The Deligne – Simpson Problem

via

weighted projective lines

and

deformed preprojective algebras

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Overview

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Sheaves

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The Deligne – Simpson Problem

Differential equations

Set $D = \{a_1 = \infty, a_2, \ldots, a_n\}$, so $U = \mathbb{P}^1 - D$ is an open affine subset with co-ordinate algebra $\mathbb{C}[U] = \mathbb{C}[z]_{(z-a_2)\cdots(z-a_n)}$.

We consider homogeneous linear differential equations on U

$$g^{(n)} = c_{n-1}g^{(n-1)} + \cdots + c_1 g + c_0, \quad c_i \in \mathbb{C}[U]$$

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$$g^{(n)} = c_{n-1}g^{(n-1)} + \cdots + c_1 g + c_0, \quad c_i \in \mathbb{C}[U]$$

We can write this in matrix form

$$\frac{df}{dz} = Af \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{n-1} \end{pmatrix}$$

and $f = (g, g^{(1)}, \ldots, g^{(n-1)})^t \in \mathbb{C}[U]^n$.

Connections

Let $\Omega^1 = \mathbb{C}[U]dz$ be the module of Kähler differentials. We can reformulate in terms of connections

$$\nabla \colon \mathbb{C}[U]^n \to \mathbb{C}[U]^n \otimes \Omega^1, \quad \nabla(f) = df - (Af)dz$$

Solutions to the differential equation thus form the kernel of ∇.

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Definition (Deligne, 1970)

The system (U, ∇) has **regular singularities** provided there exists a locally free sheaf $E \in \operatorname{loc} \mathbb{P}^1$ with a log connection ∇' extending ∇ on U.

This is more flexible than the earlier Fuchsian system, which corresponds to taking $E = \mathcal{O}^n$.

Thus connections on U with regular singularities correspond to log connections on sheaves on $\mathbb{P}^1$.

Monodromy

Fix base point pt $\in U$. The space of germs of local holomorphic solutions near pt is an n-dimensional $\mathbb{C}$-vector space Sol.

Given a loop γ at pt, analytic continuation of solutions along γ defines a linear automorphism of Sol depending only on the homotopy class of γ.

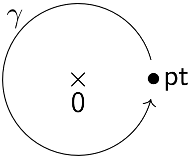
We obtain an action of $\pi_1(U, \mathrm{pt})$ on Sol, the **monodromy representation**.

Example

For example, $D = \{\infty, 0\}$, $\mathbb{C}[U] = \mathbb{C}[z]_z$, and take $A = \alpha/z$ with $\alpha \in \mathbb{C}$.

Differential equation is $z\dfrac{df}{dz} = \alpha f$.

Have $\pi_1(U, \mathrm{pt}) = \mathbb{Z}$, generator γ.



Solution λz^α is transformed into $\lambda e^{2\pi i\alpha} z^\alpha$.

Monodromy representation is $\pi_1(U, \mathrm{pt}) \to \mathbb{C}^\times$, $\gamma \mapsto e^{2\pi i\alpha}$.

Example

We keep $D = \{\infty, 0\}$ but take

$$z\frac{df}{dz} + Af = 0 \quad \text{for } A = \begin{pmatrix} 0 & -z^k \\ 0 & k \end{pmatrix}.$$

The solutions are

$$f = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} \log z \\ z^{-k} \end{pmatrix}, \quad a, b \in \mathbb{C}$$

and the monodromy representation is

$$\pi_1(U, p) \to \mathrm{GL}_2(\mathbb{C}), \quad \gamma \mapsto \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}.$$

Hypergeometric equation

The hypergeometric equation with parameters $(a, b, c) \in \mathbb{C}^3$ is

$$z(z-1)\frac{df}{dz} = Af \quad \text{for} \quad A = \begin{pmatrix} 0 & z(z-1) \\ ab & (a+b+1)z - c \end{pmatrix}$$

This has regular singularities at $D = \{\infty, 0, 1\}$.

