Bi-Frobenius Algebra structures on quantum complete intersections

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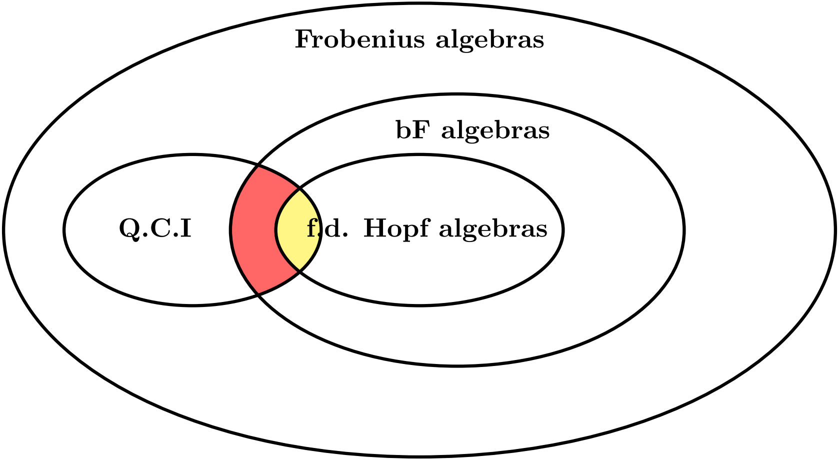
Frobenius algebras
bF algebras
Q.C.I
f.d. Hopf algebras

Frobenius algebras

Frobenius Algebra We call a finite dimensional algebra over field k is a Frobenius algebra, if there is a left A-mod isomorphism

$$A \cong A^* := \mathrm{Hom}_k(A, k), \quad \text{or equivently a right } \ A\text{-mod isomorphism} \quad A \cong A^*.$$

If there is a A-bimodule isomorphism $A \cong A^*$, then we say A is **symmetric**.

A Frobenius algebra can be represented by a pair $(A, \ \phi)$, where $\phi \in A^*, \ \ A^* = A \rightharpoonup \phi$ ($\rightharpoonup$ is the left action of A on A^*); or equivalently, $A^* = \phi \leftharpoonup A$ ($\leftharpoonup$ is the right action). Then ϕ is called a **Frobenius homomorphism** of A.

Nakayama automorphism Suppose that $(A, \ \phi)$ is a Frobenius algebra. Then exist a unique algebra isomorphism $\mathcal{N}_\phi : A \longrightarrow A$ such that

$$\phi(xy) = \phi(y\mathcal{N}_\phi(x)), \ \forall \ x, \ y \in A,$$

which is called the Nakayama automorphism of A corresponding to ϕ.

Let $\phi' \in A^*$. Then ϕ' is a Frobenius homomorphism $\Longleftrightarrow$ $\exists$ invertible element z_1 such that $\phi' = z_1 \rightharpoonup \phi$; $\Longleftrightarrow$ $\exists \ z_2$ such that $\phi' = \phi \leftharpoonup z_2$. Then $\mathcal{N}_{\phi'} : \ x \mapsto z_1\mathcal{N}_\phi(x)z_1^{-1}$: and $\mathcal{N}_{\phi'} : \ x \mapsto \mathcal{N}_\phi(z_2 x z_2^{-1})$.

Frobenius coalgebra

Suppose that $(C,\ \Delta,\ \varepsilon)$ is a k-coalgebra. Using Sweedler notation: $\Delta(c) = \sum c_1 \otimes c_2$. Then $C^* = \mathrm{Hom}_k(C,k)$ is a k-algebra: $(fg)(c) = \sum f(c_1)g(c_2)$, and C is a C^* bimodule. Denote the left action of C^* on C by $f \rightharpoonup c$. Then $f \rightharpoonup c = \sum c_1 f(c_2)$. Similarly, $c \leftharpoonup f = \sum f(c_1)c_2$.

