Minimal Models in Algebra and Operad Theory

Guodong zhou
East China Normal University

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Plan

- Koszul duality and minimal models for associative algebras
- Koszul duality and minimal models for operads
- Deformation theory and minimal models
- Koszul duality and minimal models for operated algebras

Part I: Computing $\mathrm{Tor}_*^A(\mathbf{k}, \mathbf{k})$ and $\mathrm{Ext}_A^*(\mathbf{k}, \mathbf{k})$ via projective resolutions

Let $\mathbf{k}$ be a field.
Let $A = T(V)/(R)$ be an algebra, where V is a vector space,

$$T(V) = \mathbf{k} \oplus V \oplus \cdots \oplus V^{\otimes n} \otimes \cdots$$

is the tensor algebra generated by V. Then $\mathbf{k}$ is a simple A-module.

Problem

Compute $\mathrm{Tor}_*^A(\mathbf{k}, \mathbf{k})$ *and* $\mathrm{Ext}_A^*(\mathbf{k}, \mathbf{k})$.

Answer

Find a (minimal) projective resolution $P_* \to \mathbf{k} \to 0$ *of* $\mathbf{k}$ *as* A*-modules then*

$$\mathrm{Tor}_*^A(\mathbf{k}, \mathbf{k}) = H_*(\mathbf{k} \otimes_A P_*), \ \mathrm{Ext}_A^*(\mathbf{k}, \mathbf{k}) = H^*(\mathrm{Hom}_A(P_*, \mathbf{k})).$$

Part I: Reconstruction problem

Problem (Reconstruction problem)

Can we reconstruct A from the $\mathrm{Ext}_A^*(\mathbf{k},\mathbf{k})$ *or* $\mathrm{Tor}_*^A(\mathbf{k},\mathbf{k})$*?*

Answer

Yes, but with the A_∞-algebra structure on $\mathrm{Ext}_A^*(\mathbf{k},\mathbf{k})$ *or the A_∞-coalgebra structure on* $\mathrm{Tor}_*^A(\mathbf{k},\mathbf{k})$

Definition

- *A (nonunital) associative* $\mathbb{k}$*-algebra is a pair* (A, μ)*, where* A *is a* $\mathbb{k}$*-vector space,*

$$\mu : A \otimes A \to A,\ a \otimes b \mapsto a \cdot b = ab$$

is a linear map, called multiplication, which satisfy the associativity axiom:

$$(ab)c = a(bc)$$

or equivalently,

$$(\mu \otimes \mathrm{id}_A)\mu = (\mathrm{id}_A \otimes \mu)\mu$$

- *A graded algebra is a graded space* $A = \oplus_{n\in\mathbb{Z}} A_n$ *endowed with an associative product such that* $A_p \cdot A_q \subseteq A_{p+q}, \forall p, q \in \mathbb{Z}$.
- *A differential graded algebra is a graded algebra* $A = \oplus_{n\in\mathbb{Z}} A_n$ *equipped with a square zero linear map* $d : A \to A$ *of degree* -1 *subject to*

$$d(ab) = d(a)b + (-1)^{|a|} a d(b),$$

for any $a, b \in A$ *homogeneous, where* $|a|$ *denotes the degree of* a.

Part I: From associative algebras to A_∞-algebras

Definition (Stasheff 1963)

An A_∞-algebra structure on a graded space V consists a family of operators $\{m_n\}_{n\geqslant 1}$ with $m_n : V^{\otimes n} \to V, |m_n| = n-2$, and the family $\{m_n\}_{n\geqslant 1}$ satisfies the following **Stasheff identities**:

$$\sum_{i+j+k=n, i,k\geq 0, j\geq 1} (-1)^{i+jk} m_{i+1+k} \circ (\mathrm{Id}^{\otimes i} \otimes m_j \otimes \mathrm{Id}^{\otimes k}) = 0, \forall n \geqslant 1.$$

Example

- $n=1$, $m_1 \circ m_1 = 0$ *with* $|m_1| = -1$, *i.e.* m_1 *is a differential;*
- $n=2$, $m_1 \circ m_2 = m_2 \circ (\mathrm{Id} \otimes m_1 + m_1 \otimes \mathrm{Id})$, *i.e.* m_1 *is a derivation with respect to the multiplication* m_2*;*
- $n=3$, $m_2 \circ (m_2 \otimes \mathrm{Id}) - m_2 \circ (\mathrm{Id} \otimes m_2) = -(m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathrm{Id}^{\otimes 2} + \mathrm{Id} \otimes m_1 \otimes \mathrm{Id} + m_1 \otimes \mathrm{Id}^{\otimes 2}))$, *i.e.* m_2 *is associative up to homotopy.*

J. D. Stasheff, *Homotopy associativity of H -spaces. I, II*. Trans. Amer. Math. Soc. **108** (1963), 275-292; ibid. **108** (1963) 293-312.

