+) \cong M_r(k[s, s^-])$, where $r = \operatorname{rank} F$.

Then $\operatorname{tr}(f^+\gamma) \in k[s, s^-]$, and $\operatorname{res}\operatorname{tr}(f^+\gamma) \in k$ (coefficient of s^-).

Serre duality

Obtain pairing

$$\langle -,- \rangle \colon \operatorname{Hom}(E,F) \times \operatorname{Ext}(F,E(-2)) \to k,$$
$$\langle f, \eta_\gamma \rangle = \operatorname{res}\operatorname{tr}(f^+\gamma).$$

for $F \in \operatorname{loc}\mathbb{P}^1$.

Theorem

This extends to a bifunctorial and shift invariant perfect pairing

$$\langle -,- \rangle \colon \operatorname{Hom}(E,F) \times \operatorname{Ext}(F,E(-2)) \to k$$

on all of $\operatorname{coh}\mathbb{P}^1$*, called the* **Serre pairing***.*

Weighted projective lines

A **weighted projective line** $\mathbb{X}$ consists of a set of points $a_1, \ldots, a_n \in \mathbb{P}^1$ having weights $w_1, \ldots, w_n \in \mathbb{N}$.

We construct a category $\operatorname{coh} \mathbb{X}$ sharing many of the nice properties of $\operatorname{coh} \mathbb{P}^1$

1. k-linear, hereditary abelian, noetherian
2. finite dimensional homomorphisms and extensions
3. split torsion pair $(\operatorname{tor} \mathbb{X}, \operatorname{loc} \mathbb{X})$
4. $\operatorname{tor} \mathbb{X} = \bigvee_{a \in \mathbb{P}} \operatorname{tor}_a \mathbb{X}$ uniserial Serre subcategory
5. Serre duality

but now $\operatorname{tor}_{a_i} \mathbb{X}$ has w_i simple objects.

Periodic functors

Fix representatives $a_i = (\sigma_i) \in \operatorname{Proj} k[u, v]$. Set $h_i = \deg \sigma_i$.

Let $\mathbb{Z}^n$ have standard basis x_i. Poset where $d \geq 0$ provided $d = \sum_i d_i x_i$ and $d_i \geq 0$ for all i.

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A functor $\mathcal{E} \colon (\mathbb{Z}^n)^{\mathrm{op}} \to \operatorname{coh} \mathbb{P}^1$ is given by

1. a sheaf $E_d \in \operatorname{coh} \mathbb{P}^1$ for all $d \in \mathbb{Z}^n$
2. a unique morphism $\phi_{d,e} \colon E_{d+e} \to E_d$ for all $d, e \in \mathbb{Z}^n$ with $e \geq 0$.

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Call $\mathcal{E}$ **periodic** with respect to (σ, w) if

1. $E_{d-w_i x_i} = E_d(h_i)$
2. $\phi_{d-w_i x_i, e} = \phi_{d,e}$ as maps $E_{d+e}(h_i) \to E_d(h_i)$
3. $\phi_{d, w_i x_i} = \sigma_i \colon E_d(-h_i) \to E_d$.

Periodic functors

A morphism $f\colon \mathcal{E} \to \mathcal{F}$ is a natural transformation of functors.

Call f **periodic** if $f_{d-w_i x_i} = f_d$.

Define $\operatorname{coh}\mathbb{X}$ to be the subcategory of periodic functors with periodic morphisms. Independent of choice of representatives σ_i

This is a k-linear abelian category with finite dimensional homomorphism and extension spaces.

Examples

Take $n = 1$, $\sigma \in k[u, v]$ irreducible, degree h, $w = 2$.
A coherent sheaf $\mathcal{E} \in \operatorname{coh} \mathbb{X}$ is given by

$$\cdots \xrightarrow{\phi_0} E_2 = E_0(-h) \xrightarrow{\phi_1} E_1 \xrightarrow{\phi_0} E_0 \xrightarrow{\phi_1} E_{-1} = E_1(h) \xrightarrow{\phi_0} \cdots$$

such that $\phi_1\phi_0 = \sigma = \phi_0\phi_1$ Matrix factorisations

Special case

$n = 1$, $\sigma = u^2 + v^2$ irreducible over k, $w = 2$.

$$\cdots \longrightarrow \mathcal{O}(-2)^2 \xrightarrow{\begin{pmatrix} u & v \\ -v & u \end{pmatrix}} \mathcal{O}(-1)^2 \xrightarrow{\begin{pmatrix} u & -v \\ v & u \end{pmatrix}} \mathcal{O}^2 \xrightarrow{\begin{pmatrix} u & v \\ -v & u \end{pmatrix}} \mathcal{O}(1)^2 \longrightarrow \cdots$$

This is indecomposable with endomorphism ring $k[t]/(t^2 + 1)$.