Frobenius coalgebra We say a finite dimensional coalgebra $(C,\ \Delta,\ \varepsilon)$ is Frobenius, if there is left C^*-mod isomorphism

$$C^* \cong C, \quad \text{or equivalently, right } C^*\text{-mod isomorphism } C^* \cong C.$$

Frobenius coalgebra pair $(C,\ t)$ **:** Finite dimensional coalgebra C is Frobenius if and only if there is $t \in C$ such that $C = t \leftharpoonup C^*$; or equivalently, $C = C^* \rightharpoonup t$.

bi-Frobenius algebras

Definition(Y.Doi, M.Takauchi 2000) Let A be a finite dimensional k-algebra and k-coalgebra. $t \in A$, $\phi \in A^*$, $S : A \longrightarrow A$ is a k-linear map defined by

$$S(a) = \sum \phi(t_1 a) t_2, \quad \forall \ a \in A.$$

Quadruple $(A,\ \phi,\ t,\ S)$, or simply A, is called bi-Frobenius algebra, if it satisfied

(i) The counit $\varepsilon : A \longrightarrow k$ is a algebra map; the unit 1 is a group-like element;

(ii) $(A,\ \phi)$ is a Frobenius algebra; $(A,\ t)$ is a Frobenius coalgebra;

(iii) S an anti-homomorphism of algebra, and an anti-homomorphism of coalgebra (i.e., $\varepsilon \circ S = \varepsilon$, $\Delta(S(a)) = \sum S(a_2) \otimes S(a_1)$, $\forall \ a \in A$).

If this is the case, S is called the antipode of bi-Frobenius algebra A.

Remark (1) S is a bijection.

(2) Finite dimensional Hopf algebras are bi-Frobenius algebras; the converse is not true (**however such examples are not easy to get**).

(3) Bi-Frobenius algebras are Hopf if and only if the comultiplication is an algebra map.

(4) Previous works: Y. Doi,M. Takeuchi,M. Haim, D. Simson, Libin. Li, Yinhuo. Zhang, Yanhua. Wang, Zhihua. Wang, Xiao-Wu. Chen, Pu. Zhang $\cdots$

Quantum complete intersections

Suppose that $\mathbf{a} = (a_1, \cdots, a_n)$ is an integer vector satisfying $a_i \geq 2,\ n \geq 2$; $\mathbf{q} = (q_{ij})$ is a multiplicatively antisymmetric matrix over k, i.e., a square matrix satisfying $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$ for $1 \leq i, j \leq n$. Then k-algebra

$$A(\mathbf{q}, \mathbf{a}) = k\langle x_1, \cdots, x_n\rangle / \langle x_i^{a_i},\ x_j x_i - q_{ij} x_i x_j,\ 1 \leq i,\ j \leq n\rangle$$

is called a **quantum complete intersection**.

When all $a_i = 2$, it is called quantum exterior algebra.

Originate from by Yu. I. Manin, 1987

Reach their present form by L. L. Avramov, V. N. Gasharov, I. V. Peeva, 1997.

Quantum complete intersections

- **Representation type:** (P.A.Bergh - K.Erdmann 2011; C.M.Ringel 1974) The representation of quantum complete intersection $A = A(\mathbf{q}, \mathbf{a})$ is tame or wild; and it is tame if and only if $n = 2 = a_1 = a_2$.

- **Happel's question** (1989): If the Hochschild cohomology groups of a finite dimensional algebra A over a field k vanish for all sufficiently large n, is the global dimension of A finite?
R.-O Buchweitz, E. L. Green, D. Madsen, Ø. Solberg, 2005 using a quantum exterior algebra

$$A = k\langle x, y\rangle / \langle x^2, y^2, xy + qyx\rangle$$

shows negativity (when q is not a root of unity) to this question.

- C. M. Ringel - P. Zhang (Algebra & Number Theory, 2020) use a quotient of quantum complete intersection:

$$k\langle x, y, z\rangle / \langle x^2,\ y^2,\ z^2,\ yz,\ xy + qyx,\ xz - zx,\ zy - zx\rangle$$

show the independence of the total reflexivity conditions of the Gorenstein projective modules (L. Avramov - A. Martsinkovsky, 2002).

