# Lattices and thick subcategories

ICRA 2024, Shanghai

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August 3, 2024

# Part 3: Approximating triangulated categories by spaces

Part 1: Background
Part 2: The many shapes of lattices of thick subcategories.
Part 3: Approximating triangulated categories by spaces

- A fully functorial approximation
- The non-tensor spectrum
- Comparison maps to known spectra

# Part 3: Approximating triangulated categories by spaces

## Goal

Find a space that approximates $T(K)$ via a universal functorial construction.

## Motivation

The presence of a space governing the thick subcategories allows the transfer of geometric and topological techniques to the study of a triangulated category.

# Realising $T$ as a functor

Consider the category tcat with

- objects: essentially small triangulated categories;
- morphisms: exact functors.

To any morphism $F\colon K \to L$ in tcat we associate a map

$$T(F)\colon T(K) \to T(L)$$

which sends

$$M \mapsto \langle F(M) \rangle_L.$$

# Realising $T$ as a functor

**Lemma**

*Let $F\colon K \to L$ be a morphism in* tcat*. The map $T(F)\colon T(K) \to T(L)$ preserves the order and arbitrary joins.*

This uses the following Lemma:

**Lemma**

*Let $F\colon K \to L$ be a morphism in* tcat*. For any collection of objects $C \subseteq K$ there is an equality*

$$\langle F\langle C\rangle_K\rangle_L = \langle F(C)\rangle_L.$$

# Realising $T$ as a functor

**Caution**

Even for very nice morphisms $F\colon K \to L$ in tcat, the map $T(F)$ may not preserve meets.

Take $A = \mathbb{K} \bullet \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \bullet$ and invert $\alpha$ to get $B = \mathbb{K}[x]$.

This induces a localisation $F\colon \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, B)$, which sends both of the indecomposable projectives $P_1$ and $P_2$ to $B$. We obtain

$$\langle F(P_1)\rangle \cap \langle F(P_2)\rangle = \langle B\rangle = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, B)$$

but

$$\langle F(\langle P_1\rangle \cap \langle P_2\rangle)\rangle = \langle F(0)\rangle = 0.$$

# Realising $T$ as a functor

Let CjSLat be the category with

- objects: complete lattices, viewed as complete join semi-lattices;
- morphisms: maps preserving the order and arbitrary joins.

We get a functor

$$T\colon \text{tcat} \to$$

mapping $K \in \text{tcat}$ to $T(K)$ and an exact functor $F\colon K \to L$ to $T(F)\colon T(K) \to T(L)$.

# Building a functor tcat $\to$ $\mathsf{Sob}^{\mathrm{op}}$

$$
\begin{array}{ccccccc}
 & & & & \mathsf{CjSLat} & \xleftarrow{\;T\;} & \mathsf{tcat} \\
 & & & \nearrow^{U} & & & \\
\mathsf{Sob}^{\mathrm{op}} & \underset{\mathsf{pt}}{\overset{\mathcal{O}}{\rightleftarrows}} & \mathsf{SFrm} & \hookrightarrow & \mathsf{CLat} & &
\end{array}
$$

# Building a functor tcat $\to$ Sob$^{\mathrm{op}}$

**Theorem (The special adjoint functor theorem (SAFT))**

*The following conditions are sufficient for a limit-preserving functor $R\colon C \to D$ to be a right adjoint:*

- *$C$ is complete, locally small, well-powered and has a small cogenerating set;*
- *$D$ is locally small.*

# Building a functor tcat $\to$ Sob$^{\mathrm{op}}$

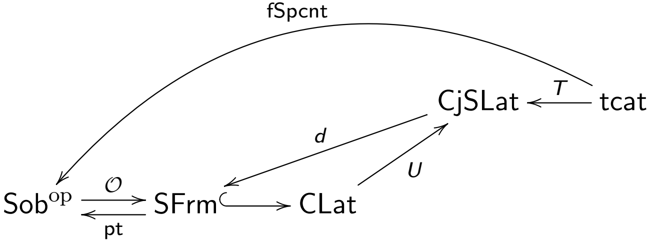
### SAFT applies

The forgetful functor SFrm $\to$ CjSLat has a left adjoint *d* by SAFT.

