Groupoids from moduli space of quadratic differentials on Riemann surfaces

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Motivations

Theorem (Bridgeland-Smith)

Let $\mathcal{D}$ be the Calabi-Yau 3 triangulated categories associated with a marked surface $(\mathbf{S}, \mathbf{M})$. There is an isomorphism of complex manifolds/orbifolds

$$\mathrm{Stab}^{\circ}(\mathcal{D}) / \mathrm{Aut}^{\circ}(\mathcal{D}) \cong \mathrm{Quad}_{\heartsuit}(\mathbf{S}, \mathbf{M})$$

Aim: Understand the topology/fundamental groupoid of $\mathrm{Stab}^{\circ}(\mathcal{D}) / \mathrm{Aut}^{\circ}(\mathcal{D})$ via moduli space of (GMN) meromorphic quadratic differentials on Riemann surface.

Recent developments: [Haiden-Katzarkov- Kontsevich], [King-Qiu],[Barbieri- Möller-Qiu-So], [Christ-Haiden-Qiu],[Qiu24].

Quadratic differentials

Definition

Let S be a Riemann surface. A *meromorphic quadratic differential* ϕ on S is a meromorphic section of the line bundle $\omega_S^{\otimes 2}$, where ω_S is the canonical bundle of S. In a local coordinate z on S, we have that

$$\phi = \psi(z) dz^{\otimes 2}$$

where $\psi(z)$ is a meromorphic function.

Example

A differential on Riemann sphere $\mathbb{P}^1$,

$$\frac{(az^2 + bz + c)}{z^2(z-1)^2} dz^2, a, c \in \mathbb{C}^*$$

there are 3 double poles at $0, 1, \infty$ and 2 zeros.

The moduli space

Remark

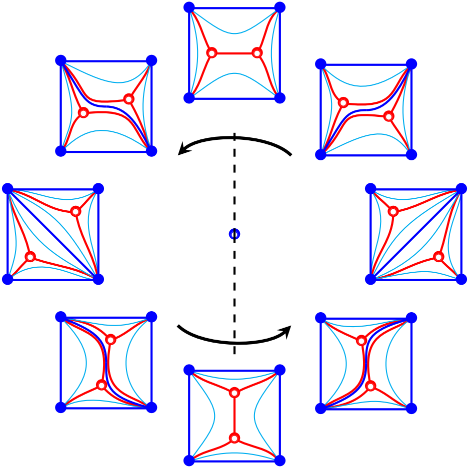
Two quadratic differentials ϕ_1, ϕ_2 on surfaces S_1 and S_2 are equivalent if there is a biholomorphism f: $S_1 \to S_2$ such that $f*(\phi_2) = \phi_1$.

For Riemann surfaces of genus g, we denote the moduli space (under equivalence of quadratic differentials) quadratic differentials by $\mathcal{Q}_g$. There is a stratification of $\mathcal{Q}_g$ by the orders of zeros and poles

$$\mathcal{Q}_g = \bigcup \mathcal{Q}_g(k_1, \ldots, k_m, -l_1, \ldots - l_n)$$

subject to $\sum k_i - \sum l_j = 4g - 4$.

Some strata $(z^2 - a)dz^2$



Horizontal foliation of quadratic differentials

Definition

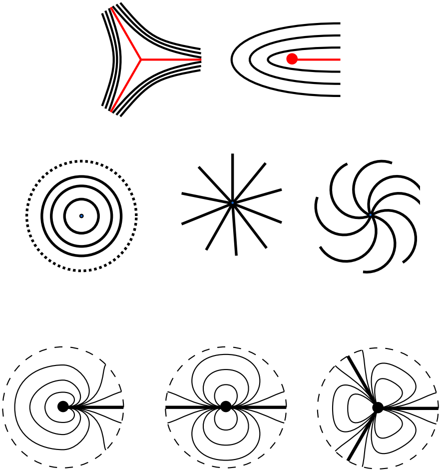
The horizontal trajectories of ϕ on S is arc of $S \setminus (Z(\phi) \cup P(\phi))$ defined by

$$\Im \int^{z} \sqrt{\phi(z)}dz = constant$$

In general, to determine a horizontal trajectory of a quadratic differential through some point is to solve the differential equation

$$\phi(z(t))z'(t)^2 > 0, \text{ or } z'(t) = \frac{\overline{\sqrt{\phi(z)}}}{|\sqrt{\phi(z)}|^2}.$$