Quantum complete intersections

Notations: Let $V = \{\mathbf{v} = (v_1, \cdots, v_n) \in \mathbb{N}_0^n \mid v_i \leq a_i - 1,\ 1 \leq i \leq n\}$. Then

$$\{x_{\mathbf{v}} := x_1^{v_1} \cdots x_n^{v_n} \mid \mathbf{v} \in V\}$$

is a k-basis of $A(\mathbf{q}, \mathbf{a})$. Specially,

$$x_{\mathbf{a}-\mathbf{1}} = x_1^{a_1-1} \cdots x_n^{a_n-1}$$

For any $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n$, denote $\mathbf{q}^{\langle \mathbf{u}|\mathbf{v}\rangle} = \prod_{1 \leq i < j \leq n} q_{ij}^{u_j v_i}$. Then $x_{\mathbf{u}} x_{\mathbf{v}} = \mathbf{q}^{\langle \mathbf{u}|\mathbf{v}\rangle} x_{\mathbf{u}+\mathbf{v}}$.

Lemma (Bergh 2009) Quantum complete intersections are local Frobenius algebras, $x^*_{\mathbf{a}-\mathbf{1}}$ is a Frobenius homomorphism, its Nakayama automorphism is

$$\mathcal{N}_{x^*_{\mathbf{a}-\mathbf{1}}}(x_{\mathbf{v}}) = \frac{\mathbf{q}^{\langle \mathbf{a}-\mathbf{1}-\mathbf{v}|\mathbf{v}\rangle}}{\mathbf{q}^{\langle \mathbf{v}|\mathbf{a}-\mathbf{1}-\mathbf{v}\rangle}} x_{\mathbf{v}},\ \forall\ \mathbf{v} \in V.$$

It is symmetric if and only if $\mathcal{N}_{x^*_{\mathbf{a}-\mathbf{1}}}$ is identity. We call $\mathcal{N}_{x^*_{\mathbf{a}-\mathbf{1}}}$ the canonical Nakayama automorphism, denoted by $\mathcal{N}$.

Let $h_{\mathbf{v}} = \frac{\mathbf{q}^{\langle \mathbf{a}-\mathbf{1}-\mathbf{v}|\mathbf{v}\rangle}}{\mathbf{q}^{\langle \mathbf{v}|\mathbf{a}-\mathbf{1}-\mathbf{v}\rangle}}$. Then $\mathcal{N}(x_{\mathbf{v}}) = h_{\mathbf{v}} x_{\mathbf{v}},\ \forall\ \mathbf{v} \in V$.

Hopf algebra structures on quantum complete intersections

Main theorem I Let $A = A(\mathbf{q}, \mathbf{a}) = k\langle x_1, \cdots, x_n\rangle / \langle x_i^{a_i},\ x_j x_i - q_{ij} x_i x_j,\ 1 \le i,\ j \le n\rangle$. Then the following statements are equivalent:

(1) A has Hopf algebra structure;

(2) A has bi-algebra structure;

(3) A is commutative, and each a_i is a positive power of p, where $p = \mathrm{char} k$ is a prime.

Corollary Quantum exterior algebra $A = k\langle x_1, \cdots, x_n\rangle / \langle x_i^2,\ x_j x_i - q_{ij} x_i x_j,\ 1 \le i,\ j \le n\rangle$ admits Hopf algebra structure if and only if A is commutative $k\langle x_1, \cdots, x_n\rangle / \langle x_i^2,\ x_j x_i - x_i x_j,\ 1 \le i,\ j \le n\rangle$ and $\mathrm{char} k = 2$.

Sketch of the proof of Main Theorem I : Kummer's Theorem

(1)$\Longrightarrow$ (2) is trivial; Key point of (2) $\Longrightarrow$ (3) is using $\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1 + \text{h}\textit{igher degree terms}$. Then by

$$\Delta(x_1)^{a_1} = \Delta(x_1^{a_1}) = 0$$

one gets $\binom{a_1}{m} = 0,\ 1 \leq m \leq a_1 - 1$. This is impossible when char k$= 0$.