### Idea

- The forgetful functor Frm $\to$ Set creates limits, and SFrm is closed under limits in Frm.
- $\{*\}$ generates Sob (any two parallel morphisms must differ at some point). By Stone duality, the frame **2** cogenerates SFrm

# The fully functorial non-tensor spectrum



# A free universal approximation

**Theorem (G.-Stevenson)**

*For all* $K \in \mathrm{tcat}$ *there exists a sober space* $\mathrm{fSpcnt}(K)$ *and map*

$$f_K \colon T(K) \to \mathcal{O}(\mathrm{fSpcnt}(K))$$

*in* CjSLat *which is universal:*
*For every map* $g \colon T(K) \to F$ *in* CjSLat, *where* $F$ *is a spatial frame, there exists a unique factorisation* $h \colon \mathcal{O}(\mathrm{fSpcnt}) \to F$ *in* SFrm*:*

$$\begin{array}{ccc} T(K) & \xrightarrow{\quad f_K \quad} & \mathcal{O}(\mathrm{fSpcnt}(K)) \\ & {\scriptstyle \forall g} \searrow \quad \swarrow {\scriptstyle \exists ! h} & \\ & F & \end{array}$$

# Concrete computation

Given a complete lattice $L$, we can associate to it a sober topological space $\mathrm{spt}(L) = \mathrm{CjSLat}(L, \mathbf{2})$ with subbasis of open subsets

$$U_\ell = \{p \in \mathrm{spt}(L) \mid p(\ell) = 1\}, \text{ for all } \ell \in L.$$

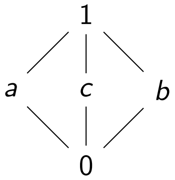
## Computing fSpcnt

Given an essentially small triangulated category $K$, fSpcnt can be explicitly computed as

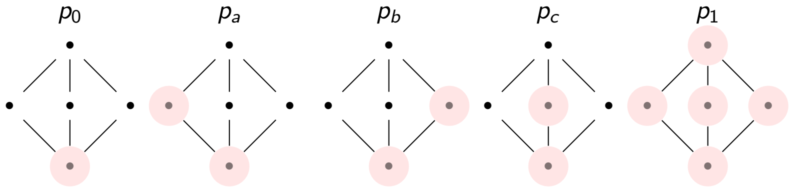
$$\mathrm{fSpcnt}(K) = \mathrm{spt}(T(K)).$$

# fSpcnt($\mathrm{D}^{\mathrm{b}}$(mod $kA_2$))

For $K = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}(\mathbb{K}A_2))$ the lattice $T(K)$ is isomorphic to



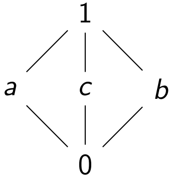

It has the following semipoints, where we colour the elements that get sent to 0.

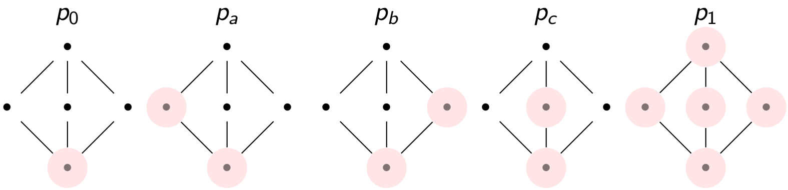


# fSpcnt($\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\mathbb{K}A_2)$)

For $K = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\mathbb{K}A_2)$ the space spt($T(K)$) has a subbasis of opens given by:

- $U_0 = \varnothing$
- $U_a = \{p_0, p_b, p_c\}$
- $U_b = \{p_0, p_a, p_c\}$
- $U_c = \{p_0, p_a, p_b\}$
- $U_1 = \{p_0, p_a, p_b, p_c\}$

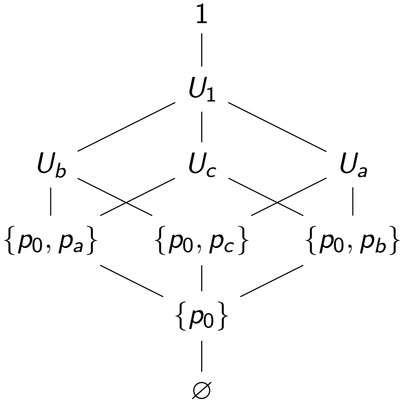


# $\mathrm{fSpcnt}(D^b(\mathrm{mod}\,\mathbb{K}A_2))$

Let $K = \mathrm{D^b}(\mathrm{mod}\,\mathbb{K}A_2)$. We have $\mathrm{fSpcnt}(K) = \{p_0, p_a, p_b, p_c, p_1\}$.