Part I: A_∞-algebras via bar construction

Let $A = \Bbbk 1_A \oplus \overline{A}$ be an augmented differential graded algebra. Its bar construction $\mathrm{B}(A)$ is defined to be the tensor coalgebra

$$T^c(s\overline{A}) = \Bbbk \oplus s\overline{A} \oplus (s\overline{A})^{\otimes 2} \oplus \cdots$$

via the deconcartenation coproduct

$$\Delta(sa_1 \otimes \cdots \otimes sa_n) = \sum_{i=0}^{n} (sa_1 \otimes \cdots \otimes sa_i) \otimes (sa_{i+1} \otimes \cdots \otimes sa_n)$$

The differential d_A of A induces a differential d_1 on $\mathrm{B}(A)$ via

$$d_1([a_1|a_2|\cdots|a_n]) = \sum_{i=1}^{n} \pm[a_1|\cdots|d_A(a_i)|\cdots|a_n].$$

The product $\mu_A : A^{\otimes}2 \to A$ on A also induces a differential d_2 on $\mathrm{B}(A)$ via

$$d_2([a_1|a_2|\cdots|a_n]) = \sum_{i=1}^{n} \pm[a_1|\cdots|\mu_A(a_i \otimes a_{i+1})|\cdots|a_n].$$

Now $(\mathrm{B}(A), d = d_1 + d_2)$ is a (coaugmented) differential graded (conilpotent cofree) coalgebra.

Theorem

A graded vector space A is an A_∞-algebra iff $\mathrm{B}(A)$ is a (coaugmented) differential graded (conilpotent cofree) coalgebra via its canonical coalgebra structure.

$$m_n : A^{\otimes n} \to A \leftrightarrow d_n : (s\overline{A})^{\otimes n} \to s\overline{A}$$

Part I: From coalgebras to A_∞-coalgebras

Definition

A noncounital coalgebra is a pair (C, Δ) *where* C *is a vector space,* $\Delta : C \to C \otimes C$ *is a linear map, called the comultiplication which satisfies the coassociativity axion*

$$(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta$$

or equivalently there exists a commutative diagram

$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow{\mathrm{Id}_C \otimes \Delta} & C \otimes C \\ \Delta \otimes \mathrm{Id}_C \uparrow & & \uparrow \Delta \\ C \otimes C & \xleftarrow[\Delta]{} & C \end{array}.$$

We can also define graded coalgebras, differential graded coalgebras etc. An A_∞*-coalgebra structure on a graded space* V *consists a family of higher comultiplications* $\Delta_n : V \to V^{\otimes n}, |\Delta_n| = n - 2$ *subject to* **coStasheff identities**.

Let $C = \mathbb{k}1_C \oplus \overline{C}$ be a coaugmented differential graded coalgebra. Its cobar construction $\Omega(C)$ is defined to be the tensor algebra $T(s^{-1}\overline{C})$ with concatenation product.
We shall write an element of $\Omega(A)$ as $\langle a_1|a_2|\cdots|a_n\rangle \in (s^{-1}\overline{A})^{\otimes n}$ for $a_1, \cdots, a_n \in \overline{A}$.
The differential d_C of C induces a differential d_1 on $\Omega(C)$ via

$$d_1(\langle c_1|\cdots|c_n\rangle) = \sum_{i=1}^{n}(-1)^{i+|c_1|+\cdots+|c_{i-1}|}\langle c_1|\cdots|d_C(c_i)|\cdots|c_n\rangle.$$

The coproduct Δ_C induced induces a differential d_2 on $\Omega(C)$ via

$$d_2(\langle c_1|\cdots|c_n\rangle) = \sum_{i=1}^{n}(-1)^{i+|c_1|+\cdots+|c_{i-1}|+|c_{i(1)}|}\langle c_1|\cdots|c_{i(1)}|c_{i(2)}|\cdots|c_n\rangle.$$

The augmented differential graded free algebra $(\Omega(C), d = d_1 + d_2)$ is called the *cobar construction* of the coaugmented differential graded coalgebra C.

