Self-orthogonal modules and Tachikawa's second conjecture

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Contents

Global notation

Joint work with Changchang Xi (CNU).

In the talk:
A: Artin algebra (e.g. finite-dim. k-algebra over a field k);
A-mod: the category of finitely generated (left) A-modules;
D: the usual duality over A-mod (e.g. $D = \mathrm{Hom}_k(-, k)$);
$D(A)$: usual duality A-A-bimodule.

§ 1. Classical homological conjectures

(NC) **Nakayama Conjecture** [Nakayama, 1958]:
If A has infinite dominant dimension $\Longrightarrow$ A: self-injective.

Dominant dimension of A:

$$\mathrm{domdim}(A) := \sup\{n \mid I^j : \text{projective } \ \forall\, 0 \leq j < n\}$$

where $0 \to {}_AA \to I^0 \to I^1 \to \cdots \to I^{n-1} \to I^n \to \cdots$ is a minimal injective coresolution.

A is **self-injective** if projectives = injectives.

How to describe infinite dominant dimension?

Morita-Tachikawa; Mueller(1968):

Theorem

Let Λ *be an algebra. Then* $\mathrm{domdim}(\Lambda) = \infty \Longleftrightarrow \Lambda = \mathrm{End}_A(M)$, *where* $M \in A\text{-mod}$*:*

- *generator (i.e.* $A \in \mathrm{add}(M)$*);*
- *cogenerator (i.e.* $D(A_A) \in \mathrm{add}(M)$*);*
- *self-orthogonal (i.e.* $\mathrm{Ext}_A^n(M, M) = 0$ *for any* $n \geq 1$*).*

Tachikawa's conjectures

(TC1) **Tachikawa's First Conjecture** [Tachikawa, 1973]:
If $\mathrm{Ext}_A^n(D(A), A) = 0 \ \forall\, n \geq 1 \Longrightarrow A$: self-injective.

In (TC1), the A-module $A \oplus D(A)$ is self-orthogonal.

(TC2) **Tachikawa's Second Conjecture** [Tachikawa, 1973]:
Let A be self-injective and $M \in A$-mod.
If ${}_AM$ is self-orthogonal $\Longrightarrow M$: projective.

Proposition

(NC) holds for all algebras $\Leftrightarrow$ (TC1)+(TC2) hold for all algebras.

Tachikawa's Second Conjecture (TC2)

TC2

Let A be self-injective and $M \in A$-mod.
If ${}_AM$ is self-orthogonal $\Longrightarrow$ M: projective.

In (TC2), we can assume M is a generator (e.g. ${}_AM = A \oplus M_0$).

Lemma

The pair (A, M) *satisfies* (TC2) $\Leftrightarrow$ $\mathrm{End}_A(M)$ satisfies (NC).

(TC2) holds for (A, M) where ${}_AM$ is **arbitrary**, but A is

- symmetric alg./local self-injective alg. with radical$^3 = 0$ [Hoshino, 1984];
- group alg. of a finite group [Schulz, 1986];
- self-injective alg. of finite represent. type [Schulz, 1986].

§ 2. Tachikawa's Second Conjecture

Aim of the talk

Understand (TC2) via the exploration of homological properties of self-orthogonal modules over self-injective algebras.

- Provide equivalent characterizations of (TC2).
- Introduce two new homological conditions.
  - Introduce Gorenstein-Morita algebras.
  - Gorenstein-Morita algebras satisfy Nakayama Conjecture.

Notation

A: arbitrary self-injective algebra (i.e. Projectives = Injectives);

A-Mod: the cat. of (left) A-modules;

Nakayama functor: $\nu_A = {}_AD(A) \otimes_A - : A\text{-Mod} \xrightarrow{\simeq} A\text{-Mod}$.

$M \in A$-mod: generator;

$\mathrm{add}\,({}_AM)$ (resp., $\mathrm{Add}\,({}_AM)$): direct summands of finite (resp., arbitrary) direct sums of copies of M;

A-Proj: the cat. of projective A-modules.

Definition

${}_AM$ *is* **Nakayama-stable** *if* $\mathrm{add}\,({}_AM) = \mathrm{add}\,(\nu_A(M))$.

(1) If A: symmetric algebra (i.e. $D(A) \simeq {}_AA_A$), then $\nu_A \simeq \mathrm{Id}$.

(2) $D\underline{\mathrm{Hom}}_A(M, -) \simeq \underline{\mathrm{Hom}}_A(-, \nu_A(M)[-1])$ by the Auslander-Reiten formula.