When char k$= p$, we need Kummer's theorem. Suppose p is a prime, n is a non-negative integer. Writing n as the expansion in base p:

$$n = n_0 + n_1 p + \cdots + n_r p^r, \quad 0 \leq n_0, \cdots, n_r \leq p - 1.$$

For integer $1 \leq m \leq n$, write m and $t = n - m$ as the expansion in base p, we get m_j and t_j. Let

$$\epsilon_j = \begin{cases} 1, & \text{if } m_j + t_j \geq p; \\ 0, & \text{else.} \end{cases}$$

Theorem (E. Kummer, 1852) Suppose that p is a prime, n, m are integers, $0 \leq m \leq n$, $\nu_p(n)$ be the p-adic valuation of n (the largest non-negative integer s such that p^s divide n). Then $\nu_p(\binom{n}{m}) = \sum_{j \geq 0} \epsilon_j$.

Corollary Let n be a positive integer and p a prime. Assume that $p \mid n$ with $\nu_p(n) = s$. Then $p \nmid \binom{n}{p^s}$.

Proof Firstly $t = n - p^s = t_s p^s + t_{s+1} p^{s+1} + \cdots + n_{s+r} p^{s+r}$. Then $t_s \neq p - 1$ (if not, p^{s+1} will divide $n = p^s + t$). Then by Kummer's theorem $\nu_p(\binom{n}{p^s}) = 0$, i.e., $p \nmid \binom{n}{p^s}$. $\square$

Sketch of the proof of Main Theorem I : The construction

(2) $\Longrightarrow$ (3) (continue): Using this corollary we can deduce that each a_i is a positive power of p. Then by $\Delta(x_jx_i) = q_{ij}\Delta(x_ix_j)$, we get $q_{ij} = 1$ for all i, j.

(3)$\Longrightarrow$ (1) We need to give A a Hopf algebra structure. Define k-linear maps $\Delta : A \longrightarrow A \otimes A$, $\varepsilon : A \longrightarrow k$ and $S : A \longrightarrow A$ as follows:

$$\begin{cases} \varepsilon(x_{\mathbf{v}}) = \delta_{\mathbf{v},\mathbf{0}}; \\ \Delta(x_{\mathbf{v}}) = \prod_{1\leq i\leq n} \sum_{0\leq k\leq v_i} \binom{v_i}{k} x_i^k \otimes x_i^{v_i-k}; \\ S(x_{\mathbf{v}}) = (-1)^{|\mathbf{v}|} x_{\mathbf{v}}, \ \forall \ \mathbf{v} \in V. \end{cases}$$

It can be verified that this is a Hopf algebra structure on A.

Remark: Quantum complete intersections are **braided Hopf algebras** when q_{ij} and a_i are specially selected. (N. Andruskiewitsch - H. J. Schneider 1998, 2002)

bi-Frobenius quantum complete intersections

As **in most cases** a quantum complete intersection **doesn't** have Hopf algebra structures, we consider its **possible bi-Frobenius algebra structures**, briefly **bi-Frobenius quantum complete intersection**.

Main question: When does a quantum complete intersection become a bi-Frobenius quantum complete intersection?

Proposition Suppose that $(A,\ \phi,\ t,\ S)$ is a bi-Frobenius algebra structure on $A = A(\mathbf{q}, \mathbf{a})$, then

(1) $\phi = x^*_{\mathbf{a}-\mathbf{1}} \leftharpoonup z$, where $z \in A$ is an invertible element.

(2) $\varepsilon(x_{\mathbf{v}}) = \delta_{\mathbf{v},\mathbf{0}}, \quad \forall\ \mathbf{v} \in V$.

(3) $t = c x_{\mathbf{a}-\mathbf{1}}, \quad c \in k - \{0\}$.

(4) $\mathcal{N}^2 = \mathrm{Id}$, i.e., $h^2_{\mathbf{v}} = 1, \quad \forall\ \mathbf{v} \in V$.

(5) If S is a graded map, then $S^2 = \mathcal{N}, \quad S^4 = \mathrm{Id}$.

(6) If $\mathrm{char} k = 0$, then $\mathcal{N}^2_{\phi} = \mathrm{Id}, \quad S^4 = \mathrm{Id}$.

