Lattices and thick subcategories

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※Extra slides

Marking of extra slides

Slides not explicitly discussed in the Lectures are marked by ※, and have a light blue background.

Part 2: The many shapes of $T(K)$

Part 1: Background

Part 2: The many shapes of lattices of thick subcategories.

- Spatial frames
- Algebraic lattices
- Beyond distributivity

Part 3: Approximating triangulated categories by spaces

Motivation

We fix K an essentially small triangulated category.

Motivation

Understanding lattice-theoretic properties of $T(K)$.

Reminder

Recall

A lattice L is *distributive* if for all $a, b, c \in L$ we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

A lattice is *complete* if all meets and joins exist.

In particular, every complete lattice is *bounded*: It has a top 1 and a bottom 0.

Examples

- For a topological space X, the lattice $\mathcal{O}(X)$ is complete and distributive.
- $T(K)$ is complete, but may or may not be distributive.

Definition

The *dual lattice* $(L^{\mathrm{op}}, \leq_{L^{\mathrm{op}}})$ of a lattice $(L, \leq_L)$ has underlying set $L^{\mathrm{op}} = L$ with relation $a \leq_{L^{\mathrm{op}}} b$ if and only if $b \leq_L a$.

- The dual lattice of a complete and distributive lattice is again complete and distributive.

Example

Given a topological space X the lattice $\mathcal{O}(X)^{\mathrm{op}}$ is isomorphic to the lattice of closed subsets of X.

Frames

Definition

A complete lattice L is a *frame* if binary meets distribute over arbitrary joins: For all $a \in L$ and $S \subseteq L$ we have

$$a \wedge (\bigvee S) = \bigvee_{s \in S} (a \wedge s).$$

Example

For a topological space X the lattice $\mathcal{O}(X)$ is a frame: For $U \in \mathcal{O}(X)$, $\{V_i \mid i \in I\} \subseteq \mathcal{O}(X)$ we have

$$U \cap \left(\bigcup_{i \in I} V_i \right) = \bigcup_{i \in I} (U \cap V_i) \, .$$

Frames

Careful

Being a frame is not equivalent to binary joins distributing over arbitrary meets!

Take $\mathbb{R}$ with the standard topology, and take $U = \mathbb{R} \setminus \{0\}$ and $V_i = (-\frac{1}{i}, \frac{1}{i})$. We have

$$U \vee \left(\bigwedge_{i \in I} V_i \right) = U \cup \operatorname{int}(\{0\}) = U \cup \varnothing = U = \mathbb{R} \setminus \{0\}.$$

However,

$$\bigwedge_{i \in I} (U \vee V_i) = \bigwedge_{i \in I} \mathbb{R} = \mathbb{R}.$$

Complete distributive lattices

Consequence

There exist complete distributive lattices which are not frames, for example $\mathcal{O}(\mathbb{R})^{\mathrm{op}}$.

From topological spaces to frames

Denote by Frm the category with

- objects: frames;
- morphisms: order preserving maps which preserve all joins and finite meets.

Denote by Top the category with

- objects: topological spaces;
- morphisms: continuous maps.

We have a functor

$$\mathcal{O}\colon \mathsf{Top}^{\mathrm{op}} \to \mathsf{Frm}\,.$$

sending a topological space X to

$$\mathcal{O}(X)$$

and a continuous map $f\colon X \to Y$ to the map of frames

$$\begin{array}{rcl} \mathcal{O}(f)\colon \mathcal{O}(Y) & \to & \mathcal{O}(X) \\ U & \mapsto & f^{-1}(U). \end{array}$$

Spatial frames

Not every frame is of the form $\mathcal{O}(X)$.

Definition

A frame F is *spatial* if there exists a topological space X and an isomorphism of frames $F \cong \mathcal{O}(X)$.

Towards an intrinsic definition via points

Let F be a frame.

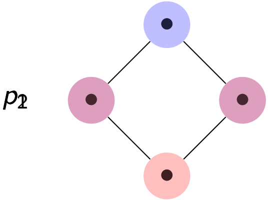
Definition

A point p of F is an element of $\mathrm{Frm}(F, \mathbf{2}) = \mathrm{pt}(F)$, where

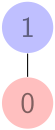
$$\mathbf{2} = \begin{array}{c} 1 \\ | \\ 0 \end{array}$$

Example

The square



has two points.



$$F \cong \mathcal{O}(\{p_1, p_2\})$$

Enough points

Definition

A frame F has *enough points* if

$$\text{for all } a, b \in F \text{ with } a \nleq b$$

there exists a point $p \in \mathrm{pt}(F)$ such that

- $p(a) = 1$ and
- $p(b) = 0$.

"Example"

For a topological space X, the lattice of opens $\mathcal{O}(X)$ has enough points: For $U, V \in \mathcal{O}(X)$ with $U \not\subseteq V$ we pick $x \in X$ such that

$$x \in U \text{ and } x \notin V.$$

We define $p_x \colon \mathcal{O}(X) \to \mathbf{2}$ by

$$p_x \colon W \mapsto \begin{cases} 1 & \text{if } x \in W \\ 0 & \text{if } x \notin W. \end{cases}$$

The point p_x separates U and V.

