Frobenius extensions about centralized matrix rings

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Notations

$\mathbb{F}$: a field

R, S : a unitary (associative) ring

$Z(R)$: the center of R

$M_n(R)$: the ring of $n \times n$ matrices over R

$e_{i,j}$: the matrix units of $M_n(R)$

$J_n(\lambda)$: the Jordan block $\lambda I_n + \sum_{i=1}^{n-1} e_{i,i+1}$ with $\lambda \in R$

$\bigoplus_i J_{n_i}(\lambda_{n_i})$: $diag(J_{n_1}(\lambda_{n_1}), \cdots, J_{n_i}(\lambda_{n_i}), \cdots, J_{n_s}(\lambda_{n_s}))$

- Centralized matrix rings
- Frobenius extensions

Introduction: centralized matrix ring

Definition

For $A \in M_n(R)$, we define the centralized matrix ring of A by

$$S_n(A,R) := \{B \in M_n(R) | AB = BA\}.$$

Examples

- Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $S_n(A,R) = R \ltimes R$ is a trivial extension of R by R.
- Let $A = \bigoplus_i J_{n_i}(\lambda)$ be a Jordan matrix with $J_{n_i}(\lambda) \in M_{s-i+1}(\mathbb{F})$, then $S_n(A,\mathbb{F})$ is isomorphic to the algebra given by the quiver with relations (Xi, Zhang, 2021):

$$\underset{1}{\bullet} \underset{\alpha_1}{\overset{\beta_1}{\rightleftarrows}} \underset{2}{\bullet} \underset{\alpha_2}{\overset{\beta_2}{\rightleftarrows}} \bullet \cdots \bullet \underset{\alpha_{s-2}}{\overset{\beta_{s-2}}{\rightleftarrows}} \bullet \underset{\alpha_{s-1}}{\overset{\beta_{s-1}}{\rightleftarrows}} \underset{s}{\bullet}, \beta_{s-1}\alpha_{s-1} = 0, \alpha_i\beta_i = \beta_{i-1}\alpha_{i-1}.$$

Examples

- Let $A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$, then

$$S_n(A,R) = \{B \in M_n(R) | b_{ij} = b_{n+1i,n+1j}, 1 \le i,j \le n\}$$

is the centrosymmetric matrix algebra which have significant applications in Markov processes, engineering problems and quantum physics (Datta, Morgera 1989; Weaver, 1985).

In particular, $S_2(A,R) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $S_3(A,R) = \begin{bmatrix} a & b & c \\ d & e & d \\ c & b & a \end{bmatrix}$

Introduction: Frobenius extension

Definition

- An extension of rings R/S is Frobenius if $\exists E \in \mathrm{Hom}_{S-S}(R,S)$, x_i, $y_i \in R$ s.t. $\forall r \in R$

$$\sum_i x_i E(y_i r) = r = \sum_i E(r x_i) y_i.$$

(E, x_i, y_i) is called a Frobenius system of the extension.

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- Moreover, R/S is s separable Frobenius extension if $\exists d \in R$ such that $dS = Sd, \sum_i x_i d y_i = 1$.

Example (Kadison, 1999)

- $R/\mathbb{F}$ is a Frobenius extension with R a Frobenius $\mathbb{F}$-algebra.
- $\mathbb{Z}G/\mathbb{Z}H$ is a Frobenius extension with H a subgroup of G.
- The G-Galois extension is a separable Frobenius extension.

Theorem (Xi, et al)

- *$M_n(R)/S_n(A,R)$ is a separable Frobenius extension with*

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

- *If R has no zero-divisors and $A \in M_n(R)$ is similar to a Jordan-block matrix with all eigenvalues in $Z(R)$, then $M_n(R)/S_n(A,R)$ is a separable Frobenius extension.*
- *For any $A \in M_n(\mathbb{F})$, $M_n(\mathbb{F})/S_n(A,\mathbb{F})$ is always a separable Frobenius extension.*

Introduction: Motivation

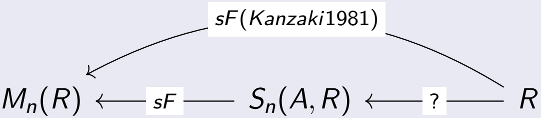
In fact, R can be identified with the subring of $M_n(R)$ consisting of the "scalar matrices".

- $M_n(R)/S_n(A,R)$ and $S_n(A,R)/R$ are separable Frobenius extensions with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ [Xi, Zhang,2020 and Kanzaki, 1981].
- Let S be the subalgebra of $M_4(\mathbb{F})$ with $\mathbb{F}$-basis consisting of the idempotents and matrix units: $e_1 = e_{11} + e_{44}, e_2 = e_{22} + e_{33}, e_{21}, e_{31}, e_{41}, e_{42}, e_{43}$.Then $M_4(\mathbb{F})/S$ is a separable Frobenius extension, but $S/\mathbb{F}$ is not a Frobenius extension [Kadison, 1999].

Introduction: Motivation

Question

Is $S_n(A,R)/R$ always a separable Frobenius extension (short for "sF")?