Full subcategories of A-Mod determined by M

$^{\perp>0}M := \{X \in A\text{-Mod} \mid \mathrm{Ext}_A^n(X, M) = 0, \forall n > 0\};$

$M^{\perp>0} := \{X \in A\text{-Mod} \mid \mathrm{Ext}_A^n(M, X) = 0, \forall n > 0\};$

$\mathscr{G} := {}^{\perp>0}M \cap M^{\perp>0};$

$\mathscr{G}^{\mathbf{fin}} := \mathscr{G} \cap A\text{-mod};$

$\varinjlim \mathscr{G}^{\mathrm{fin}}$: filtered colimits in A-Mod of modules from $\mathscr{G}^{\mathrm{fin}}$.

If $M = A$, then $\mathscr{G}^{\mathrm{fin}} = A$-mod and $\mathscr{G} = A\text{-Mod} = \varinjlim \mathscr{G}^{\mathrm{fin}}$.

Relative stable category

Definition

The M-stable category $A\text{-Mod}/[M]$ *of* $A\text{-Mod}$*:*
Objects*: A-modules;*
Morphisms*:* $\forall\, X, Y \in A\text{-Mod}$,

$$\underline{\mathrm{Hom}}_{M}(X,Y) := \mathrm{Hom}_{A}(X,Y)/\mathcal{M}(X,Y)$$

where $\mathcal{M}(X,Y)$ *consists of homos. factorizing through objects in* $\mathrm{Add}(M)$.

$A\text{-Mod}/[A] = A\text{-}\underline{\mathrm{Mod}}$ (stable module category of A)

Gorenstein projective modules

Definition

A module Y over an Artin algebra B is **Gorenstein projective** *if $\exists$ exact complex of projective B-modules*

$$P^\bullet: \quad \cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow P^2 \longrightarrow \cdots$$

s.t. $\mathrm{Im}\,(d^0) = Y$ *and the complex* $\mathrm{Hom}^\bullet_B(P^\bullet, B)$ *is exact.*

Notation: B-GProj (resp., B-Gproj): the cat. of (resp., finitely generated) Gorenstein-projective B-modules.

B-GProj is a triangulated category!

Compact objects in categories

$\mathscr{C}$: additive category with set-indexed coproducts.

Definition

An object $X \in \mathscr{C}$ *is* *compact* *if* $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathscr{C} \to \mathbb{Z}\text{-Mod}$ *commutes with coproducts.*

$\mathscr{C}^c$: *the subcat. of* $\mathcal{C}$ *consisting of compact objects.*

For example, $A\text{-}\underline{\mathrm{Mod}}^c = A\text{-}\underline{\mathrm{mod}}$ and $B\text{-}\underline{\mathrm{Gproj}} \subseteq B\text{-}\underline{\mathrm{GProj}}^c$.

What happens if M is orthogonal?

$\mathscr{G} := {}^{\perp>0}M \cap M^{\perp>0}$ and $\mathscr{G}^{\mathbf{fin}} := \mathscr{G} \cap A\text{-mod}$.

Proposition

If ${}_AM$: self-orthogonal (i.e. $M \in {}^{\perp>0}M$)

$\Longrightarrow$

(1) $\mathscr{G}$ *(resp., $\mathscr{G}^{\mathrm{fin}}$) is a* **Frobenius category**. *Its full subcategory of projective-injective objects equals* $\mathrm{Add}\,(M)$ *(resp.,* $\mathrm{add}\,(M)$*).*

(2) *Let* $\Lambda := \mathrm{End}_A(M)$.

$\Longrightarrow$

$\mathrm{Hom}_A(M, -) : A\text{-Mod} \to \Lambda\text{-Mod}$ *induces triangle equivalences*

$$\mathscr{G}/[M] \xrightarrow{\simeq} \Lambda\text{-}\underline{\mathrm{GProj}} \quad \text{and} \quad \mathscr{G}^{\mathrm{fin}}/[M] \xrightarrow{\simeq} \Lambda\text{-}\underline{\mathrm{Gproj}}.$$

Relative compact and filtered modules

Let ${}_AM$: self-orthogonal generator.

Definition

(1) *An A-module X is M-compact if it is a compact object in* $\mathscr{G}/[M]$.