Example

$n = 2$, σ, τ irreducible of degree 1, $w_1 = 3$, $w_2 = 2$.
A sheaf $\mathcal{E} \in \operatorname{coh} \mathbb{X}$ is a periodic array

$$
\begin{array}{ccccccccc}
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & E_0(-2) & \longrightarrow & E_{2x}(-1) & \longrightarrow & E_x(-1) & \longrightarrow & E_0(-1) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & E_y(-1) & \longrightarrow & E_{2x+y} & \longrightarrow & E_{x+y} & \longrightarrow & E_y & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & E_0(-1) & \longrightarrow & E_{2x} & \longrightarrow & E_x & \longrightarrow & E_0 & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}
$$

$$\sigma : E_0(-2) \to E_0(-1), \qquad \tau : E_0(-1) \to E_0$$

Sheaves on $\mathbb{X}$

The definition of $\operatorname{coh}\mathbb{X}$ is over-specified. The forgetful functor restricting a sheaf $\mathcal{E}$ to the axes in $\mathbb{Z}^n$ is fully faithful.

So, just need to specify a sheaf $E_0 \in \operatorname{coh}\mathbb{P}^1$ and an n-tuple of functors $\mathcal{E}_i \colon \mathbb{Z}^{\mathrm{op}} \to \operatorname{coh}\mathbb{P}^1$, periodic with resepect to (σ_i, w_i), satisfying $E_{i,0} = E_0$.

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The advantage is that have all **shifts**.

For $d' \in \mathbb{Z}^n$ define $\mathcal{E}(d')$ with $(\mathcal{E}(d'))_d = E_{d-d'}$. Note

$$\mathcal{E}(d')(d'') = \mathcal{E}(d' + d'')$$

Also have shift $\mathcal{E}(c)$ with $(\mathcal{E}(c))_d = E_d(1)$. Then $\mathcal{E}(w_i x_i) = \mathcal{E}(h_i c)$.

These give **shift group**

$$\mathbb{L} = \mathbb{Z}^n \oplus \mathbb{Z}c/(w_i x_i - h_i c).$$

Group homomorphism $\mathbb{L} \to \operatorname{Aut}(\operatorname{coh} \mathbb{X})$

Recollement

Have exact functor

$$\pi\colon \operatorname{coh}\mathbb{X} \to \operatorname{coh}\mathbb{P}^1, \quad \mathcal{E} \mapsto E_0.$$

Admits exact and fully faithful left and right adjoints, $\pi_!$ and π_*.

Set $\bar{w} = \sum_i (w_i - 1)x_i \in \mathbb{Z}^n$. Then for $0 \leq d \leq \bar{w}$ have

$$(\pi_! E)_{-d} = E \quad \text{and} \quad (\pi_* E)_d = E$$

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$$(\pi_! E)_{-d} = E \quad \text{and} \quad (\pi_* E)_d = E$$

Theorem

We have

$$(\pi_! E)(\bar{w}) = \pi_* E.$$

Also for all $i \geq 0$ have

$$\operatorname{Ext}^i_{\mathbb{X}}(\pi_! E, \mathcal{F}) \cong \operatorname{Ext}^i_{\mathbb{P}}(E, F_0) \quad \textit{and} \quad \operatorname{Ext}^i_{\mathbb{X}}(\mathcal{F}, \pi_* E) \cong \operatorname{Ext}^i_{\mathbb{P}}(F_0, E)$$

Locally free sheaves

1. Sheaf $\mathcal{E}$ is **locally free** if each $E_d \in \operatorname{loc} \mathbb{P}^1$.
   $\operatorname{loc} \mathbb{X}$ is an exact subcategory.
2. If $\mathcal{E} \in \operatorname{loc} \mathbb{X}$, then rank E_d is constant.
3. An **invertible sheaf** is a locally free sheaf of rank one.
   For example $\mathcal{O} = \mathcal{O}_{\mathbb{X}} = \pi_! \mathcal{O}_{\mathbb{P}}$.