Permutation antipode and compatible permutations

Definition Suppose quantum complete intersection $A = A(\mathbf{q}, \mathbf{a})$ admits a bi-Frobenius algebra structure $(A,\ \phi,\ t,\ S)$. Its antipode S is called **permutation antipode**, if there exists a permutation π of V, and $c_{\mathbf{v}} \in k,\ \forall\ \mathbf{v} \in V$, such that

$$S(x_{\mathbf{v}}) = c_{\mathbf{v}} x_{\pi(\mathbf{v})},\ \forall\ \mathbf{v} \in V.$$

All bi-Frobenius quantum complete intersections founded so far have **permutation antipodes.**

Proposition If all $a_i \geq 3,\ q_{ij} \neq 1$, and the antipode S of bi-Frobenius quantum complete intersection $A = A(\mathbf{q}, \mathbf{a})$ is a graded map, then S is a permutation antipode.

Definition Permutation $\pi \in S_n$ is called a **compatible permutation** of quantum complete intersection $A(\mathbf{q}, \mathbf{a})$, if

$$a_{\pi(i)} = a_i, \quad q_{\pi(i)\pi(j)} = q_{ji}, \quad 1 \leq i, j \leq n.$$

Proposition The antipode S of a bi-Frobenius quantum complete intersection is a permutation antipode if and only if there is a compatible permutation $\pi \in S_n$ and $c_i \in k,\ 1 \leq i \leq n$, such that $S(x_i) = c_i x_{\pi(i)},\ \ 1 \leq i \leq n$.

Main question: When does a quantum complete intersection become a bi-Frobenius quantum intersection with permutation antipode?

Necessity

Proposition Suppose that $A = A(\mathbf{q}, \mathbf{a})$ is a bi-Frobenius algebra with permutation antipode (S, π). Then

(1) $S(x_{\mathbf{a}-\mathbf{1}}) = x_{\mathbf{a}-\mathbf{1}}$; $\pi(\mathbf{a}-\mathbf{1}) = \mathbf{a}-\mathbf{1}$; $\pi^2 = \mathrm{Id}$; S is a graded map, $S^2 = \mathcal{N}$ and $S^4 = \mathrm{Id}$.

(2) π induce an compatible permutation, still denoted by π, satisfies

$$\pi(\mathbf{v}) = (v_{\pi^{-1}(1)}, \cdots, v_{\pi^{-1}(n)}), \qquad x_{\pi(\mathbf{v})} = x_1^{v_{\pi^{-1}(1)}} \cdots x_n^{v_{\pi^{-1}(n)}}, \ \ \forall\ \mathbf{v} \in V;$$

$$S(x_i) = c_i x_{\pi(i)}, \quad \text{where each} \quad c_i \in k, \quad \text{satisfies}$$

$$c_i c_{\pi(i)} = h_{\mathbf{e}_i}, \ \forall\ 1 \le i \le n; \qquad q_\pi \prod_{1 \le i \le n} c_i^{a_i - 1} = 1$$

where $q_\pi = \prod_{1 \le j < k \le n} (\mathbf{q}^{\langle \pi(\mathbf{e}_k) | \pi(\mathbf{e}_j) \rangle})^{(a_k-1)(a_j-1)}$.

bi-Frobenius quantum complete intersection with permutation antipode

Main Theorem 2 Quantum complete intersection $A = A(\mathbf{q}, \mathbf{a})$ admits a bi-Frobenius algebra structure with permutation antipode if and only if there is a compatible permutation $\pi \in S_n$ and $c_i \in k,\ 1 \le i \le n$, such that

$$\pi^2 = \mathrm{Id}, \qquad c_i c_{\pi(i)} = h_{\mathbf{e}_i}, \qquad q_\pi \prod_{1 \le i \le n} c_i^{a_i - 1} = 1.$$