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Riemann (1857) computed the monodromy when $c, a-b, c-a-b \notin \mathbb{Z}$:

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i c} \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} \frac{\mu-\theta}{\mu-1} & \frac{\mu(1-\theta)}{(\mu-1)^2} \\ \theta-1 & \frac{\mu\theta-1}{\mu-1} \end{pmatrix}$$

where $\quad \theta = e^{2\pi i(c-a-b)} \quad$ and $\quad \mu = \dfrac{\sin(\pi(c-a))\sin(\pi(c-b))}{\sin(\pi a)\sin(\pi b)}$

The Riemann – Hilbert Problem (Hilbert 21)

Question (RHP)

Can every finite dimensional $\pi_1(U, \mathrm{pt})$-representation be realised as the monodromy of a differential equation with regular singularities?

Restrict to regular singularities, otherwise too many possibilities:

If $D = \{\infty\}$, then $\pi_1(U, \mathrm{pt}) = 0$.

Every polynomial $c(z)$ gives a differential equation $\frac{df}{dz} = c(z)f$ whose solution is entire, so with vanishing monodromy.

Only $c = 0$ has regular singularities.

Deligne's Solution

Key Lemma

Given (U, ∇) there exists a locally free **analytic** sheaf E^{an} on $\mathbb{P}^1$ and a log connection extending ∇ on U.

Idea of proof

Locally one has punctured discs, with no patching conditions. Get log connection on an analytic sheaf E^{an}.

Applying Serre's GAGA Theorem (géométrie algébrique et géométrie analytique) we obtain an **algebraic** sheaf $E \in \operatorname{coh} \mathbb{P}^1$ with log connection extending ∇ on U.

The Riemann – Hilbert Correspondence

Last time: log connections correspond to sections of the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(-2) & \longrightarrow & A(E) & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \| & & \\ 0 & \longrightarrow & E(n-2) & \longrightarrow & M(E) & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

where $\sigma \in \mathbb{C}[u, v]$ has simple zeros at $a_1, \ldots, a_n$.

Get abelian category $\log\operatorname{conn}\mathbb{P}^1$ having objects (E, ∇) where $E \in \operatorname{coh}\mathbb{P}^1$ and ∇ is a log connection on E.

The Riemann – Hilbert Correspondence

Computed earlier the example $z\frac{df}{dz} = \alpha f$, yielding monodromy

$$\mathbb{Z} \to \mathbb{C}^\times, \quad 1 \mapsto \exp(2\pi i\alpha)$$

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Yesterday computed log connections (using $\sigma = uv$) for the same example:

For all $m \in \mathbb{Z}$ have log connection ∇ on $\mathcal{O}(m)$ given by

$$\nabla^{+}(s^{r}) = (\alpha + r)s^{r}, \quad \nabla^{-}(s^{-r}) = (\alpha + m - r)s^{-r}$$

having residues $\operatorname{Res}_0 \nabla = \alpha$ and $\operatorname{Res}_\infty \nabla = -(\alpha + m)$.

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Have map

$$\{(E, \nabla) \in \log\operatorname{conn}\mathbb{P}^1 \mid E \in \operatorname{loc}\mathbb{P}^1\} \to \operatorname{mod}\pi_1(U, \mathrm{pt})$$
$$(E, \nabla) \mapsto \exp(2\pi i \operatorname{Res}_0(\nabla))$$

The Riemann – Hilbert Correspondence

Fit a transversal T to $\mathbb{Z}$ in $\mathbb{C}$. Define $\log \operatorname{conn}_T \mathbb{P}^1$ to be subcategory of $\log \operatorname{conn} \mathbb{P}^1$ consisting of those (E, ∇) such that

1. E locally free
2. eigenvalues of $\operatorname{Res}_i \nabla$ lie in T

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Theorem

Have equivalence of categories

$$\log\operatorname{conn}_T \mathbb{P}^1 \cong \operatorname{mod} \pi_1(U, pt), \quad (E, \nabla) \mapsto \big(\exp(2\pi i \operatorname{Res}_j \nabla)\big)$$

In particular, $\log\operatorname{conn}_T \mathbb{P}^1$ *is an abelian category.*

This is the most basic version. More generally one compares

flat connections with regular singularities to local systems, or

regular holonomic D-modules to perverse sheaves.