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   For example $\mathcal{O} = \mathcal{O}_{\mathbb{X}} = \pi_! \mathcal{O}_{\mathbb{P}}$.
4. Every locally free sheaf is filtered by invertible sheaves.
5. Every invertible sheaf is uniquely $\mathcal{O}(d)$ for $d \in \mathbb{L}$.

Torsion sheaves

1. Sheaf $\mathcal{E}$ is **torsion** if each $E_d \in \operatorname{tor} \mathbb{P}^1$.
   $\operatorname{tor} \mathbb{X}$ is a Serre subcategory.
2. For $a \in \mathbb{P}^1$ have Serre subcategory

$$\operatorname{tor}_a \mathbb{X} = \{\mathcal{E} \mid E_d \in \operatorname{tor}_a \mathbb{P}^1\}.$$

   It is a uniserial length category.

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   $$\operatorname{tor}_a \mathbb{X} = \{\mathcal{E} \mid E_d \in \operatorname{tor}_a \mathbb{P}^1\}.$$
   It is a uniserial length category.
3. $\operatorname{tor}_{a_i} \mathbb{X}$ has w_i simple objects S_{ip}
   $$0 \longrightarrow \mathcal{O}((p-1)x_i) \longrightarrow \mathcal{O}(px_i) \longrightarrow S_{ip} \longrightarrow 0$$
   Otherwise $\operatorname{tor}_a \mathbb{X} \cong \operatorname{tor}_a \mathbb{P}^1$, with unique simple $\pi_! S_a$.
4. Have decomposition $\operatorname{tor} \mathbb{X} = \bigvee_{a \in \mathbb{P}} \operatorname{tor}_a \mathbb{X}$.

Split torsion pair

Have $\mathrm{Hom}(\mathrm{tor}\,\mathbb{X}, \mathrm{loc}\,\mathbb{X}) = 0 = \mathrm{Ext}(\mathrm{loc}\,\mathbb{X}, \mathrm{tor}\,\mathbb{X})$.

Have functorial short exact sequence

$$0 \longrightarrow \mathcal{E}_{\mathrm{tor}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_{\mathrm{loc}} \longrightarrow 0$$

The category $\mathrm{coh}\,\mathbb{X}$ is **hereditary** and **noetherian**.

Examples

Previous example: $n = 1$, $\sigma = u^2 + v^2$, $w = 2$.
Have short exact sequence

$$0 \longrightarrow \pi_! \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow \pi_* \mathcal{O} \longrightarrow 0$$

$$\begin{array}{cccccccccc}
\text{pos} & & & & & & & & & \\
2x & 0 & \longrightarrow & \mathcal{O}(-2) & \xrightarrow{(0,1)^t} & \mathcal{O}(-2)^2 & \xrightarrow{(1,0)} & \mathcal{O}(-2) & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle \begin{pmatrix} u & v \\ -v & u \end{pmatrix}} & & \downarrow{\scriptstyle \sigma} & & \\
x & 0 & \longrightarrow & \mathcal{O}(-2) & \xrightarrow{(v,u)^t} & \mathcal{O}(-1)^2 & \xrightarrow{(u,-v)} & \mathcal{O} & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \begin{pmatrix} u & -v \\ v & u \end{pmatrix}} & & \| & & \\
0 & 0 & \longrightarrow & \mathcal{O} & \xrightarrow{(0,1)} & \mathcal{O}^2 & \xrightarrow{(1,0)} & \mathcal{O} & \longrightarrow & 0
\end{array}$$

Examples

$n = 1$, σ of degree h, $w = 3$.