If this is th case, $(A,\ x^*_{\mathbf{a-1}},\ x_{\mathbf{a-1}},\ S)$ is a bi-Frobenius algebra; counit, comultiplication and antipode:

$$\begin{cases} \varepsilon(x_{\mathbf{v}}) = \delta_{\mathbf{v},\mathbf{0}} \quad \text{for} \quad \mathbf{v} \in V; \\ \Delta(1_A) = 1_A \otimes 1_A; \\ \Delta(x_{\mathbf{v}}) = 1_A \otimes x_{\mathbf{v}} + x_{\mathbf{v}} \otimes 1_A \quad \text{for} \quad \mathbf{v} \in V - \{\mathbf{0}, \mathbf{a-1}\}; \\ \Delta(x_{\mathbf{a-1}}) = \sum_{\mathbf{v} \in V} g_{\mathbf{a-1-v},\pi(\mathbf{v})}\ x_{\mathbf{a-1-v}} \otimes x_{\pi(\mathbf{v})}; \\ S(x_{\mathbf{v}}) = \prod_{1 \le i \le n} c_i^{v_i} \prod_{1 \le j < k \le n} (\mathbf{q}^{\langle \pi(\mathbf{e}_k) | \pi(\mathbf{e}_j) \rangle})^{v_j v_k} x_{\pi(\mathbf{v})},\ \forall\ \mathbf{v} \in V. \end{cases}$$

where

$$g_{\mathbf{a-1-v},\pi(\mathbf{v})} = \frac{1}{\mathbf{q}^{\langle \mathbf{a-1-v} | \mathbf{v} \rangle}} \prod_{1 \le i \le n} c_i^{v_i} \prod_{1 \le j < k \le n} (\mathbf{q}^{\langle \pi(\mathbf{e}_k) | \pi(\mathbf{e}_j) \rangle})^{v_j v_k},\ \forall\ \mathbf{v} \in V.$$

Specifically, $S(x_i) = c_i x_{\pi(i)}, 1 \le i \le n$.

Corollaries

Corollary Suppose that $\mathrm{char}k = 2$. Then $A = A(\mathbf{q}, \mathbf{a})$ admits a bi-Frobenius algebra structure with permutation antipode if and only if there is compatible permutation $\pi \in S_n$ such that $\pi^2 = \mathrm{Id}$, $\mathcal{N}^2 = \mathrm{Id}$.

Corollary Suppose that $A = A(\mathbf{q}, \mathbf{a})$ is symmetric. Then A admits a bi-Frobenius algebra structure with permutation antipode if and only if there is a compatible permutation $\pi \in S_n$ such that $\pi^2 = \mathrm{Id}$.

Intrinsic characterization

Suppose that A is a bi-Frobenius algebra with permutation antipode. Follows propositions before, $h_{\mathbf{e}_i}^2 = 1,\ 1 \le i \le n$; then by Main Theorem 2, there is a compatible permutation $\pi \in S_n$ such that $\pi^2 = \mathrm{Id}$. Hence, define index sets

$$I = \{i \mid 1 \le i \le n,\ \pi(i) = i\}, \quad J = \{i \mid 1 \le i \le n,\ \pi(i) \ne i\}.$$

Intrinsic characterization

Suppose that A is a bi-Frobenius algebra with permutation antipode. Follows propositions before, $h_{\mathbf{e}_i}^2 = 1,\ 1 \leq i \leq n$; then by Main Theorem 2, there is a compatible permutation $\pi \in S_n$ such that $\pi^2 = \mathrm{Id}$. Hence, define index sets

$$I = \{i \mid 1 \leq i \leq n,\ \pi(i) = i\}, \quad J = \{i \mid 1 \leq i \leq n,\ \pi(i) \neq i\}.$$

By considering the parity of a_i and $h_{\mathbf{e}_i} = \pm 1$, define index sets:

$$\begin{aligned}
I_1 &= \{i \in I \mid h_{\mathbf{e}_i} = 1,\ a_i \text{ is even}\}, & J_1 &= \{i \in J \mid h_{\mathbf{e}_i} = 1,\ a_i \text{ is even}\},\\
I_2 &= \{i \in I \mid h_{\mathbf{e}_i} = 1,\ a_i \text{ is odd}\}, & J_2 &= \{i \in J \mid h_{\mathbf{e}_i} = 1,\ a_i \text{ is odd}\},\\
I_3 &= \{i \in I \mid h_{\mathbf{e}_i} = -1,\ a_i \text{ is even}\}, & J_3 &= \{i \in J \mid h_{\mathbf{e}_i} = -1,\ a_i \text{ is even}\},\\
I_4 &= \{i \in I \mid h_{\mathbf{e}_i} = -1,\ a_i \text{ is odd}\}, & J_4 &= \{i \in J \mid h_{\mathbf{e}_i} = -1,\ a_i \text{ is odd}\}.
\end{aligned}$$

When $\mathrm{char} k \neq 2$, these sets have no intersection. Denote the number of elements in set I by $|I|$.