The Deligne – Simpson Problem

Recall: $D = \{a_1, \ldots, a_n\}$, so

$$\pi_1(U, \mathrm{pt}) = \langle g_1, \ldots, g_n \mid g_1 \cdots g_n = 1 \rangle$$

An r-dimensional representation is

$$(M_1, \ldots, M_n) \in \mathrm{GL}_r(\mathbb{C})^n \quad \text{such that} \quad M_1 \cdots M_n = 1$$

Question (DSP)

Given conjugacy classes $\mathcal{C}_j \subset \mathrm{GL}_r(\mathbb{C})$ can we find matrices $M_j \in \mathcal{C}_j$ with

$$M_1 \cdots M_n = 1?$$

Call a solution **irreducible** if no common invariant subspace $0 \subset V \subset \mathbb{C}^r$.

The Deligne – Simpson Problem

Note: we have fixed a transversal T. So, given a conjugacy class $\mathcal{C} \subset \mathrm{GL}_r(\mathbb{C})$, there exists a unique conjugacy class $\mathcal{C}' \subset \mathrm{GL}_r(\mathbb{C})$ with eigenvalues in T such that

$$R \in \mathcal{C}' \iff \exp(2\pi i R) \in \mathcal{C}$$

In terms of connections the Deligne – Simpson Problem therefore asks:

what are the possible conjugacy classes of the residues?

Describing conjugacy classes

Let $\mathcal{C}$ be a conjugacy class in $M_r(\mathbb{C})$.

There exist scalars $\zeta_1, \ldots, \zeta_w \in \mathbb{C}$ and integers $d_1, \ldots, d_w \geq 0$ such that

$$M \in \mathcal{C} \quad \Longleftrightarrow \quad \operatorname{rank}(M - \zeta_1) \cdots (M - \zeta_p) = d_p \quad \forall\ p$$

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Proposition

The following are equivalent

1. $M \in \mathcal{C}$
2. *there exists flag* $0 = V_w \subseteq \cdots \subseteq V_1 \subseteq V_0 = \mathbb{C}^r$, $\dim V_p = d_p$ *such that*

$$(M - \zeta_p)(V_{p-1}) = V_p \quad \forall\ p$$

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Proposition

The following are equivalent

1. $M \in \mathcal{C}$ $\qquad M \in \bar{\mathcal{C}}$
2. *there exists flag* $0 = V_w \subseteq \cdots \subseteq V_1 \subseteq V_0 = \mathbb{C}^r$, $\dim V_p = d_p$ *such that*

$$(M - \zeta_p)(V_{p-1}) = V_p \quad \forall\ p \qquad (M - \zeta_p)(V_{p-1}) \subseteq V_p \quad \forall\ p$$

Parabolic sheaves revisited

Fix $\lambda_{jp} \in \mathbb{C}$. Take $\zeta_{jp} \in T$ with $\exp(2\pi i \zeta_{jp}) = \lambda_{jp}$,
and corresponding $\chi \colon K_0(\operatorname{coh} \mathbb{X}) \to \mathbb{C}$.

Theorem

Have forgetful functor

$$\operatorname{conn}_\zeta \operatorname{par} \mathbb{P}^1 \to \log \operatorname{conn}_T \mathbb{P}^1, \quad (E, V, \nabla) \mapsto (E, \nabla)$$

Get induced functor

$$\operatorname{conn}_\chi \operatorname{loc} \mathbb{X} \to \operatorname{mod} \pi_1(U, p)$$

The image is abelian, consisting of representations $(M_1, \ldots, M_n)$ *with*

$$(M_j - \lambda_{j1}) \cdots (M_j - \lambda_{jw_j}) = 0 \quad \forall\, j, p$$

Note: have fixed the eigenvalues λ_{jp} but not the integers (r, d_{jp}).

Reinterpreting DSP

The forgetful functor

$$F\colon \operatorname{conn}_\chi \operatorname{loc} \mathbb{X} \to \operatorname{mod} \pi_1(U, \mathrm{pt})$$

admits fully faithful left and right adjoints from its image.

It is almost a recollement, but $\operatorname{conn}_\chi \operatorname{loc} \mathbb{X}$ is only exact, not abelian.

Take left adjoint L, with counit $LF \to \mathrm{id}$ on $\operatorname{conn}_\chi \operatorname{loc} \mathbb{X}$.

Call $(\mathcal{E}, \nabla)$ **strict** if the counit is an isomorphism.