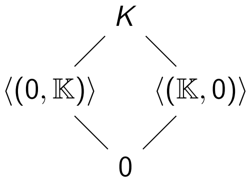
- $U_0 = \varnothing$
- $U_a = \{p_0, p_b, p_c\}$
- $U_b = \{p_0, p_a, p_c\}$
- $U_c = \{p_0, p_a, p_b\}$
- $U_1 = \{p_0, p_a, p_b, p_c\}$



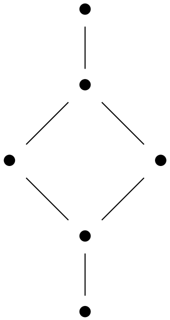
# $D^b(\operatorname{mod} \mathbb{K} \times \mathbb{K})$

For $K = D^b(\operatorname{mod} \mathbb{K} \times \mathbb{K})$ we have

$T(K)$




$\mathcal{O}(\mathrm{fSpcnt}(K))$



# Pros and cons

fSpcnt is

- free
- universal
- functorial with respect to all exact functors

However: If $T(K)$ is distributive, we generally have $T(K) \subsetneq \mathcal{O}(\mathrm{fSpcnt}(K))$.

**Question**

Is there an alternative construction which treats distributive lattices faithfully?

# Confluent functor

**Definition**

A map $F\colon K \to L$ in tcat is called *confluent* if $T(F)$ preserves finite meets.

**Open challenge**

Is there an intrinsic way to describe confluent functors in tcat?

# $T_\wedge$

**Lemma**

*The composition of two confluent functors is again confluent.*

Denote by $\mathrm{tcat}_\wedge$ the category with

- objects: essentially small triangulated categories;
- morphisms: confluent functors.

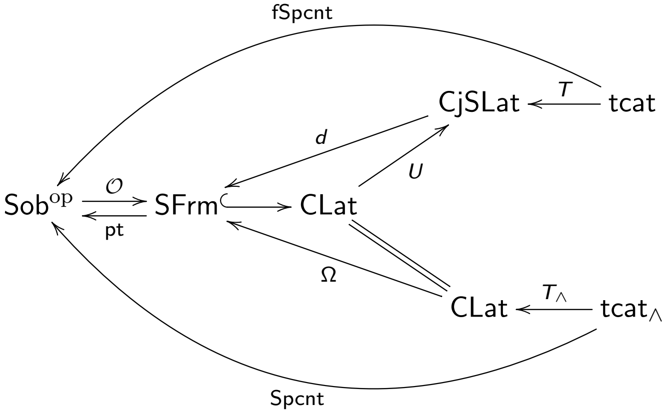
Denote by $T_\wedge \colon \mathrm{tcat}_\wedge \to \mathrm{CLat}$ the functor mapping $K \in \mathrm{tcat}$ to

$$T_\wedge(K) = T(K)$$

and a confluent $F \colon K \to L$ to the map of complete lattices

$$T_\wedge(F) = T(F) \colon T_\wedge(K) \to T_\wedge(L)$$

# The non-tensor spectrum



# Concrete computation

Given a complete lattice $L$, we can associate to it a sober topological space $\mathrm{pt}(L) = \mathrm{CLat}(L, \mathbf{2})$ with open subsets

$$U_\ell = \{p \in \mathrm{pt}(L) \mid p(\ell) = 1\}, \text{ for all } \ell \in L.$$

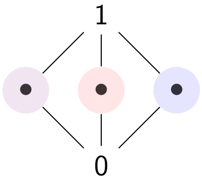
## Computing Spcnt

Given an essentially small triangulated category $K$, Spcnt can be explicitly computed as

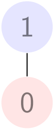
$$\mathrm{Spcnt}(K) = \mathrm{pt}(T(K)).$$

# $\mathrm{fSpcnt}(\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, kA_2))$

For $K = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}(\mathbb{K}A_2))$ the lattice $T(K)$ is isomorphic to




It has no points!




**Consequence**

We have $\mathrm{Spcnt}(\mathrm{D}^{\mathrm{b}}(\mathbb{K}A_2)) = \varnothing$.