Intrinsic characterization

Main theorem 3 Suppose that char$k \neq 2$ and $\sqrt{-1} \in k$. Then $A = A(\mathbf{q}, \mathbf{a})$ admits a bi-Frobenius algebra structure with permutation antipode if and only if $\mathcal{N}^2 = \mathrm{Id}$, and there is compatible permutation $\pi \in S_n$ satisfying $\pi^2 = \mathrm{Id}$, such that $|I_1| + |I_3| \neq 0$ or $\frac{|J_3|}{2}$ is even.

Intrinsic characterization

Main theorem 3 Suppose that char$k \neq 2$ and $\sqrt{-1} \in k$. Then $A = A(\mathbf{q}, \mathbf{a})$ admits a bi-Frobenius algebra structure with permutation antipode if and only if $\mathcal{N}^2 = \mathrm{Id}$, and there is compatible permutation $\pi \in S_n$ satisfying $\pi^2 = \mathrm{Id}$, such that $|I_1| + |I_3| \neq 0$ or $\frac{|J_3|}{2}$ is even.

Main theorem 4 Suppose that char$k \neq 2$ and $\sqrt{-1} \notin k$. Then $A = A(\mathbf{q}, \mathbf{a})$ admits a bi-Frobenius algebra structure with permutation antipode if and only if $\mathcal{N}^2 = \mathrm{Id}$, and there is compatible permutation $\pi \in S_n$ satisfying $\pi^2 = \mathrm{Id}$, such that:

(1) $|I_3| + |I_4| = 0$;
(2) $|I_1| \neq 0$ or $\frac{|J_3|}{2}$ is even.

Example 1: symmetric

Let $\mathbf{a}=(2,2,2)$, $\mathbf{q}=\begin{pmatrix} 1 & b & \frac{1}{b} \\ \frac{1}{b} & 1 & b \\ b & \frac{1}{b} & 1 \end{pmatrix}$, where $0\neq b\in k$. Then $A=k\langle x_1,x_2,x_3\rangle/\langle x_1^2,\ x_2^2,\ x_3^2,\ x_2x_1-bx_1x_2,\ x_3x_1-\frac{1}{b}x_1x_3,\ x_3x_2-bx_2x_3\rangle$,

$$V=\{(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,0),\ (1,0,1),\ (0,1,1),\ (1,1,1)\},$$

$h_{\mathbf{e}_1}=h_{\mathbf{e}_2}=h_{\mathbf{e}_3}=1$. Thus A is symmetric. Let $\pi=(2,3)\in S_3$. Then $q_{\pi(i)\pi(j)}=q_{ji}$, $1\leq i,j\leq 3$. Let $c_1=c_2=c_3=1$. Define

$$\begin{cases} \varepsilon(x_{\mathbf{v}})=\delta_{\mathbf{v},\mathbf{0}} \quad \text{for all} \quad \mathbf{v}\in V; \\ \Delta(1)=1\otimes 1; \\ \Delta(x_{\mathbf{v}})=1\otimes x_{\mathbf{v}}+x_{\mathbf{v}}\otimes 1, \quad \forall \quad \mathbf{v}\in V-\{(0,0,0),(1,1,1)\}, \\ \Delta(x_1x_2x_3)=1\otimes x_1x_2x_3+x_1x_2x_3\otimes 1 \\ \qquad\qquad +x_2x_3\otimes x_1+\frac{1}{b}x_1x_3\otimes x_3+x_1x_2\otimes x_2 \\ \qquad\qquad +\frac{1}{b}x_3\otimes x_1x_3+x_2\otimes x_1x_2+x_1\otimes x_2x_3; \\ S(1)=1; \quad S(x_1)=x_1; \quad S(x_2)=x_3; \quad S(x_3)=x_2; \\ S(x_1x_2)=\frac{1}{b}x_1x_3; \quad S(x_1x_3)=bx_1x_2; \quad S(x_2x_3)=x_2x_3; \\ S(x_1x_2x_3)=x_1x_2x_3. \end{cases}$$

By the Main theorem, $(A,\ (x_1x_2x_3)^*,\ x_1x_2x_3,\ S)$ is a bi-Frobenius algebra with permutation antipode.