Reinterpreting DSP

Recall: have conjugacy classes $\mathcal{C}_j \subset \mathrm{GL}_r(\mathbb{C})$, with corresponding data (λ_{jp}, d_{jp}).

Have $\zeta_{jp} \in T$ and $\chi\colon K_0(\mathrm{coh}\,\mathbb{X}) \to \mathbb{C}$. Form dimension vector

$$d = re_* + \sum_{jp} d_{jp} e_{jp} \in K_0(Q_*).$$

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Theorem

$\exists\ (M_1, \dots, M_n) \in \operatorname{mod} \pi_1(U, p)$, $M_j \in \bar{\mathcal{C}}_j$ $\iff$ $\exists\ (\mathcal{E}, \nabla) \in \operatorname{conn}_\chi \operatorname{loc} \mathbb{X}$, $\underline{\dim}\, \mathcal{E} = d$

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$\exists (M_1, \dots, M_n) \in \operatorname{irrep} \pi_1(U, p)$, $M_j \in \mathcal{C}_j$ $\iff$ $\exists$ *simple* $(\mathcal{E}, \nabla) \in \operatorname{conn}_\chi \mathbb{X}$, $\underline{\dim}\, \mathcal{E} = d$

Change of perspective

Have therefore changed perspective.

Rather than fixing the conjugacy classes $\mathcal{C}_i$ themselves, we just consider their eigenvalues (with multiplicities).

In this way we obtain a length category $\mathrm{conn}_\chi \mathbb{X}$, and the question becomes:

> *what are the dimension vectors of the simple objects?*

Existence of connections

Consider the first part of the theorem.

We know that $\mathcal{E} \in \operatorname{loc} \mathbb{X}$ admits a χ-connection if and only if $\chi[\mathcal{E}'] = 0$ for all indecomposable direct summands $\mathcal{E}'$ of $\mathcal{E}$.

Next, we have

$$-\chi(a\partial + d) = a + d_* \sum_j \zeta_{j1} + \sum_{j,p} d_{jp}(\zeta_{jp+1} - \zeta_{jp})$$

Thus for each $d \in K_0(Q_*)$ there exists $a \in \mathbb{Z}$ with $\chi(a\partial + d) = 0$ if and only if $\chi(d) \in \mathbb{Z}$, in which case a is unique.

We therefore need to know the dimension vectors of the indecomposable locally free sheaves.

Analogue of Kac's Theorem

We write $\Phi(Q_*) \subset K_0(Q_*)$ for the set of (real and imaginary) roots.
This has a combinatorial description involving the Weyl group

Theorem (Kac)

Take $d \in K_0(Q_*)$ positive.
There exists an indecomposable $\mathbb{C}Q_*$-module of dimension vector d

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Theorem (Kac)

Take $d \in K_0(Q_*)$ positive.
There exists an indecomposable $\mathbb{C}Q_*$-module of dimension vector d

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Theorem (Crawley-Boevey)

Let $d \in K_0(Q_*)$ and $a \in \mathbb{Z}$.
There exists an indecomposable locally free sheaf on $\mathbb{X}$ of class $a\partial + d$

$$\iff \quad d \in \Phi(Q_*) \text{ with } d_* > 0.$$

Existence of connections

Putting this together we recover a result of Crawley-Boevey.

Proposition

There exists $(\mathcal{E}, \nabla) \in \mathrm{conn}_\chi \operatorname{loc} \mathbb{X}$ *of dimension vector* d

$$\iff \quad d = d_1 + \cdots + d_m \text{ with } d_i \in \Phi(Q_*),\ d_{i*} > 0,\ \chi(d_i) \in \mathbb{Z}.$$

Simple objects and DSP

Recall

$$-\chi(d) = d_* \sum_j \zeta_{j1} + \sum_{j,p} d_{jp}(\zeta_{jp+1} - \zeta_{jp})$$

Using that $\lambda_{jp} = \exp(2\pi i \zeta_{jp})$ we set

$$\lambda^{[d]} = \prod_j \lambda_{j1}^{d_*} \cdot \prod_{j,p} (\lambda_{jp+1}/\lambda_{jp})^{d_{jp}}$$

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Then

$$\chi(d) \in \mathbb{Z} \quad \Longleftrightarrow \quad \lambda^{[d]} = 1$$

We write Φ_χ or Φ_λ for those $d \in \Phi(Q_*)$, $d_* > 0$ which satisfy this condition.