Theorem

A graded vector space C is an A_∞-coalgebra iff $\Omega(C)$ is an (augmented) differential graded (free) algebra via its canonical algebra structure.

$$\Delta_n : V \to V^{\otimes n} \leftrightarrow d_n : s^{-1}V \to (s^{-1}V)^{\otimes n}$$

Part I: Computing A_∞-structures via homotopy transfer theory

Theorem (Kadeishivili)

Let A be a a differential graded algebra. Then its homology $H_*(A)$ has an A_∞-algebra structure which can be computed via homotopy transfer theory

$$H_*(A) \rightleftarrows A \qquad \circlearrowleft$$

Theorem

Let C be a a differential graded algebra. Then its homology $H_*(C)$ has an A_∞-coalgebra structure which can be computed via homotopy transfer theory

$$H_*(C) \rightleftarrows C \qquad \circlearrowleft$$

Part I: Computing A_∞-structures via homotopy transfer theory

Problem

Compute the A_∞-structure on $\mathrm{Ext}^*_A(\mathbf{k},\mathbf{k})$ and $\mathrm{Tor}^A_*(\mathbf{k},\mathbf{k})$.

Answer

Form the bicomplex $\mathrm{Hom}_A(P_*,P_*)$ and take its total complex, still denoted by $\mathrm{Hom}_A(P_*,P_*)$. We have

$$\mathrm{Ext}^*_A(\mathbf{k},\mathbf{k}) = H^*(\mathrm{Hom}_A(P_*,P_*)).$$

Moreover, we have a deformation retract:

$$\mathrm{Ext}^*_A(\mathbf{k},\mathbf{k}) \underset{p}{\overset{i}{\rightleftarrows}} \mathrm{Hom}_A(P_*,P_*) \quad \circlearrowleft h$$

Observe that $\mathrm{Hom}_A(P_*,P_*)$ is a differential graded algebra. Via homotopy transfer theory, one gets an A_∞-algebra structure on $\mathrm{Ext}^*_A(\mathbf{k},\mathbf{k})$.
Similarly, one gets an A_∞-coalgebra structure on $\mathrm{Tor}^A_*(\mathbf{k},\mathbf{k})$.

Part I: Computing A_∞-structures via homotopy transfer theory

Problem (Reconstruction problem)

Can we reconstruct A from the $\mathrm{Ext}^*_A(\mathbf{k},\mathbf{k})$ or $\mathrm{Tor}^A_*(\mathbf{k},\mathbf{k})$?

Answer

Under certain conditions of finiteness and minimality, there is a sequence of quasi-isomorphism of differential graded algebras

$$\Omega(\mathrm{Tor}^A_*(\mathbf{k},\mathbf{k})) \to \Omega(B(A)) \to A.$$

In this case, $\Omega(\mathrm{Tor}^A_*(\mathbf{k},\mathbf{k}))$ is a minimal model of A.

Definition

A cofibrant resolution (resp. a minimal model) of A is a quasi-isomorphism of differential graded algebras

$$(T(V'), d) \to A$$

(resp. subject to a certain minimality condition).
Equivalently, A cofibrant resolution (resp. a minimal model) of A is an A_∞-coalgebra $A^¡ = s^{-1}V'$ such that $\Omega(A^¡)$ is quasi-isomorphic to A (resp. subject to a certain minimality condition).

Definition

When the minimal model exists, $A^¡$ will be called the Koszul dual A_∞-coalgebra of A.

Fact

From a cofibrant resolution (resp. a minimal model), one can construct the (minimial) projective resolution $P_* \to A \to 0$ *via*

$$P_* = A \otimes_\tau A^{\mathrm{i}},$$

where $\tau : A^{\mathrm{i}} \to A$ *is a twisting cochain and* $A \otimes_\tau A^{\mathrm{i}}$ *is the associated twisted tensor product.*

Corollary

Cofibrant resolutions (resp. the minimal model) of an algebra are equivalent to (resp. minimal) projective resolutions of the trivial module.