Local foliations/trajectories



The global trajectories

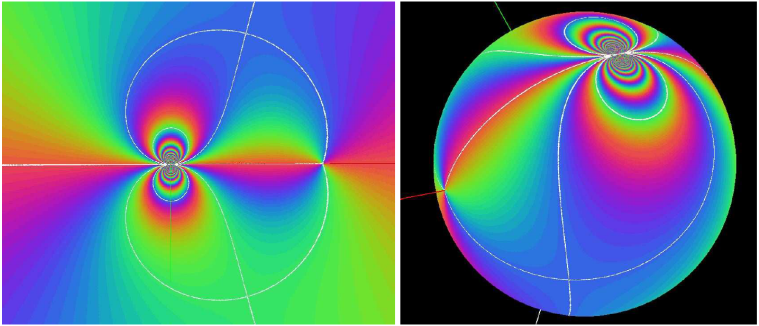


Figure: The trajectories of $\phi(z) = \frac{(z^2-z+1)^2}{z^4(z-1)^2}dz^2$

A.Alvarez-Parrilla etc.
On the geometry, flows and visualization of singular complex analytic vector fields on Riemann surfaces, (arXiv:1811.04157)

The global trajectories

The global trajectory of a quadratic differential ϕ is given by one of the following[Strebel, Bridgeland-Smith]:

(1) *saddle trajectories* tend zeros of ϕ in both directions;

(2) *separating trajectories* tend a zero or simple pole and the other to a pole in $Pol_{\geq 2}(\phi)$;

(3) *generic trajectories* tend a pole in $Pol_{\geq 2}(\phi)$ in both directions;

(4) *closed trajectories* are simple closed curves in S°.

(5) *divergent trajectories* are recurrent in at least one direction.

GMN differentials

Definition (Bridgeland-Smith)

A *GMN(Gaiotto-Moore-Neitzke) differential* (S, ϕ) is a meromorphic quadratic differential ϕ on S such that

(1) ϕ has simple zeros,

(2) ϕ has at least one pole,

(3) ϕ has at least one finite critical point.

Example

The quadratic differential $\phi_1(z) = \frac{(z-1)(z+1)}{z^2} dz^{\otimes 2}$ is a GMN differential on Riemann sphere. But the differential $\phi_2(z) = \frac{(z-1)^2}{z} dz^{\otimes 2}$ is not a GMN differential, since $z = 1$ is a double zero.

Moduli spaces of GMN differentials

A collection of orders of poles is called a polar type.

Definition (Brigeland-Smith)

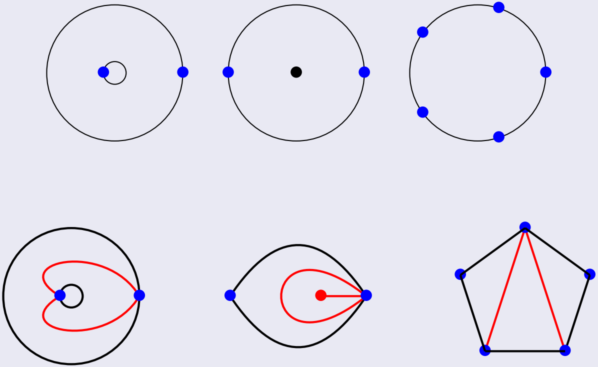
Given a connected compact Riemann surface S of genus g and a polar type $\mathbf{m} = \{m_i\}$, there is a moduli space $\mathrm{Quad}(g, \mathbf{m})$

$$\{[S, \phi] \mid \phi \text{ GMN differential with polar type } \mathbf{m}\},$$

where $[S, \phi]$ is the equivalence-class of (S, ϕ).

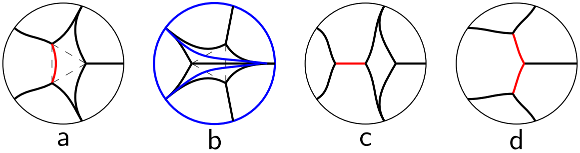
Marked surfaces and triangulation

Example



Example

The differential $(z^3 + uz + v)dz^2$ on Riemann sphere has 3 zeros (counting multiplicity) and a pole of order 7 at ∞.