$$
\begin{array}{ccccccccc}
& 0 & \longrightarrow & \mathcal{O}(-x) & \longrightarrow & \mathcal{O} & \longrightarrow & S_{\sigma,0} & \longrightarrow & 0
\end{array}
$$

$$
\begin{array}{cccccccccc}
\text{pos} & & & & & & & & & \\
3x & 0 & \longrightarrow & \mathcal{O}(-2h) & \xrightarrow{\sigma} & \mathcal{O}(-h) & \longrightarrow & S_{\sigma} & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle \sigma} & & \| & & \downarrow & & \\
2x & 0 & \longrightarrow & \mathcal{O}(-h) & = & \mathcal{O}(-h) & \longrightarrow & 0 & \longrightarrow & 0 \\
& & & \| & & \| & & \downarrow & & \\
x & 0 & \longrightarrow & \mathcal{O}(-h) & = & \mathcal{O}(-h) & \longrightarrow & 0 & \longrightarrow & 0 \\
& & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \\
0 & 0 & \longrightarrow & \mathcal{O}(-h) & \xrightarrow{\sigma} & \mathcal{O} & \longrightarrow & S_{\sigma} & \longrightarrow & 0
\end{array}
$$

Examples

$n = 1$, σ of degree h, $w = 3$.

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(x) \longrightarrow S_{\sigma,1} \longrightarrow 0$$

$$\begin{array}{cccccccccc}
\text{pos} & & & & & & & & & \\
3x & 0 & \longrightarrow & \mathcal{O}(-h) & = & \mathcal{O}(-h) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & \| & & \| & & \downarrow & & \\
2x & 0 & \longrightarrow & \mathcal{O}(-h) & = & \mathcal{O}(-h) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \\
x & 0 & \longrightarrow & \mathcal{O}(-h) & \xrightarrow{\sigma} & \mathcal{O} & \longrightarrow & S_{\sigma} & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \sigma} & & \| & & \downarrow & & \\
0 & 0 & \longrightarrow & \mathcal{O} & = & \mathcal{O} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Examples

$n = 1$, σ of degree h, $w = 3$.

$$0 \longrightarrow \mathcal{O}(x) \longrightarrow \mathcal{O}(2x) \longrightarrow S_{\sigma,2} \longrightarrow 0$$

$$\begin{array}{lccccccccc}
\text{pos} & & & & & & & & & \\
3x & 0 & \longrightarrow & \mathcal{O}(-h) & = & \mathcal{O}(-h) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle\sigma} & & \downarrow & & \\
2x & 0 & \longrightarrow & \mathcal{O}(-h) & \xrightarrow{\sigma} & \mathcal{O} & \longrightarrow & S_{\sigma} & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle\sigma} & & \| & & \downarrow & & \\
x & 0 & \longrightarrow & \mathcal{O} & = & \mathcal{O} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & \| & & \| & & \downarrow & & \\
0 & 0 & \longrightarrow & \mathcal{O} & = & \mathcal{O} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Grothendieck group

Recall: Every locally free sheaf has a well-defined rank. Every torsion sheaf has rank zero.

We have $K_0(\operatorname{coh}\mathbb{X}) = \mathbb{Z}^{2+\sum_i (w_i-1)}$, where

$$[\mathcal{E}] = (\operatorname{rank}\mathcal{E}, \deg E_0, \deg E_{px_i}).$$

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Alternative basis ∂, e_*, e_{ip} such that

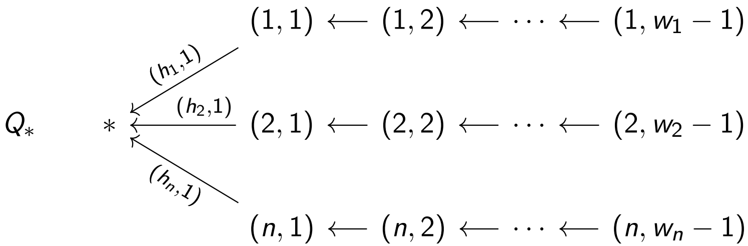
$$[\mathcal{E}] = (\deg E_0)\partial + \underline{\dim}\,\mathcal{E},$$

where

$$\underline{\dim}\,\mathcal{E} = (\operatorname{rank}\mathcal{E})e_* + \sum_{i,p} \tfrac{1}{h_i}\big(\deg E_{px_i} - \deg E_{w_i x_i}\big)e_{ip}$$

Euler form

We view $\underline{\dim}\,\mathcal{E}$ as a dimension vector for (valued) quiver



Theorem

The **Euler form** $\{\mathcal{E},\mathcal{F}\} = \dim \operatorname{Hom}(\mathcal{E},\mathcal{F}) - \dim \operatorname{Ext}(\mathcal{E},\mathcal{F})$ *descends to a bilinear form on* $\mathbb{Z}\partial \oplus K_0(Q_*)$ *given by*

$$\{a\partial + x, b\partial + y\} = \{x,y\}_{Q_*} + x_* b - a y_*.$$

Standard resolution

Recall: restriction to axes is fully faithful.