Part I: Koszul case

If $A = T(V)/(R)$ with $R \subset V \otimes V$, can define its Koszul dual algebra

$$A^! = T(V^*)/(R^\perp)$$

and its Koszul dual coalgebra

$$C(sV, s^2R) = \Bbbk \oplus V \oplus R \oplus \cdots \oplus \Big(\bigcap_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \Big) \oplus \cdots .$$

Theorem

A is a Koszul algebra iff $A^{¡}$ is a graded coalgebra iff $A^{¡} = C(sV, s^2R)$. In this case, the cobar construction of its Koszul dual coalgebra is a minimal model A.

Example

- *$T(V)$ is Koszul*
- *The polynomial algebra $S(V) = T(V)/(v \otimes w - w \otimes v, v, w \in V)$ is Koszul*
- *The algebra $A = T(V)/(v \otimes w, v, w \in V) = \mathbf{k} \oplus V$ is Koszul*

Part I: Non-Koszul case

Theorem

When A is not Koszul, $A^{¡} = \mathrm{Tor}_*^A(\mathbf{k}, \mathbf{k})$ is a genuine A_∞-coalgebra and there is a quasi-isomorphism $\Omega(A^{¡}) \to A$.

Remark

Tomaroff found the A_∞-coalgebra structure on $A^{¡} = \mathrm{Tor}_*^A(\mathbf{k}, \mathbf{k})$ for monomial algebras.

Part II: A crash course on operads

Operad theory is a tool to describe algebraic operations.

Roughly speaking, a nonsymmetric operad $\mathcal{P}$ is a sequence of spaces $\mathcal{P}(n), n \geq 1$, where $\mathcal{P}(n)$ is considered as a space of n-ary operations and where it is asked there are compositions of operations subject to associativity axioms etc.

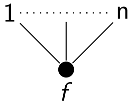
Part II: A crash course on operads

Example

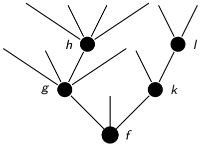
Let V be a vector space. The endomorphism operad End_V is defined to be $\mathrm{End}_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$, the space of n-linear maps on V.
For $f \in \mathrm{End}_V(n), g \in \mathrm{End}_V(m)$ and $1 \leq i \leq n$, define partial composition

$$f \circ_i g = f(\mathrm{id}^{\otimes i-1} \otimes g \otimes \mathrm{id}^{n-i-1}).$$

An n-ary operation $f \in \mathrm{End}_V(m)$ will be presented by



For example, the 10-ary operation $(((f \circ_1 g) \circ_3 h) \circ_9 k) \circ_{10} l$ can be represented by the following tree



Part II: The operad of associative algebras

An associative algebra is a pair (A, μ) such that

$$\mu \circ (\mu \otimes \mathrm{id}_A) - \mu \circ (\mathrm{id}_A \otimes \mu)$$

Definition

The operad of associative algebras is defined to be

$$\mathcal{A}ss = \mathcal{F}(M)/I$$

where $M = \mathbf{k}\mu$, $\mathcal{F}(M)$ is the free operad generated by M, I is the operadic ideal generated by

$$\mu \circ (\mu \otimes \mathrm{id}_A) - \mu \circ (\mathrm{id}_A \otimes \mu).$$

Associative algebras are exactly algebras over $\mathcal{A}ss$.

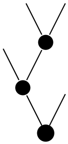
Definition

Let $\mathcal{P}$ be an operad. An $\mathcal{P}$-algebra structure on a vector space V is given by an operad map $\mathcal{P} \to \mathrm{End}_V$.

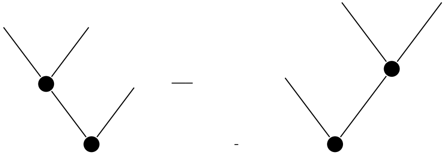
Part II: The operad of associative algebras

Present μ as the corolla with two leaves

Then elements of $\mathcal{F}(M)$ are presented by all plane binary rooted trees, such as



The associativity axiom can be presented as



Definition

A cofibrant resolution (resp. a minimal model) of an operad $\mathcal{P}$ is a quasi-isomorphism of differential graded operads

$$(\mathcal{F}(M), d) \to \mathcal{P}$$

(resp. subject to a certain minimality condition).
Equivalently, A cofibrant resolution (resp. a minimal model) of $\mathcal{P}$ is a homotopy cooperad $\mathcal{P}^{\text{i}} = s^{-1}M$ such that $\Omega(\mathcal{P}^{\text{i}})$ is quasi-isomorphic to $\mathcal{P}$ (resp. subject to a certain minimality condition).