Framed quadratic differentials

Definition

An $(\mathbf{S}, \mathbf{M})$-*framed quadratic differential* (X, ϕ, f) is a Riemann surface X with GMN differential ϕ, equipped with a diffeomorphism $f \colon \mathbf{S} \to X^{\phi}$, where X^{ϕ} is the associated smooth marked surface of (X, ϕ), preserving the marked points and punctures.

$\mathrm{FQuad}(\mathbf{S}, \mathbf{M})$ is the space of equivalent classes of $(\mathbf{S}, \mathbf{M})$-framed quadratic differentials.

Theorem(Bridgeland-Smith, King-Qiu, Allegretti)

The space $\mathrm{FQuad}(\mathbf{S}, \mathbf{M})$ is a complex manifold.

Differentials associated with marked surfaces

Remark

The quadratic differentials in $\mathrm{FQuad}(\mathbf{S},\mathbf{M})$ can have different polar types. Because there are different choices, simple pole or double pole, at each puncture.

$$\mathrm{FQuad}(\mathbf{S},\mathbf{M}) = \bigcup_{(g,\mathbf{m})} \mathrm{FQuad}(g,\mathbf{m})$$

where $g = g(\mathbf{S})$ and $m_i \in \{1,2\}$ for each puncture and m_i is 2 plus the number of marked points on a boundary. If there are no punctures, then

$$\mathrm{FQuad}(\mathbf{S},\mathbf{M}) = \mathrm{FQuad}(g,\mathbf{m})$$

Quad(⬟) and FQuad(⬟)

Remark

$$\text{Quad}(⬟) \cong \{(z^3 + az + b)dz^2 \mid 4a^2 + 27b^3 \neq 0\}/\mathbb{Z}_5.$$

Note that $\text{MCG}(⬟) = \mathbb{Z}_5$.

$$\text{FQuad}(⬟) \cong \{(z^3 + az + b)dz^2 \mid 4a^2 + 27b^3 \neq 0\}.$$

Stratification of FQuad$(\mathbf{S}, \mathbf{M})$

Let r_ϕ be the the number of divergent trajectories , s_ϕ the number of saddle trajectories, and t_ϕ the number of separating trajectories, $r_\phi + 2s_\phi + t_\phi = k := 3|Zer(\phi)|$. Define subsets

$$B_l = \{\phi \in \mathrm{FQuad}(g, \mathbf{m}) \mid t_\phi \geq k - l\}$$

Bridgeland-Smith

The subsets $B_l \subset \mathrm{FQuad}(g, \mathbf{m})$ form an increasing chain of dense open subsets

$$B_0 = B_1 \subset B_2 \subset \cdots \subset B_k = \mathrm{FQuad}(g, \mathbf{m}) \subset \mathrm{FQuad}(\mathbf{S}, \mathbf{M}).$$

Define $F_p = B_p \setminus B_{p-1}$ for $p \geq 1$ and $F_0 = B_0$, then

$$\mathrm{FQuad}(g, \mathbf{m}) = \bigsqcup F_p.$$

An example of *Fquad*(**S**, **M**)

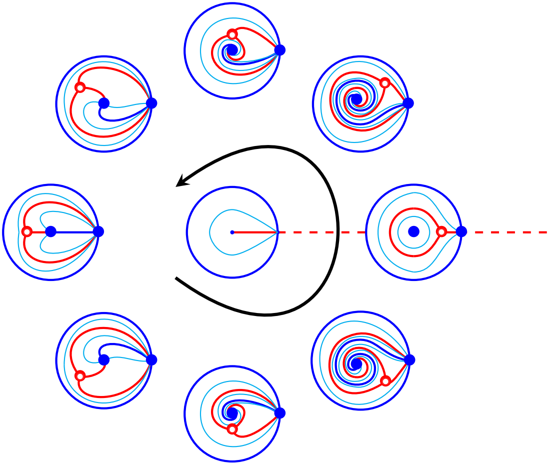


Figure: Differentials $\frac{z-a}{z^2}dz^2$ in FQuad$(0,\{2,3\})$

A flip of ordinary triangles

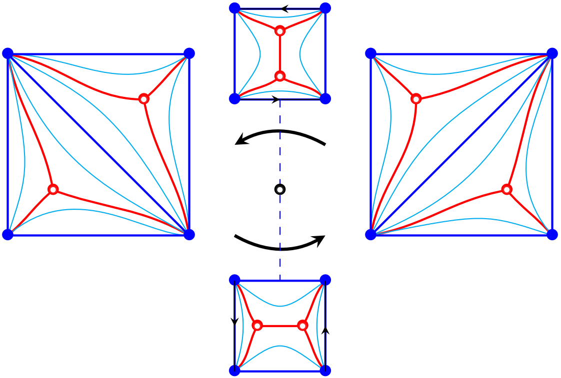


Figure: Two flips (some orientation on walls)