Simple objects and DSP

Thus: if $d \in \Phi_\chi$, then there exists $(\mathcal{E}, \nabla) \in \operatorname{conn}_\chi \operatorname{loc} \mathbb{X}$ with $\mathcal{E}$ indecomposable and $\underline{\dim}\, \mathcal{E} = d$.

This is not enough to ensure a simple object.

We need a further restriction involving the Euler form $\{-,-\}$ on $K_0(Q_*)$ (cf. Crawley-Boevey's description of simples for def. preproj. algebras).

Simple objects and DSP

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Set $p(d) = 1 - \{d, d\}$. Write Σ_χ (or Σ_λ) for the set of $d \in \Phi_\chi$ such that

$$p(d) > p(d_1) + \cdots + p(d_m)$$

whenever

$$d = d_1 + \cdots + d_m \quad \text{with} \quad d_i \in \Phi_\chi$$

Simple objects and DSP

Theorem (Crawley-Boevey, H)

There exists simple $(\mathcal{E}, \nabla) \in \mathrm{conn}_\chi \mathbb{X}$ *of dimension vector d*

$$\iff \quad d \in \Sigma_\chi.$$

There exists $(M_1, \ldots, M_n) \in \mathrm{irrep}\,\pi_1(U, pt)$ *with* $M_i \in \mathcal{C}_i$

$$\iff \quad \lambda^{[d]} = 1.$$

Hypergeometric equation

Recall the hypergeometric equation

$$z(z-1)\frac{df}{dz} = Af \quad \text{for} \quad A = \begin{pmatrix} 0 & z(z-1) \\ ab & (a+b+1)z-c \end{pmatrix}$$

having regular singularities at $D = \{0, 1, \infty\}$.

Assume $c, a-b, c-a-b \notin \mathbb{Z}$. Monodromy (M_0, M_1, M_∞) satisfies

$$M_0 \sim \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i c} \end{pmatrix}, \quad M_1 \sim \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(c-a-b)} \end{pmatrix}, \quad M_\infty \sim \begin{pmatrix} e^{2\pi i a} & 0 \\ 0 & e^{2\pi i b} \end{pmatrix}$$

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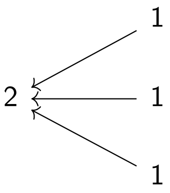
Assume $c, a-b, c-a-b \notin \mathbb{Z}$. Monodromy (M_0, M_1, M_∞) satisfies

$$M_0 \sim \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi ic} \end{pmatrix}, \quad M_1 \sim \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(c-a-b)} \end{pmatrix}, \quad M_\infty \sim \begin{pmatrix} e^{2\pi ia} & 0 \\ 0 & e^{2\pi ib} \end{pmatrix}$$

Corresponding quiver Q_*, dimension vector d, scalars λ_{jp} are

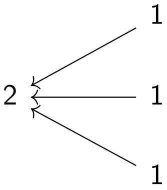


$$\lambda_{11} = 1 \qquad \lambda_{12} = e^{-2\pi ic}$$

$$\lambda_{21} = 1 \qquad \lambda_{22} = e^{2\pi i(c-a-b)}$$

$$\lambda_{31} = e^{2\pi ia} \qquad \lambda_{32} = e^{2\pi ib}$$

Hypergeometric equation



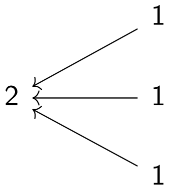
$\lambda_1 = 1$ $\qquad \mu_1 = e^{-2\pi i c}$

$\lambda_2 = 1$ $\qquad \mu_2 = e^{2\pi i(c-a-b)}$

$\lambda_3 = e^{2\pi i a}$ $\qquad \mu_3 = e^{2\pi i b}$

Irrep provided $d = (2, 1, 1, 1)$ lies in Σ_λ. Check: $\lambda^{[d]} = \prod_j \lambda_j \mu_j = 1$.

Hypergeometric equation



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Irrep provided $d = (2, 1, 1, 1)$ lies in Σ_λ. Check: $\lambda^{[d]} = \prod_j \lambda_j \mu_j = 1$.