Definition

When the minimal model exists, $\mathcal{P}^{\text{i}}$ will be called the Koszul dual homotopy coopera of $\mathcal{P}$.

Part III: Koszul case

Assume the operad $\mathcal{P}$ is Koszul.

Example

- *associative algebras*
- *commutative associative algebras*
- *Lie algebras*
- *Poisson algebras*
- *pre-Lie algebras*
- *Leibniz algebras*
- *Lie triple systems*
- *etc*

Part II: Minimal models in Koszul cases

Answer

$\mathcal{P}_\infty = \Omega(\mathcal{P}^{\text{i}})$ *is the minimal model of* $\mathcal{P}$, *where* $\mathcal{P}^{\text{i}}$ *is a genuine cooperad.*

V. Ginzburg, M. Kapranov, *Koszul duality for operads.* Duke Math. J. **76** (1994), no. 1, 203-272.

E. Getzler and D. S. J. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, hep-th/9403055 (1994).

Theorem (Ginzburg-Kapranov 90)

The operad Ass is a Koszul operad and the cobar construction of $Ass^{¡}$ *is the operad governing* A_∞*-algebras. The latter is a minimal model of* Ass.

V. Ginzburg, M. Kapranov, *Koszul duality for operads.* Duke Math. J. **76** (1994), no. 1, 203ï¿½C272.

Part III: History about algebraic deformation theory

- 1945, G. Hochschild introduced a cohomology theory of associative rings

G. Hochschild, *On the cohomology groups of an associative algebra.* Ann. of Math. (2) **46** (1945), 58-67.

- 1945, G. Hochschild introduced a cohomology theory of associative rings

G. Hochschild, *On the cohomology groups of an associative algebra.* Ann. of Math. (2) **46** (1945), 58-67.

- 1963-64, M. Gerstenhaber discovered a dg Lie algebra structure over Hochschild cochain complex and developed algebraic deformation theory

M. Gerstenhaber, *The cohomology structure of an associative ring*. Ann. Math. (2) **78** (1963) 267-288.

M. Gerstenhaber, *On the deformation of rings and algebras*. Ann. Math. (2) **79** (1964) 59-103.

- 1963, J. Stasheff defined A_∞-algebras

J. Stasheff, *Homotopy associativity of H -spaces. I, II*. Trans. Amer. Math. Soc. **108** (1963), 275-292; ibid. 108 1963 293ï¿½C312.

Part III: History about algebraic deformation theory

- 1966, A. Nijenhuis and R. W. Richardson investigated deformations and cohomologies of graded Lie algebras.

A. Nijenhuis, R. W. Richardson, *Cohomology and deformations in graded Lie algebras*. Bull. Amer. Math. Soc. **72** (1966), 1-29.

- 1990, J. Stasheff introduced L_∞-algebras

J. Stasheff, *Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras*. Quantum groups (Leningrad, 1990), pp. 120-137, Lecture Notes in Mathematics, 1510. Springer, Berlin (1992)

Part III: History about algebraic deformation theory

- 1993, P. Deligne proposed Deligne conjecture via operad theory

P. Deligne, *Letter to Stasheff, Gerstenhaber, May, Schechtman, Drinfeld*, 1993.

- 1997 M. Kontsevich proved his deformation quantization theorem for Poisson manifolds

M. Kontsevich, *Deformation quantization of Poisson manifolds*. Lett. Math. Phys. **66**(2003), 157-216.

Part III: Philosophy of Deformation Theory after Deligne, Drinfeld, Kontsevich,...

- (Deligne 1986): "The deformation theory of any mathematical object, e.g., an associative algebra, a complex manifold, etc., can be discribed starting from a certain dg Lie algebra associated to the mathematical obejct in question."

Part III: Philosophy of Deformation Theory after Deligne, Drinfeld, Kontsevich,...