Flips of self-folded trianlge

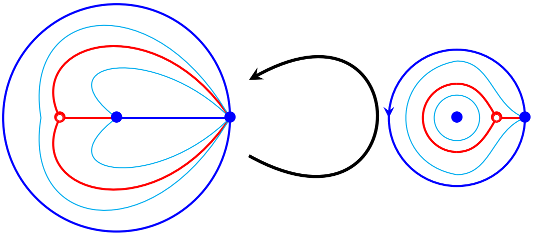
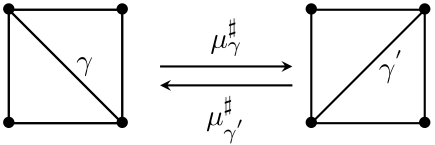


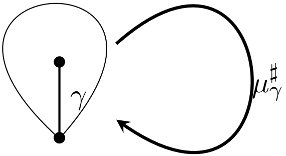
Figure: a pop

Pop-n-flip groupoid

Definition

The *pop-n-flip graph* of $(\mathbf{S}, \mathbf{M})$ is an oriented graph whose vertices are triangulations of $(\mathbf{S}, \mathbf{M})$ and whose arrows are the forward flips and pops between triangulations. We denote by $\mathrm{PFG}(\mathbf{S}, \mathbf{M})$ the pop-flip graph of $(\mathbf{S}, \mathbf{M})$.





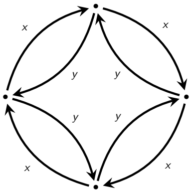
The pop-n-flip groupoid of a marked surface

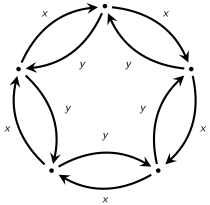
Definition

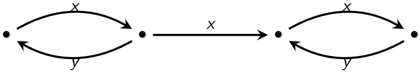
Given a marked surface $(\mathbf{S}, \mathbf{M})$, the *pop-n-flip groupoid* $\mathcal{PFG}(\mathbf{S}, \mathbf{M})$ of $(\mathbf{S}, \mathbf{M})$ is the quotient of the path groupoid of $\mathrm{PFG}(\mathbf{S}, \mathbf{M})$ by the following relations:

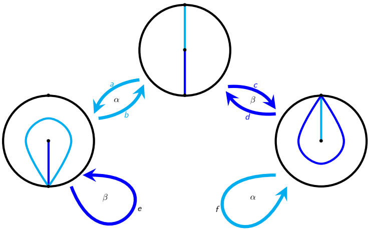
1. Square relations $x^2 = y^2$.
2. Pentagon relations $x^2 = y^3$.
3. Dumbbell relations $x^2 y = y x^2$.
4. Hexagon relations $aeb = cfd$.

The relations in the groupoid









The cell structure

The open subset B_0 is the disjoint union of the cells $\mathrm{U}(\mathbf{T})$ for $\mathbf{T} \in \mathrm{PFG}(\mathbf{S}, \mathbf{M})$,

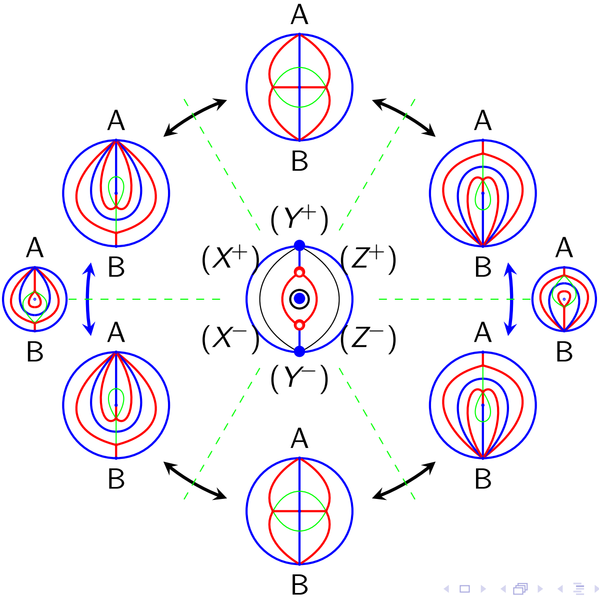
$$B_0 = \bigsqcup_{\mathbf{T} \in \mathrm{PFG}(\mathbf{S}, \mathbf{M})} U(\mathbf{T}),$$