(2) *An A-module X is M-filtered if it has a countable filtration in A*-Mod

$$0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$$

such that $X = \bigcup_{n=0}^{\infty} X_n$ *and* X_{n+1}/X_n *is isomorphic to a finite direct sum of A-modules in the set* $\{{}_AA\} \cup \{\Omega_A^{-i}(M) \mid i \in \mathbb{N}\}$.

- $\forall\ X \in \mathscr{G}^{\mathbf{fin}} := \mathscr{G} \cap A$-mod is M-compact.
- A-compact modules = A-mod (up to stable equivalence); A-filtered modules are projective.

Minimal left approximations

$\mathcal{C}$: additive category, $\mathcal{B}$: full subcat. of $\mathcal{C}$.

Definition

A morphism $f : X \to B$ (or the object B) in $\mathcal{C}$ is called a minimal left $\mathcal{B}$-approximation of X if

- $B \in \mathcal{B}$,
- $\operatorname{Hom}_{\mathcal{C}}(f, B') : \operatorname{Hom}_{\mathcal{C}}(B, B') \to \operatorname{Hom}_{\mathcal{C}}(X, B')$ *is surjective* $\forall\, B' \in \mathcal{B}$,
- $g \in \operatorname{End}_{\mathcal{C}}(B)$ *is an isomorphism whenever* $f = fg$.

Minimal left $\mathscr{G}$-approximations of modules

Assumptions:
A: self-injective Artin algebra,
M: self-orthogonal, Nakayama-stable generator for A-mod.
Let

$\Omega_A^-(M) \to W$: minimal left $\mathscr{G}$-approximation of $\Omega_A^-(M)$,
$\mathscr{M} := \{X \in A\text{-mod} \mid M\text{-}resdim(X) < \infty\}$.

Definition

M-resdim$(X) < \infty \Leftrightarrow \exists$ exact sequence in A-mod

$$0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

with $M_i \in \operatorname{add}(M)$ for $0 \leq i \leq n$.

Remk: $\mathscr{M} \subseteq \mathscr{G}^{\perp>0} \subseteq W^{\perp 1}$ and $\varinjlim \mathscr{G}^{\text{fin}} \subseteq \mathscr{G}$.

Equivalent characterizations of TC2

The same assumptions:

A: self-injective Artin algebra,

M: self-orthogonal, Nakayama-stable generator for A-mod.

Theorem

The following statements are equivalent:

(1) ${}_AM$ *is projective (i.e. TC2 holds for* (A, M)*).*

(2) $\mathscr{G} = \varinjlim \mathscr{G}^{\text{fin}}$.

(3) *Any* M*-compact and* M*-filtered* A*-module lies in* $\operatorname{Add}(M)$.

(4) $W \in \varinjlim \mathscr{G}^{\text{fin}}$.

(5) $\operatorname{Ext}^1_A(W, \bigoplus_{i \in \mathbb{N}} M_i) = 0$ *for all* $M_i \in \mathscr{M}$.

Comments on the theorem

Remark

In the theorem:

- (2) holds $\Leftrightarrow$ $\operatorname{End}_A(M)$ is virtually Gorenstein.
- Any *finitely generated*, M-compact and M-filtered A-module lies in $\operatorname{add}(M)$. But (3) asks for the situation of *countably generated* modules.
- (4) and (5) hold if W is the direct sum of finitely generated A-modules (i.e. W is *pure-projective*).

Definition (Beligiannis, 2005)

B is **virtually Gorenstein** *if* $B\text{-GProj}^{\perp>0} = {}^{\perp>0}B\text{-GInj}$.

B-GInj: the cat. of Gorenstein-injective B-modules.

Modules of finite projective dimension

$$\mathscr{P}^{<\infty}(B) := \{Y \in B\text{-mod} \mid \text{proj.dim}\,(Y) < \infty\}$$

$$\text{fin.dim}\,(B) := \sup\{\text{proj.dim}\,(Y) \mid Y \in \mathscr{P}^{<\infty}(B)\}.$$

Finitistic Dimension Conjecture (FDC) [Rosenberg, Zelinsky; Bass, 1960]: $\forall$ Artin algebra B, $\text{fin.dim}\,(B) < \infty$.

Remark

For a given algebra, (FDC) $\Longrightarrow$(NC).