- (Deligne 1986): `"The deformation theory of any mathematical object, e.g., an associative algebra, a complex manifold, etc., can be discribed starting from a certain dg Lie algebra associated to the mathematical obejct in question."`

Theorem (Lurie, Pridham)

In characteristic 0, there exists an equivalence between the ∞-category of formal moduli problems and the ∞-category of DG Lie algebras (L_∞-algebras).

J. Lurie, DAG X: Formal moduli problems.

J. P. Pridham, *Unifying derived deformation theories*, Adv. Math. **224** (2010), no. 3, 772-826.

Part III: Differential graded Lie algebras

Throughout this talk, let k be a field of characteristic zero.

Definition

A differential graded Lie algebra (aka dg Lie algebra) is a graded space $L = \oplus_{i \in \mathbb{Z}} L_i$ together with two operations:

$$l_1 : L_i \to L_{i-1}$$

of degree -1 *and*

$$l_2 : L_i \otimes L_j \to L_{i+j}$$

of degree zero such that

(i) $l_1 : L_i \to L_{i-1}$ *is a differential,*

(ii) $l_2 : L_i \otimes L_j \to L_{i+j}$ *is a Lie bracket,*

(IV) l_1 *is a derivation for* l_2 *,i.e.*

$$l_1 l_2(a \otimes b) = l_2(l_1(a) \otimes b) + (-1)^{|a|} l_2(a \otimes l_1(b))$$

for $a, b \in L$ *homogeneous.*

Part III: Maurer-Cartan elements in dg Lie algebra

Definition

Let L be a dg Lie algebra. An element $\alpha \in L_{-1}$ *is a Maurer-Cartan element if*

$$l_1(\alpha) - \frac{1}{2} l_2(\alpha \otimes \alpha) = 0.$$

Proposition (Twisting procedure)

Let L be a dg Lie algebra. Given a Maurer-Cartan element $\alpha \in L_{-1}$*, one can produce a new dg Lie algebra by imposing*

$$l_1^{\alpha}(x) = l_1(x) - l_2(\alpha \otimes x)$$

and

$$l_2^{\alpha}(x \otimes y) = l_2(x \otimes y)$$

Part III: L_∞-algebras

Definition

Let $L = \bigoplus_{i \in \mathbb{Z}} L_i$ *be a graded space over* k*. Assume that* L *is endowed with a family of linear operators* $l_n : L^{\otimes n} \to L, n \geq 1$ *with* $|l_n| = n - 2$ *satisfying the following conditions:* $\forall \sigma \in S_n, x_1, \ldots, x_n \in L,$

(i) *(Skew-symmetry)*

$$l_n(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}) = \chi(\sigma, x_1, \ldots, x_n) l_n(x_1, \ldots, x_n),$$

(ii) *(Higher Jacobi identities)*

$$\sum_{i=1}^{n} \sum_{\sigma \in Sh(i,n-i)} \chi(\sigma, x_1, \ldots, x_n)(-1)^{i(n-i)} l_{n-i+1}(l_i(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(i)}) \otimes x_{\sigma(i+1)} \otimes \ldots \otimes x_{\sigma(n)}) = 0,$$

where $Sh(i, n-i)$ *is the set of* $(i, n-i)$ *shuffles, i.e.,* $Sh(i, n-i) = \{\sigma \in S_n \text{ such that } \sigma(1) < \sigma(2) < \cdots < \sigma(i), \text{ and } \sigma(i+1) < \sigma(i+2) < \ldots \sigma(n)\}.$

Then $(L, \{l_n\}_{n \geq 1})$ *is called a* L_∞*-algebra.*

Example

Let $(L, \{l_n\}_{n\geq 1})$ *be a* L_∞*-algebra.*

(i) $n = 1$, l_1 *is a differential,*

(ii) $n = 2$, l_1 *is a derivation for* l_2,

(IV) $n = 3$, l_2 *satisfies the Jacobi identity up to homotopy.*

Part III: Maurer-Cartan elements and L_∞-algebras

Definition

Let $(L, \{l_n\}_{n\geq 1})$ be an L_∞-algebra and $\alpha \in L_{-1}$. Then α is called a Maurer-Cartan element if it satisfies equation:

$$\sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}} l_n(\alpha^{\otimes n}) = 0,$$

whenever this infinite sum exists.