the subset $U(\mathbf{T})$ consists of differentials in $B_0 \subseteq \mathrm{FQuad}(\mathbf{S}, \mathbf{M})$ which have the triangulation $\mathbf{T}$ on $(\mathbf{S}, \mathbf{M})$.
The periods of the saddle classes corresponding to the edges γ give the coordinate $(u_\gamma)_{\gamma \in \mathbf{T}}$, and

$$\mathrm{U}(\mathbf{T}) \cong \mathbb{H}^n,$$

where $\mathbb{H}$ is the strict upper half plane and $n = |\mathbf{T}|$.

An example of cell structure



The embedding in FQuad($\mathbf{S}, \mathbf{M}$)

Lemma

There is a canonical embedding

$$\rho_{\mathbf{S}} \colon \mathsf{PFG}(\mathbf{S}, \mathbf{M}) \to \mathsf{FQuad}(\mathbf{S}, \mathbf{M})$$

whose image is dual to $B_2(\mathbf{S})$. More precisely, the embedding is unique up to homotopy, satisfying

- for each triangulation $\mathbf{T} \in \mathsf{PFG}(\mathbf{S})$, the point $\rho_{\mathbf{S}}(\mathrm{T})$ is in $\mathrm{U}(\mathbf{T})$,
- for each flip $\mu \colon \mathbf{T} \to \mathbf{T}^{\sharp}_{\gamma}$, the path $\rho_{\mathbf{S}}(\mu)$ is in $\mathrm{U}(\mathrm{T}) \cup \partial^{\sharp}_{\gamma}\mathrm{U}(\mathbf{T}) \cup \mathrm{U}(\mathbf{T}^{\sharp}_{\gamma})$, connecting $\rho_{\mathbf{S}}(\mathrm{T})$ to $\rho_{\mathbf{S}}(\mathbf{T}^{\sharp}_{\gamma})$ and intersecting $\partial^{\sharp}_{\gamma}\mathrm{U}(\mathrm{T})$ at exactly one point.

Different groupoids

The inclusions $B_2 \subseteq B_4 \subseteq \text{FQuad}(\mathbf{S}, \mathbf{M})$ induce maps between their fundamental goupoids

$$\iota_4 \colon \Pi_1(B_4) \to \Pi_1(\text{FQuad}(\mathbf{S}, \mathbf{M})),$$

and

$$\iota_4^{(2)} \colon \Pi_1(B_2) \to \Pi_1(B_4).$$

Note that the path groupoid of $\text{PFG}(\mathbf{S}, \mathbf{M})$ is equivalent to the fundamental groupoid $\Pi_1(B_2)$.

Lemma (after Bridgeland-Smith)

The embedding $\rho_{\mathbf{S}}$ induce a surjective map

$$\eta \colon \Pi_1(B_2) \to \mathcal{PFG}(\mathbf{S}, \mathbf{M})).$$

The main result

Theorem(King-Qiu-H)

There is an equivalence between pop-n-flip groupoid $\mathcal{PFG}(\mathbf{S}, \mathbf{M})$ and groupoid $\Pi_1 B_4$.

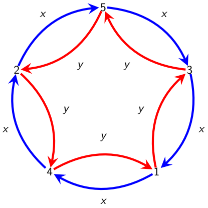
Theorem (need to prove)

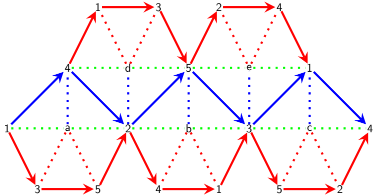
There is an equivalence of groupoids $\Pi_1 \operatorname{FQuad}(\mathbf{S}, \mathbf{M}) \cong \mathcal{PFG}(\mathbf{S}, \mathbf{M})$.

The groupoid of FQuad(⬟)

Example

$$\mathrm{FQuad}(⬟) = \{\phi(z) = (z^3 - 3vz + 2u)dz^2 \mid u^2 - v^3 \neq 0\}, (p, q) = (u/v, u^2/v^3)$$





Thank You!