Have $p(d') = 0 \ \forall \ d' \in \Phi(Q_*)$. So $d \in \Sigma_\lambda \Leftrightarrow d' \notin \Phi_\lambda \ \forall \ d' < d$.

Hypergeometric equation

Smaller roots are $(1,1,1,1)$, $(1,1,1,0)$, $(1,1,0,0)$, $(1,0,0,0)$ and permutations of last three entries. Requirement is

$$\mu_1\mu_2\mu_3, \quad \mu_1\mu_2\lambda_3, \quad \mu_1\lambda_2\lambda_3, \quad \lambda_1\lambda_2\lambda_3$$

and permutations all different from 1. Equivalently, $a, b, c-a, c-b \notin \mathbb{Z}$.

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and permutations all different from 1. Equivalently, $a, b, c - a, c - b \notin \mathbb{Z}$.

Thus generically, if $a, b, c, a - b, a - c, b - c, a + b - c \notin \mathbb{Z}$, there is an irreducible monodromy representation (M_0, M_1, M_∞) with

$$M_0 \sim \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i c} \end{pmatrix}, \quad M_1 \sim \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(c-a-b)} \end{pmatrix}, \quad M_\infty \sim \begin{pmatrix} e^{2\pi i a} & 0 \\ 0 & e^{2\pi i b} \end{pmatrix}$$

and the corresponding differential equation is exactly the hypergeometric equation.

Local description

Let $\mathscr{A} \subset \operatorname{coh} \mathbb{X}$ be a thick abelian length subcategory, with finitely many simples $\mathcal{A}_1, \ldots, \mathcal{A}_m$.

Theorem

There exists a finite quiver Q having m vertices such that

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Idea of proof

The natural constructions yield fully faithful $\mathscr{A} \to \operatorname{mod}_0 \mathbb{C}Q$ which is an equivalence on objects having Loewy length at most 2.

Since both categories are hereditary abelian, we get that the categories are equivalent.

Local description

We want to lift the equivalence $\mathscr{A} \cong \mathsf{mod}_0\,\mathbb{C}Q$ to the full subcategory $\mathsf{conn}_\chi\,\mathscr{A} \subseteq \mathsf{conn}_\chi\,\mathbb{X}$, having objects those $(\mathcal{E}, \nabla)$ with $\mathcal{E} \in \mathscr{A}$.

The equivalence $\mathscr{A} \cong \mathsf{mod}_0\,\mathbb{C}Q$ yields an isomorphism of their Grothendieck groups

$$\mathbb{Z}^m = K_0(Q) \cong K_0(\mathscr{A}) \subseteq K_0(\mathsf{coh}\,\mathbb{X}).$$

Thus χ determines a map $\lambda\colon \mathbb{Z}^m \to \mathbb{C}$.

We can use λ to define a deformed preprojective algebra $\Pi^\lambda Q$.

Deformed preprojective algebras

Let the quiver Q have arrows a. The **double quiver** $\bar{Q}$ has the same vertices, but we adjoin new arrows a^* by reversing the orientation of a.

We identify $\lambda\colon \mathbb{Z}^m \to \mathbb{C}$ with an element of $\mathbb{C}^m \subseteq \mathbb{C}\bar{Q}$.

The **deformed preprojective algebra** is

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The **deformed preprojective algebra** is

$$\Pi^\lambda Q = \mathbb{C}\bar{Q} / \big(\sum (aa^* - a^*a) - \lambda \big)$$

Normally, modules for $\Pi^\lambda Q$ are regarded as tuples of matrices (M_a, M_{a^*}) satisfying the required condition.

This is of no use if we want to relate them to $\operatorname{conn}_\chi \mathscr{A}$. We need a new description.

Modules for preprojective algebras

Suppose $\mathbb{C}Q$ is finite dimensional. The (ordinary) preprojective algebra

$$\Pi Q = \mathbb{C}\bar{Q}/\big(\sum (aa^* - a^*a)\big)$$

is isomorphic to the tensor algebra $T_{\mathbb{C}Q}(\tau^- \mathbb{C}Q)$.