Observations

$$\mathscr{P}^{<\infty}(B) \cap B\text{-Gproj} = B\text{-proj},$$

$$B\text{-}\underline{\text{Gproj}} \subseteq B\text{-}\underline{\text{GProj}}^{c}.$$

Extend "finitely generated" modules to "compactly generated" modules

A generalization of objects in $\mathscr{P}^{<\infty}(B)$:

Definition

*A B-module X is compactly filtered if it has a countable filtration in B-*Mod

$$0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$$

such that $X = \bigcup_{n=0}^{\infty} X_n$ *and* $X_{n+1}/X_n \in \mathscr{P}^{<\infty}(B)$, $\forall n \in \mathbb{N}$.

A generalization of objects in B-$\underline{\text{Gproj}}$:

Definition

A B-module X is **compactly Gorenstein-projective** *if it is compact in B-$\underline{\text{GProj}}$.*

Two new homological conditions

Notation:
B-CF: compactly filtered B-modules.
B-Proj_ω: countably generated projective B-modules.
B-GProj^c: compactly Gorenstein-projective B-modules.
B-GProj_ω: countably generated Gorenstein-projective B-modules.

(HC1): $\mathrm{Ext}_B^{>0}\big(B\text{-}\mathrm{GProj}_\omega, \bigoplus_{i\in\mathbb{N}} M_i\big) = 0$ for all $M_i \in \mathscr{P}^{<\infty}(B)$.

(HC2): $B\text{-CF} \cap B\text{-}\mathrm{GProj}^c = B\text{-}\mathrm{Proj}_\omega$.

Remk: $\mathscr{P}^{<\infty}(B) \subseteq B\text{-}\mathrm{GProj}_\omega^{\perp>0}$.

Implications of homological conditions

Lemma

(1) $\operatorname{fin.dim}(B) < \infty \Longrightarrow$ (HC1) $\Longrightarrow$ (HC2).

(2) $B\text{-GProj}^{\perp>0} \subseteq B\text{-Mod}$: *closed under direct sums* $\Longrightarrow$ (HC1).

(3) *If B is* **virtually Gorenstein**, *then* (HC1) *holds.*

(4) (HC2) *is preserved under*

- *derived equivalences,*
- *stable equivalences of Morita type,*
- *certain singular equivalences of Morita type with level.*

Gorenstein-Morita algebras

Definition

An algebra B is a **Gorenstein-Morita** *algebra if*

- $B = \text{End}_A(M)$ *where*
  - *A: self-injective algebra;*
  - *M: Nakayama-stable generator for A-mod.*
- B *satisfies* (HC2) : $B\text{-CF} \cap B\text{-GProj}^c = B\text{-Proj}_\omega$.

Example: **gendo-symmetric algebras** (that is, endomorphism algebras of generators over symmetric algebras, introduced by Ming Fang and Steffen Koenig in 2011) which are *virtually Gorenstein* or have *finite finitistic dimension.*

Nakayama Conjecture for Gorenstein-Morita algebras

Corollary

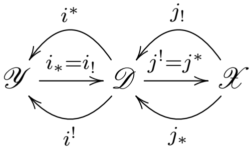
Let B be a Gorenstein-Morita algebra.
If $\operatorname{domdim}(B) = \infty$*, then B is self-injective.*
In particular, any gendo-symmetric, virtually Gorenstein algebra with infinite dominant dimension is symmetric.

Open questions:
Does (HC1) (resp., (HC2)) hold for all Artin algebras?

A counterexample to (HC1) or (HC2) will lead to a counterexample to (FDC).

§ 3. Recollements of (relative) stable categories

Beilinson, Bernstein and Deligne [1982]:
A recollement $(\mathscr{Y}, \mathscr{D}, \mathscr{X})$:



- 6 triangle functors;
- 4 adjoint pairs: $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*);
- 3 fully faithful functors (pointing to $\mathscr{D}$, e.g. i_*);
- 3 zeros of composition (along the same level, e.g. $i^*j_! = 0$);
- 2 triangles: $\forall\, X \in \mathscr{D}$, $\exists$ triangles in $\mathscr{D}$:

$$j_!j^!(X) \xrightarrow{counit} X \xrightarrow{unit} i_*i^*(X) \longrightarrow j_!j^!(X)[1],$$
$$i_!i^!(X) \xrightarrow{counit} X \xrightarrow{unit} j_*j^*(X) \longrightarrow i_!i^!(X)[1].$$

Self-orthogonal generators over self-injective algebras

Assumptions:
A: self-injective Artin algebra;
M: self-orthogonal, Nakayama-stable generator for A-mod.