A morphism $f\colon \mathcal{E} \to \mathcal{F}$ is completely determined by commutative diagrams

$$\begin{array}{ccccccccc}
E_0(-h_i) & \longrightarrow & \cdots & \longrightarrow & E_{2x_i} & \longrightarrow & E_{x_i} & \longrightarrow & E_0 \\
\downarrow{\scriptstyle f_0} & & & & \downarrow{\scriptstyle f_{2x_i}} & & \downarrow{\scriptstyle f_{x_i}} & & \downarrow{\scriptstyle f_0} \\
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\end{array}$$

Obtain exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{X}}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathbb{P}}(E_0, F_0) \oplus \bigoplus_{i,p} \mathrm{Hom}_{\mathbb{P}}(E_{px_i}, F_{px_i})$$

$$\longrightarrow \bigoplus_{i,q} \mathrm{Hom}_{\mathbb{P}}(E_{qx_i}, F_{(q-1)x_i})$$

where $0 < p < w_i$ and $0 \leq q < w_i$.

Standard resolution

Can reinterpret this as $\mathrm{Hom}(-, \mathcal{F})$ applied to a standard presentation of $\mathcal{E}$. This is analogous to standard resolution for quiver representations.

Theorem

Have standard presentation

$$\bigoplus_{i,q} \pi_! E_{qx_i}((q-1)x_i) \longrightarrow \pi_! E_0 \oplus \bigoplus_{i,p} \pi_! E_{px_i}(px_i) \longrightarrow \mathcal{E} \longrightarrow 0$$

Can describe kernel explicitly

Serre duality

Have $\pi_! E(\bar{w}) = \pi_* E$, so

$$\mathrm{Ext}_{\mathbb{X}}(\mathcal{F}, \pi_! E(-2c + \bar{w})) = \mathrm{Ext}_{\mathbb{X}}(\mathcal{F}, \pi_* E(-2c)) \cong \mathrm{Ext}_{\mathbb{P}}(F_0, E(-2))$$

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As $\mathrm{coh}\,\mathbb{X}$ is hereditary, apply $\mathrm{Hom}_{\mathbb{X}}(\mathcal{F}, -)$ to standard presentation for $\mathcal{E}(-2c + \bar{w})$ to get surjection

$$\Psi\colon\ \mathrm{Ext}_{\mathbb{P}}(F_0, E_0(-2)) \oplus \bigoplus_{i,p} \mathrm{Ext}_{\mathbb{P}}(F_{px_i}, E_{px_i}(-2)) \twoheadrightarrow \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{F}, \mathcal{E}(-2c + \bar{w})).$$

Serre duality

Set $\omega = -2c + \bar{w}$ in $\mathbb{L}$. Have

$$\mathrm{Ext}_{\mathbb{P}}(F_0, E_0(-2)) \oplus \bigoplus_{i,p} \mathrm{Ext}_{\mathbb{P}}(F_{px_i}, E_{px_i}(-2)) \xrightarrow{\Psi} \mathrm{Ext}_{\mathbb{X}}(\mathcal{F}, \mathcal{E}(\omega))$$

$$\mathrm{Hom}_{\mathbb{P}}(E_0, F_0) \oplus \bigoplus_{i,p} \mathrm{Hom}_{\mathbb{P}}(E_{px_i}, F_{px_i}) \longleftarrow \mathrm{Hom}_{\mathbb{X}}(\mathcal{E}, \mathcal{F})$$