Main result

Theorem (Wang-Zhu)

Let R be an algebra over an integral domain $\mathbb{K}$ and $J = \bigoplus J_{n_i}(\lambda_i)$ a Jordan matrix in $M_n(R)$ with eigenvalues $\lambda_i \in \mathbb{K}$. Then

- *$S_n(J, R)/R$ is a Frobenius extension if and only if $n_i = n_j$ for $\lambda_i = \lambda_j$.*
- *If any $k \in \{1, \cdots, n\}$ is a unit, then $S_n(J, R)/R$ is a separable Frobenius extension if and only if $n_i = 1$ for any i.*

Sketch of the proof

To complete the proof, we introduce the notation $J^k_{(m,n)} = \sum_{i=1}^{n-k} e_{i,i+k}$ to simplify the calculation.

1. $S_n(J_n(\lambda), R)/R$ is a Frobenius extension with

$$E : S_n(J_n(\lambda), R) \to R, \text{via} \sum_{i=0}^{n-1} r_i \cdot J^i_{(n,n)} \mapsto r_{n-1},$$

$X_i = J^i_{(n,n)}$ and $Y_i = J^{n-1-i}_{(n,n)} (i = 0, 1, \dots, n-1)$.

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$X_i = J_{(n,n)}^{i}$ and $Y_i = J_{(n,n)}^{n-1-i} (i = 0, 1, \ldots, n-1)$.

2. Let $J = \bigoplus J_{n_i}(\lambda_i)$, then $S_n(J,R)/R$ is a Frobenius extension if $n_i = n_j$ for $\lambda_i = \lambda_j$.In fact,

$$S_k(J,R) = M_{k/n_j}(S_{n_j}(J_{n_j}(\lambda), R)) \text{ if } J = \bigoplus J_{n_j}(\lambda);$$

$$S_k(J,R) = \bigoplus (S_{n_j}(J_{n_j}(\lambda_j), R)) \text{ if } J = \bigoplus J_{n_j}(\lambda_j) \text{ with different } \lambda_j.$$

Sketch of the proof

3. If the Jordan matrix J has different size Jordan blocks with same eigenvalues. $S_n(J, R)/R$ is not a Frobenius extension.

Sketch of the proof

3. If the Jordan matrix J has different size Jordan blocks with same eigenvalues. $S_n(J,R)/R$ is not a Frobenius extension. For example,

set $J = J_2(\lambda) \bigoplus J_3(\lambda)$, then $A = \begin{bmatrix} O & aJ_{(2,3)}^2 \\ O & bJ_{(3,3)}^3 \end{bmatrix} \in S_5(J,R)$.

Assume that there is a Frobenius system (E, X_i, Y_i) with $X_i = [x_i(k,l)]$ and $Y_i = [y_i(k,l)]$. By $A = \sum_i E(AX_i)Y_i$, we have an equation since a, b are arbitrary

$$\begin{bmatrix} E(J_{(2,3)}^2) \\ E(J_{(3,3)}^2) \end{bmatrix} \cdot \left[\sum_i x_i(5,5)y_i(1,5),\ \sum_i x_i(5,5)y_i(3,5)\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

a contradiction.

Example

Let $\mathbb{R}$ be real number field and

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} \in M_3(\mathbb{R}), C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in M_2(\mathbb{R}).$$

Then

- $S_n(A, \mathbb{R})/\mathbb{R}$ is not a Frobenius extension since $A \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- $S_n(b, \mathbb{R})/\mathbb{R}$ is a Frobenius extension since $B \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
- $S_n(c, \mathbb{R})/\mathbb{R}$ is a separable Frobenius extension since $C \sim \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

Applications: Frobenius algebras

Follows from the main Theorem, we obtain a family of Frobenius algebras.

Corollary

Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. Assume that A is similar to $\bigoplus_i J_{n_i}(\lambda_i)$ in $M_n(\overline{\mathbb{F}})$ with $\lambda_i \in \overline{\mathbb{F}}$. Then

- $S_n(A, \mathbb{F})$ is a Frobenius algebra if and only if $n_i = n_j$ for $\lambda_i = \lambda_j$.
- $S_n(A, \mathbb{F})$ is a separable Frobenius algebra if and only if $n_i = 1$ for any i.

Corollary

Let $\mathbb{F}$ be an algebraically closed field. Then a matrix $A \in M_n(\mathbb{F})$ is diagonalizable if and only if $S_n(A, \mathbb{F})$ is a separable Frobenius algebra.

Applications: Homological invariants

Inspired by [Xi, 2021] and [Zhao, 2024], we can obtain some extensions of rings with homological invariants by the main result.

In further, we want to get more Homological invariants.

Applications: Homological invariants

Let $0 \to R_R \to I_0 \to \cdots \to I_n \to \cdots$ be a minimal injective resolution of R_R.

Definition

- A ring R is (resp. right quasi) k-Gorenstein if the right flat dimension of I_n is less than or equal to n (resp. $n+1$) for any $0 \leq n \leq k$ [Huang, 1999].
- The dominant dimension of R, which is denoted by resp. ν-) domdim(R), is by definition the minimal number n such that I_n is not (resp. ν-stably) projective [Xi, 2021].

Corollary

Let $J = \bigoplus J_s$ be a Jordan-block matrix in $M_n(R)$.

- $S_n(J, R)$ is (right quasi) k-Gorenstein if and only if R so is.
- Let R be an Artin algebra, then

$$(\nu\text{-})\ \mathrm{domdim}(S_n(J,R)) = (\nu\text{-})\ \mathrm{domdim}(R).$$

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Thank you for your attention!