**Theorem (G.-Stevenson)**

*For all $K \in \mathrm{tcat}$ there exists a sober space $\mathrm{Spcnt}(K)$ and map*

$$\alpha_K \colon T(K) \to \mathcal{O}(\mathrm{Spcnt}(K))$$

*in $\mathrm{SFrm}$ which is universal:*
*For every map $g \colon T(K) \to F$ in $\mathrm{CLat}$, where $F$ is a spatial frame, there exists a unique factorisation $h \colon \mathcal{O}(\mathrm{Spcnt}) \to F$ in $\mathrm{SFrm}$:*

$$\begin{array}{ccc} T(K) & \xrightarrow{\alpha_K} & \mathcal{O}(\mathrm{Spcnt}(K)) \\ & {\scriptstyle \forall g} \searrow \quad \swarrow {\scriptstyle \exists! h} & \\ & F & \end{array}$$

*If $T(K)$ is distributive, then $\alpha_K$ is an isomorphism.*

# Examples

There are cases where $T(K)$ is distributive.

- $K = \mathrm{Perf}(R)$ for a commutative ring $R$ [Thomason]
- $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\mathbb{K}G)$ for a finite $p$-group $G$ [Benson-Carlson-Rickard]
- $\mathrm{D}_{\mathrm{sg}}(A)$ for a complete intersection $A$ [Stevenson]

The last one is not covered by tt-geometry.

# Comparison maps

We have a functor

$$\mathrm{pt}\colon \mathrm{CLat}^{op} \to \mathrm{Sob}\,.$$

Thus, if $f\colon L \to T(K)$ is a map in CLat, we get a map

$$\mathrm{pt}(f)\colon \mathrm{Spcnt}(K) \to \mathrm{pt}(L).$$

**Comparison maps**

Compare Spcnt to known spectra from classes $\mathcal{L}$ of thick subcategories such that $\mathcal{L}$ is a complete sublattice of $T(K)$.

# Comparison to the Balmer spectrum

Let $K$ be a rigid tt-category (all ideals are radical).

**Theorem (G.-Stevenson)**

*We have a comparison map*

$$\begin{array}{ccc} \mathrm{Spcnt}(K) & \longrightarrow & (\mathrm{Spc}(K))^{\vee} \\ \| & & \| \\ \mathrm{pt}(T(K)) & & \mathrm{pt}(T^{\otimes}(K)) \end{array}$$

**Idea**

Because $K$ is rigid, an arbitrary join of $\otimes$-ideals in $T(K)$ is again an ideal. We obtain an inclusion of complete lattices $T^{\otimes}(K) \hookrightarrow T(K)$.

# Comparison to the non-commutative spectrum

Let $K$ be a monoidal triangulated category.

**Definition**

A $\otimes$-ideal $\mathcal{I}$ is a thick subcategory $\mathcal{I}$ such that for all $a \in K$ and $b \in I$ we have $a \otimes b \in \mathcal{I}$ and $b \otimes a \in \mathcal{I}$.
A $\otimes$-ideal is *semiprime* if $a \otimes r \otimes a \in \mathcal{Q}$ for all $r \in K$ then $a \in \mathcal{Q}$.

**Theorem (G.-Stevenson)**

*Assume every ideal of $K$ is semiprime. We get a map*

$$\mathrm{Spcnt}(K) \to \mathrm{pt}(T^{\otimes}(K)).$$

If all prime ideals are completely prime, this yields a comparison map to the dual of the noncommutative spectrum due to [Nakano-Vashaw-Yakimov].

# Centre of $T(K)$

Let $K \in$ tcat, and consider the Yoneda embedding
$\mathcal{Y}\colon K \to \operatorname{Mod} K = [K^{\mathrm{op}}, \mathrm{Ab}]_{\mathrm{add}}$.
If $U \hookrightarrow K$ is a thick subcategory, we get an induced adjunction

$$\operatorname{Mod} U \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \operatorname{Mod} K$$

where $\iota^*$ is fully faithful and $\iota_*$ is a quotient. Set $\Gamma_U = \iota^* \iota_*$.

# Centre of $T(K)$

**Definition**

Two thick subcategories $U, V \in T(K)$ *commute* if

$$\Gamma_U \Gamma_V \xleftarrow{\simeq} \Gamma_{U \cap V} \xrightarrow{\simeq} \Gamma_V \Gamma_U \; .$$

A thick subcategory $U \in T(K)$ is called *central* if it commutes with all $V \in T(K)$. The *centre* $T(K)$ is the subset

$$Z(T(K)) = \{U \in T(K) \mid U \text{ is central}\}.$$

# Comparison to the centre

**Theorem (Krause)**

*The centre $Z(T(K))$ is closed in $T(K)$ under joins and finite meets. Moreover, it is a spatial frame.*

**Consequence**

We get a comparison map

$$\mathrm{Spcnt}(K) \to \mathrm{pt}(Z(T(K))).$$

# Open question

**Open question**

Can we find other interesting distributive sublattices of $T(K)$?