Theorem

Have bifunctorial and shift invariant perfect pairing

$$\langle -, - \rangle_{\mathbb{X}} \colon \mathrm{Hom}_{\mathbb{X}}(\mathcal{E}, \mathcal{F}) \times \mathrm{Ext}_{\mathbb{X}}(\mathcal{F}, \mathcal{E}(\omega)) \to k$$

called the **Serre pairing**, *such that*

$$\langle f, \Psi(\eta_0, \eta_{px_i}) \rangle_{\mathbb{X}} = \langle f_0, \eta_0 \rangle_{\mathbb{P}} + \sum_{i,p} \langle f_{px_i}, \eta_{px_i} \rangle_{\mathbb{P}}.$$

Parabolic sheaves

Take $\mathcal{E} \in \operatorname{loc} \mathbb{X}$. Then $\sigma_i \colon E_0(-h_i) \rightarrowtail E_0$ is injective with cokernel the fibre $E_0[a_i]$ of E_0 at a_i. Note $E_0[a_i] \cong S_{a_i}^r$ for $r = \operatorname{rank} \mathcal{E}$.

Also, $\kappa(a_i) = \operatorname{End}(S_{a_i})$ is the residue field, and add $S_{a_i} \cong \operatorname{mod} \kappa(a_i)$.

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Set $V_{i,p}$ to be cokernel of $E_0(-h_i) \rightarrowtail E_{px_i}$. Get exact commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_0(-h_i) & \longrightarrow & E_{px_i} & \longrightarrow & V_{i,p} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_0(-h_i) & \longrightarrow & E_0 & \longrightarrow & V_{i,0} & \longrightarrow & 0 \end{array}$$

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0 & \longrightarrow & E_0(-h_i) & \longrightarrow & E_0 & \longrightarrow & V_{i,0} & \longrightarrow & 0
\end{array}$$

Yields flag of $\kappa(a_i)$-vector spaces inside fibre $E_0[a_i]$

$$0 = V_{i,w_i} \subseteq \cdots \subseteq V_{i,1} \subseteq V_{i,0} = E_0[a_i]$$

Call (E_0, V) a **parabolic sheaf** on $\mathbb{P}^1$. Arise naturally in various contexts

Parabolic sheaves

Theorem

Have an equivalence of exact categories $\operatorname{par}\mathbb{P}^1 \cong \operatorname{loc}\mathbb{X}$.

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Given (E, V), have

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with $\mathcal{F}$ torsion and $F_{px_i} = E_0[a_i]$ for $0 < p < w_i$.

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The flags V determine torsion subsheaf $\mathcal{V} \subseteq \mathcal{F}$, so can take pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_! E & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{V} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_! E & \longrightarrow & \pi_* E & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

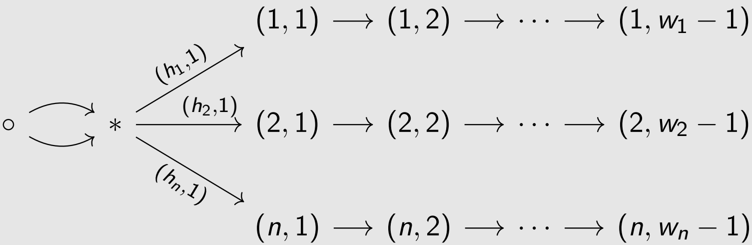
Tilting sheaves

Theorem

Have torsion sheaves $\mathcal{T}_{ip} \in \operatorname{tor}_{a_i} \mathbb{X}$ *giving tilting sheaf*

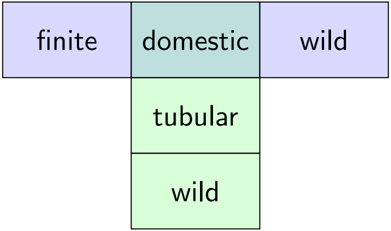
$$\mathcal{T} = \mathcal{O}(-c) \oplus \mathcal{O} \oplus \bigoplus_{i,p} \mathcal{T}_{ip}$$

Its endomorphism algebra is a squid (+ relations)



Hereditary categories

Up to derived equivalence, k-linear, hereditary abelian, noetherian categories admitting tilting object are



Horizontal line: $\operatorname{mod}\Lambda$ for a fin. dim. hereditary algebra Λ.

Finite if Dynkin. Domestic if affine.

Vertical line: $\operatorname{coh}\mathbb{X}$ for a weighted projective line $\mathbb{X}$.

Domestic if Q_* Dynkin. Tubular if Q_* affine.

Thank you !