Proposition

Let $(L, \{l_n\}_{n\geq 1})$ be an L_∞-algebra. Given a Maurer-Cartan element α in L_∞-algebra L, we can define a new L_∞ structure $\{l_n^\alpha\}_{n\geq 1}$ on graded space L, where $l_n^\alpha : L^{\otimes n} \to L$ is defined as :

$$l_n^\alpha(x_1 \otimes \ldots \otimes x_n) = \sum_{i=0}^{\infty} \frac{1}{i!}(-1)^{in+\frac{i(i-1)}{2}} l_{n+i}(\alpha^{\otimes i} \otimes x_1 \otimes \ldots \otimes x_n).$$

Part III: Problems from Deformation Theory

Given an algebraic structure governed by an operad $\mathcal{P}$, two basic problems of deformation theory:

Problem

Find a cofibrant resolution (or minimal model) $\mathcal{P}_\infty$ of $\mathcal{P}$, that is, there exists a quasi-isomorphism

$$\mathcal{P}_\infty = \Omega(\mathcal{P}^{¡}) \to \mathcal{P}.$$

In general, the Koszul dual $\mathcal{P}^{¡}$ is only a homotopy cooperad.

Problem

Define the deformation cohomology of $\mathcal{P}$-algebras and describe the dg Lie algebra (or L_∞ structure) on the deformation complex

Part III: From minimal models to L_∞-structures

Problem

Describe the L_∞ structure on the deformation complex

Answer

Given a cofibrant resolution, or in particular, the minimal model $\mathcal{P}_\infty = \Omega(\mathcal{P}^{\mathsf{i}})$ of $\mathcal{P}$, can introduce the deformation complex $\mathrm{Hom}(\mathcal{P}^{\mathsf{i}}, \mathrm{End}_V)$ and describe the L_∞-algebra structure on the deformation complex.

- M. Kontsevich and Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Conférence Moshé Flato 1999, Vol. I (Dijon), Math. Phys. Stud. 21 (2000), 255-307.
- P. Van der Laan, *Operads up to Homotopy and Deformations of Operad Maps*, arXiv 0208041.
- P. Van der Laan, *Coloured Koszul duality and strongly homotopy operads*, arXiv 0312147.

Part III: Koszul case

Assume the operad $\mathcal{P}$ is Koszul.

Example

- *associative algebras*
- *commutative associative algebras*
- *Lie algebras*
- *Poisson algebras*
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- *Leibniz algebras*
- *Lie triple systems*
- *etc*

Part II: Koszul case

Let $\mathcal{P}$ be a Koszul operad.

- Can compute Koszul dual cooperad $\mathcal{P}^{¡}$.
- The dg operad $\mathcal{P}_\infty = \Omega(\mathcal{P}^{¡})$ is the minimal model of $\mathcal{P}$.
- The homotopy version of $\mathcal{P}$-algebras are exactly $\mathcal{P}_\infty$-algebras.
- For a vector space, there is a graded Lie algebra

  $$(\mathrm{Hom}(\mathcal{P}^{¡}, \mathrm{End}_V), l_2)$$

  such that its Maurer-Cartan elements are in bijection with $\mathcal{P}$-algebra structures on V.
- Let μ be a $\mathcal{P}$-algebra structures on V. Then the underlying complex of the twisted dg Lie algebra

  $$(\mathrm{Hom}(\mathcal{P}^{¡}, \mathrm{End}_V), l_1^\mu, l_2^\mu)$$

  is the deformation complex of $\mathcal{P}$-algebra (V, μ).

Part III: Non-Koszul case

When $\mathcal{P}$ is NOT Koszul, no general answer so far.

Example

- *Rota-Baxter associative/Lie algebras*
- *differential associative/Lie algebras with nonzero weight*
- *Hom-associative algebras, Hom-Lie algebras,* $\cdots$
- *etc*

Part III: Our method for non-Koszul case

Four steps:

$$\begin{array}{ccc} \text{formal deformations} & & \\ \Downarrow & & \\ \text{deformation complex} & & \\ \Downarrow & & \\ L_\infty\text{−structure} & & \dashleftarrow \text{derived brackets (Bai, Guo, Sheng, Tangetc)} \\ \Downarrow & & \\ \text{minimal model} & & \end{array}$$