**Motivation**

Try to parametrise $T(K)$ by the space corresponding to the sublattice plus additional data.

# Acknowledging continuous and distributive parts

**Potential angle of attack**

Realising $T(K)$ as a lattice of "decorated spaces".

For $N \in \mathbb{N}$ consider the set $[N] = \{0, 1, \dots, N\}$ with the discrete topology. We have

$$\mathcal{O}([N]) = 2^{[N]}.$$

Now "decorate" the space $[N]$ by associating to each $U \in \mathcal{O}([N])$ the lattice of non-crossing partitions of $U$.

# Non-exhaustive non-crossing partitions

**Definition**

We set the *non-exhaustive non-crossing partitions* of $[N]$ to be the set $\mathrm{NNC}([N])$ defined as

$$\{(U, \mathcal{P}) \mid U \in \mathcal{O}([N]),\ \mathcal{P} \text{ a non-crossing partition of } U\}.$$

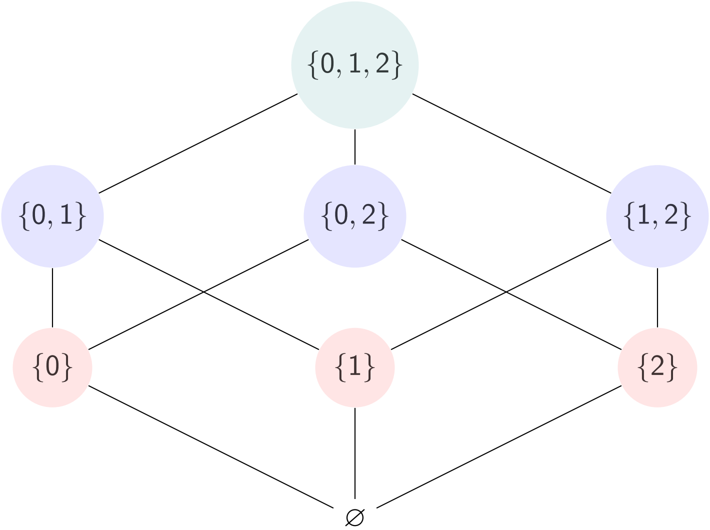
This is a lattice under the product order: $(U, \mathcal{P}) \leq (V, \mathcal{Q})$ if and only if $U \subseteq V$ and $\mathcal{P} \leq \mathcal{Q}$.

# Thick subcategories of discrete cluster categories

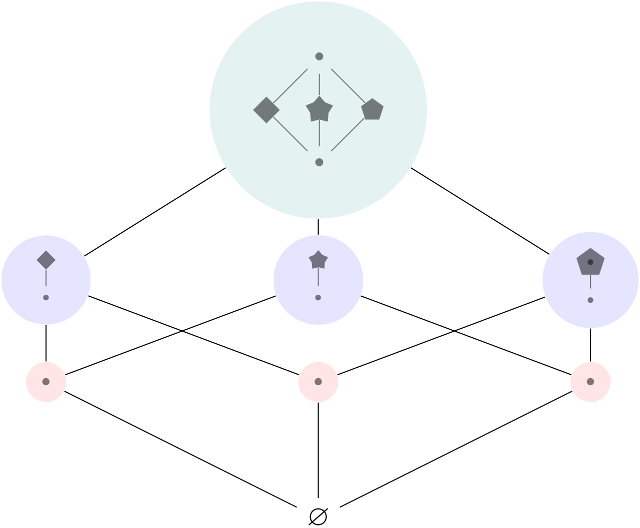
**Theorem (G.-Zvonareva)**

*The lattice* $\mathrm{NNC}([N])$ *can be realised as a lattice of thick subcategories. More precisely, it is isomorphic to the lattice of thick subcategories of a discrete cluster category* $\mathcal{C}_N$ *of infinite type A.*

# Powerset of [2]



# NNC([2])



# $\mathsf{Spcnt}(\mathcal{C}(\mathcal{Z}))$

## Computation

For $N \geq 2$ we have

$$\mathsf{Spcnt}(\mathcal{C}_N) \cong \mathsf{pt}(\mathrm{NNC}[N]) \cong [N].$$

# Open question

**Question**

When can we realise $T(K)$ as a lattice of decorated open subsets of Spcnt?

# Thank you!