Example 2: non-symmetric

Let $\mathbf{a} = (2,2,2)$, $\mathbf{q} = \begin{pmatrix} 1 & b & \frac{1}{b} \\ \frac{1}{b} & 1 & -b \\ b & -\frac{1}{b} & 1 \end{pmatrix}$, where $0 \neq b \in k$. Then $A = k\langle x_1, x_2, x_3\rangle / \langle x_1^2, x_2^2, x_3^2, x_2x_1 - bx_1x_2, x_3x_1 - \frac{1}{b}x_1x_3, x_3x_2 + bx_2x_3\rangle$,

$$V = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\},$$

$h_{\mathbf{e}_1} = 1$, $h_{\mathbf{e}_2} = h_{\mathbf{e}_3} = -1$. Thus A is not symmetric. Let $\pi = (2,3) \in S_3$, $c_1 = -1$, $c_2 = c_3 = \sqrt{-1}$. Then $q_{\pi(i)\pi(j)} = q_{ji}$, $1 \leq i,j \leq 3$,

$$c_1 c_{\pi(1)} = c_1^2 = h_{\mathbf{e}_1} = 1, \quad c_2 c_{\pi(2)} = c_3 c_{\pi(3)} = c_2 c_3 = h_{\mathbf{e}_2} = h_{\mathbf{e}_3} = -1$$

$$\prod_{1 \leq j < k \leq n} (\mathbf{q}^{\langle \pi(\mathbf{e}_k) | \pi(\mathbf{e}_j)\rangle})^{(a_k-1)(a_j-1)} = \mathbf{q}^{\langle \pi(\mathbf{e}_2)|\pi(\mathbf{e}_1)\rangle} \mathbf{q}^{\langle \pi(\mathbf{e}_3)|\pi(\mathbf{e}_1)\rangle} \mathbf{q}^{\langle \pi(\mathbf{e}_3)|\pi(\mathbf{e}_2)\rangle} = 1.$$

By the main theorem, $(A, (x_1x_2x_3)^*, x_1x_2x_3, S)$ is a bi-Frobenius algebra with permutation antipode, where

$$\begin{cases} \varepsilon(x_{\mathbf{v}}) = \delta_{\mathbf{v},\mathbf{0}} \quad \text{for all} \quad \mathbf{v} \in V; \\ \Delta(1) = 1 \otimes 1; \\ \Delta(x_{\mathbf{v}}) = 1 \otimes x_{\mathbf{v}} + x_{\mathbf{v}} \otimes 1, \quad \forall \quad \mathbf{v} \in V - \{(0,0,0),(1,1,1)\}; \\ \Delta(x_1x_2x_3) = 1 \otimes x_1x_2x_3 + x_1x_2x_3 \otimes 1 \\ \qquad - x_2x_3 \otimes x_1 - \frac{\sqrt{-1}}{b} x_1x_3 \otimes x_3 + \sqrt{-1}x_1x_2 \otimes x_2 \\ \qquad + \frac{\sqrt{-1}}{b} x_3 \otimes x_1x_3 - \sqrt{-1}x_2 \otimes x_1x_2 - x_1 \otimes x_2x_3; \\ S(1) = 1; \quad S(x_1) = -x_1; \quad S(x_2) = \sqrt{-1}x_3; \quad S(x_3) = \sqrt{-1}x_2; \\ S(x_1x_2) = -\frac{\sqrt{-1}}{b} x_1x_3; \quad S(x_1x_3) = -b\sqrt{-1}x_1x_2; \quad S(x_2x_3) = -x_2x_3; \\ S(x_1x_2x_3) = x_1x_2x_3. \end{cases}$$

Thank you !