Part IV: Rota-Baxter associative algebras

Definition

Let $(R, \mu = \cdot)$ *be an associative algebra and* $\lambda \in k$. *A linear operator* $T : R \to R$ *is said to be a Rota-Baxter operator of weight* λ *if it satisfies*

$$T(a) \cdot T(b) = T(a \cdot T(b)) + T(T(a) \cdot b) + \lambda T(a \cdot b) \qquad (1)$$

for any $a, b \in R$, *that is,*

$$\mu \circ (T \otimes T) = T \circ \mu \circ (Id \otimes T) + T \circ \mu \circ (T \otimes Id) + \lambda T \circ \mu. \qquad (2)$$

Then (R, μ, T) *is called a Rota-Baxter algebra of weight* λ.

Remark

The Rota-Baxter relation does not lie in degree two, so one cannot use the Koszul duality theory for quadratic operads to study its cohomology theory and its minimal model.

Part IV: Rota-Baxter associative algebras

Theorem

- *Can define cohomology theory of (relative) Rota-Baxter associative algebras*
- *Can found the L_∞-structure on the deformation complex*
- *Can defined homotopy Rota-Baxter associative algebras*
- *Can prove that the operad of homotopy Rota-Baxter algebras is a minimal model of the operad of Rota-Baxter algebras.*
- *Can find the Koszul dual homotopy cooperad of the operad of Rota-Baxter algebras.*

K. Wang and G. Zhou, *Deformations and homotopy theory of Rota-Baxter algebras of any weight*, arXiv:2108.06744.

K. Wang and G. Zhou, *The minimal model of Rota-Baxter operad with arbitrary weight*. Selecta Math. to appear.

Homotopy Rota-Baxter algebras are equivalent to the following data: two family of operators $m_n : V^{\otimes n} \to V \quad \text{and} \quad T_n : V^{\otimes n} \to V$. For each $n \geqslant 1$,

$$\sum_{\substack{i+j+k=n,\\ i,k\geqslant 0, j\geqslant 1}} (-1)^{i+jk} m_{i+1+k} \circ \left(\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes k}\right) = 0$$

and

$$\sum_{\substack{l_1+\cdots+l_k=n,\\ l_1,\ldots,l_k\geqslant 1}} (-1)^{\alpha} m_k \circ \left(T_{l_1} \otimes \cdots \otimes T_{l_k}\right) = \sum_{1\leqslant q\leqslant p} \sum_{\substack{r_1+\cdots+r_q+p-q=n,\\ r_1,\ldots,r_q\geqslant 1}} \sum_{\substack{i+1+k=r_1,\\ i,k\geqslant 0}} \sum_{\substack{j_1+\cdots+j_q+q-1=p,\\ j_1,\ldots,j_q\geqslant 0}}$$

$$(-1)^{\beta} \lambda^{p-q} T_{r_1} \circ \left(\mathrm{id}^{\otimes i} \otimes m_p \circ (\mathrm{id}^{\otimes j_1} \otimes T_{r_2} \otimes \mathrm{id}^{\otimes j_2} \otimes \cdots \otimes T_{r_q} \otimes \mathrm{id}^{\otimes j_q}) \otimes \mathrm{id}^{\otimes k}\right)$$

The second equation for $n = 1, 2$ gives

$$m_1 \circ T_1 = T_1 \circ m_1, \tag{3}$$

and

$$\begin{aligned} m_2 \circ (T_1 \otimes T_1) - T_1 \circ m_2 \circ (\mathrm{id} \otimes T_1) - T_1 \circ m_2 \circ (T_1 \otimes \mathrm{id}) - \lambda T_1 \circ m_2 \\ = -\big(m_1 \circ T_2 + T_2 \circ (\mathrm{id} \otimes m_1) + T_2 \circ (m_1 \otimes \mathrm{id})\big). \end{aligned} \tag{4}$$

Equation (3) implies that $T_1 : (V, m_1) \to (V, m_1)$ is a chain map, thus T_1 is well-defined on the $\mathrm{H}_\bullet(V, m_1)$; Equation (4) indicates that T_1 is a Rota-Baxter operator of weight λ with respect to m_2 up to homotopy, whose obstruction is just operator T_2. As a consequence, $(\mathrm{H}_\bullet(V, m_1), m_2, T_1)$ is a Rota-Baxter algebra.

Thank you very much