Thus finite dimensional modules can be regarded as pairs (M, g) such that $M \in \operatorname{mod} \mathbb{C}Q$ and $g\colon M \to \tau^- M$.

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This is the formulation we want to generalise. Two problems:

1. $\mathbb{C}Q$ will not be finite dimensional. What takes the role of $\tau^- M$?
2. $\lambda \neq 0$.

Modules for deformed preprojective algebras

When $\mathbb{C}Q$ is finite dimensional, we construct $\tau^- M$ by applying $D\,\mathrm{Hom}_{\mathbb{C}Q}(-, \mathbb{C}Q)$ to the standard presentation of M.

This exhibits $\tau^- M$ as the kernel of a natural map between injective modules.

We are only interested in nilpotent modules, so we pass to an appropriate subcategory of $\mathrm{Mod}\,\mathbb{C}Q$ having enough injectives.

Modules for deformed preprojective algebras

We define $\operatorname{Mod}_0 \mathbb{C}Q$ to be the subcategory of **locally nilpotent modules**, so those $M = \operatorname{Mod} \mathbb{C}Q$ which are the union of their finite dimensional nilpotent submodules.

This is a Grothendieck category, so has enough injectives. It is equivalent to the completion of $\operatorname{mod}_0 \mathbb{C}Q$ under filtered colimits.

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Note: completing $\operatorname{coh} \mathbb{X}$ under filtered colimits yields the category of **quasi-coherent sheaves**.

Modules for deformed preprojective algebras

We have a functorial projective resolution of our nilpotent M

$$0 \longrightarrow P_M^1 \longrightarrow P_M^0 \longrightarrow M \longrightarrow 0$$

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This is a functorial construction yielding modules

$$I_M^1 = \varinjlim D\operatorname{Hom}_{\mathbb{C}Q}(P_M^1,\Lambda_n) \quad \text{and} \quad I_M^0 = \varinjlim D\operatorname{Hom}_{\mathbb{C}Q}(P_M^0,\Lambda_n)$$

which are both relative injective in $\operatorname{Mod}_0 \mathbb{C}Q$, together with a natural map

$$j_M \colon I_M^1 \to I_M^0$$

Modules for deformed preprojective algebras

If $N \in \operatorname{mod}_0 \mathbb{C}Q$, then there is a natural isomorphism

$$\operatorname{Hom}_{\mathbb{C}Q}(N, I_M^0) \cong \operatorname{Hom}_{\mathbb{C}^m}(N, M)$$

The element $\lambda \in \mathbb{C}^m$ yields an endomorphism of M as $\mathbb{C}^m$-module. It acts as multiplication by λ_i on the vector space M_i. We obtain $\tilde{\lambda}_M \colon M \to I_M^0$.

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Write $\operatorname{mod}_0 \mathbb{C}\bar{Q}$ and $\operatorname{mod}_0 \Pi^\lambda Q$ for those modules whose restriction to $\mathbb{C}Q$ is nilpotent.

Theorem

There is an equivalence of categories

$$\operatorname{mod}_0 \mathbb{C}\bar{Q} \cong \{(M, f) \mid M \in \operatorname{mod}_0 \mathbb{C}Q,\ f \colon M \to I_M^1\}$$

This induces an equivalence of categories

$$\operatorname{mod}_0 \Pi^\lambda Q \cong \{(M, f) \mid M \in \operatorname{mod}_0 \mathbb{C}Q,\ f \colon M \to I_M^1,\ j_M f = \tilde{\lambda}_M\}$$

Local description

Theorem

We can lift the equivalence

$$\mathscr{A} \cong \operatorname{mod}_0 \mathbb{C}Q$$

to an equivalence

$$\operatorname{conn}_\chi \mathscr{A} \cong \operatorname{mod}_0 \Pi^\lambda Q$$

This allows us locally to use Crawley-Boevey's classification of simple $\Pi^\lambda Q$-modules.

Global description

We still need to reduce to a suitable length subcategory $\mathscr{A}$. For this we need reflection functors.

Luckily, this was done by Crawley-Boevey and Shaw (in terms of multiplicative preprojective algebras), based on middle convolution introduced by Katz when studying DSP.

Putting this all together yields the classification of simple objects in $\mathrm{conn}_\chi \mathbb{X}$, and hence the DSP.

Thank you !