Notation:
$\Gamma := \underline{\mathrm{End}}_A(M)$: the endomorphism algebra of M in A-$\underline{\mathrm{Mod}}$;
$\mathscr{G} := {}^{\perp>0}M \cap M^{\perp>0}$;
$(\mathscr{G} = \{X \in A\text{-Mod} \mid \underline{\mathrm{Hom}}_A(M[n], X) = 0, \forall\, n \neq 0, 1\})$

$\mathscr{E} := \{X \in \mathscr{G} \mid \underline{\mathrm{Hom}}_A(M, X), \underline{\mathrm{Hom}}_A(M[1], X) \in \Gamma\text{-mod}\}$;

$\underline{M}^{\perp} := \{X \in A\text{-}\underline{\mathrm{Mod}} \mid \underline{\mathrm{Hom}}_A(M, X[n]) = 0,\ \forall\, n \in \mathbb{Z}\} \subseteq \underline{\mathscr{E}}$;

$\pi : \underline{\mathscr{G}} = \mathscr{G}/[A] \to \mathscr{G}/[M]$: the quotient functor.

Thick subcategories of module categories

Definition

A full subcat. $\mathcal{U} \subseteq A\text{-Mod}$ *is* **thick** *if*

- *it is closed under direct summands in* A-Mod*;*
- $\forall\ 0 \to X \to Y \to Z \to 0$ *exact seq. in* A-Mod *with two terms in* $\mathcal{U}$*, the third term also belongs to* $\mathcal{U}$*.*

Notation:

$\mathscr{S}$: the smallest **thick** subcat. of A-Mod containing M and being closed under **direct sums**.

- $\underline{\mathscr{S}}$: localizing subcat. of A-$\underline{\text{Mod}}$ containing M.
- If $M = A$, then $\mathscr{S} = A$-Proj.

Recollements of relative stable categories

Theorem

∃ a recollement of triangulated categories:

$$\underline{M}^{\perp}\ \begin{array}{c} \xleftarrow{\ \ \widetilde{\Psi}\ \ } \\ \xrightarrow{\ \pi\circ\mathrm{inc}\ } \\ \xleftarrow[\ \ \widetilde{\Psi'}\ \ ]{} \end{array}\ \mathscr{G}/[M]\ \begin{array}{c} \xleftarrow{\ \ \mathrm{inc}\ \ } \\ \xrightarrow{\ \ \widetilde{\Phi}\ \ } \\ \xleftarrow[\ \ \Phi''\ \ ]{} \end{array}\ (\mathscr{G}\cap\mathscr{S})/[M]$$

which restricts to a recollement

$$\underline{M}^{\perp}\ \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array}\ \mathscr{E}/[M]\ \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array}\ (\mathscr{E}\cap\mathscr{S})/[M].$$

We construct the above labelled functors explicitly (via the information of $\mathrm{End}_A(M)$).

Compact objects and compact generating sets

Proposition

(1) *Let* $X \in \mathscr{E} \cap \mathscr{S}$. *Then* X *is M-compact and isomorphic in* A-$\underline{\mathrm{Mod}}$ *to an M-filtered module. If* X *is finitely generated, then* $X \in \mathrm{add}\,(M)$.

(2) *Let* $\mathcal{S}$ *be the set of iso. classes of simple objects of the heart of a torsion pair in* A-$\underline{\mathrm{Mod}}$ *defined by* M. *Then*

$$(\mathscr{G} \cap \mathscr{S})/[M] = \langle \mathrm{Add}\,(\mathcal{S})\rangle_{2n}^{\{0,1\}},$$

$$((\mathscr{G} \cap \mathscr{S})/[M])^{c} = (\mathscr{E} \cap \mathscr{S})/[M] = \langle \mathcal{S}\rangle_{2n}^{\{0,1\}}.$$

where $\mathcal{S}$ *is a finite set and* n *is the Loewy length of* Γ.

Remk: ${}_A M$ is projective $\Leftrightarrow$ $(\mathscr{G} \cap \mathscr{S})/[M] = 0$ $\Leftrightarrow$ $(\mathscr{E} \cap \mathscr{S})/[M] = 0$.

Related papers

[1] H.X.Chen and C.C.Xi, Homological theory of orthogonal modules, 1-40, arXiv:2208.14712.
[2] H.X.Chen, Ming Fang and C.C.Xi, Tachikawa's second conjecture, derived recollements, and gendo-symmetric algebras, arXiv:2211.08037.

Thank you very much!