Spatial frames

Fact

A frame is spatial if and only if it has enough points.

Proof.

$\Rightarrow$: $U, V \in \mathcal{O}(X)$ with $U \subsetneq V$ are separated by p_x for $x \in U$, $x \notin V$.
$\Leftarrow$: Assume F has enough points. We can endow $\mathrm{pt}(F)$ with a topology by declaring the open subsets to be the sets

$$U_\ell = \{p \in \mathrm{pt}(F) \mid p(\ell) = 1\}, \text{ for all } \ell \in F.$$

One checks that this yields a bijection $F \to \mathcal{O}(\mathrm{pt}(F))$ given by $\ell \mapsto U_\ell$, which preserves finite meets and arbitrary joins. □

Non-spatial frame

Definition

A *regular open set* of a topological space X is an open subset such that

$$\mathrm{int}(\overline{X}) = X,$$

where $\mathrm{int}(\overline{X})$ denotes the interior of the topological closure of X.

Every open interval of $\mathbb{R}$ is a regular open set. However, not every open is regular open:

$$\mathrm{int}(\overline{(-1,0)\cup(0,1)}) = (-1,1) \neq (-1,0)\cup(0,1).$$

The regular open subsets $\mathcal{RO}(X)$ form a frame under inclusion.

Example

A non-spatial frame

The poset of regular open subsets $\mathcal{RO}(\mathbb{R})$ is a non-spatial frame under inclusion. In fact, it has *no* points.

Set up for the proof

First note that $\mathcal{RO}(\mathbb{R})$ has no *atoms*: For every $\varnothing \neq U \in \mathcal{RO}(\mathbb{R})$ there exists a regular open U' with $\varnothing \neq U' \subsetneq U$.
Second, note that every $U \in \mathcal{RO}(\mathbb{R})$ has a *lattice complement*: There exist $U^c = \operatorname{int}(\mathbb{R} \setminus U)$ such that

$$U \vee U^c = \mathbb{R} \text{ and } U \wedge U^c = \varnothing.$$

Note that $\mathcal{RO}(\mathbb{R})$ also has no *coatoms*.

Example

A non-spatial frame

The poset of regular open subsets $\mathcal{RO}(\mathbb{R})$ is a non-spatial frame under inclusion. In fact, it has *no* points.

Proof.

Assume p is a point and set $U = \bigvee\{W \in \mathcal{RO}(\mathbb{R}) \mid p(W) = 0\}$.
We have $p(U) = 0$, and hence $U \neq \mathbb{R}$.
We can therefore pick $U \subsetneq V \subsetneq \mathbb{R}$, and find $p(V) = 1$.
In particular, this implies $p(V^c) = 0$ and therefore $V^c \leq U$.
However, $U \subseteq V$ also implies that $V^c \leq U^c$.
This implies $\varnothing \neq V^c \leq U^c \wedge U = \varnothing$; a contradiction. □

Stone duality

Denote by SFrm the full subcategory of Frm whose objects are the spatial frames. We have an adjunction

$$\mathsf{Top}^{\mathrm{op}} \underset{\mathsf{pt}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathsf{SFrm}\ .$$

Definition

A topological space X is *sober* if every irreducible closed subset has a unique generic point.
A closed subset Z is *irreducible* if $Z = Z_1 \cup Z_2$ with Z_1 and Z_2 closed implies $Z = Z_1$ or $Z = Z_2$. A generic point of Z is a point $z \in Z$ such that $Z = \overline{\{z\}}$.

Stone duality

Proposition

Let F be a spatial frame. Then $\mathrm{pt}(F)$ is a sober space.

Idea

The closure of $p \in \mathrm{pt}(F)$ is the set

$$\{q \in \mathrm{pt}(F) \mid p^{-1}(0) \subseteq q^{-1}(0)\}.$$

It turns out that the sets of this form are exactly the irreducible closed subsets.

Stone duality

Denote by Sob the full subcategory of Top which has as objects sober spaces.

Stone duality

We obtain a duality

$$\mathsf{Sob}^{\mathrm{op}} \underset{\mathsf{pt}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathsf{SFrm} .$$

Distributive versus non-distributive lattices

Lattices of thick subcategories arising from:

tt geometry $\rightsquigarrow$ spatial frames

finite dimensional algebras $\rightsquigarrow$ often non-distributive lattices

Implications

spatial frame $\Rightarrow$ frame $\Rightarrow$ complete distributive lattice

Compact elements

An element l in a lattice L is *compact* if whenever

$$l \leq \bigvee S$$

for some subset $S \subseteq L$ then there exists a finite subset $\{s_1, \ldots, s_n\} \subseteq S$ such that

$$l \leq \bigvee_{i=1}^{n} s_i .$$

Algebraic lattice

Definition

A complete lattice L is *algebraic* if its compact elements generate under joins, that is, every element of L is a join of compact elements.

Theorem (Grätzer-Schmidt)

A lattice L is algebraic if and only if it is isomorphic to a lattice of subobjects of a T-algebra for a Lawvere theory T.

Lattices of thick subcategories are algebraic

Theorem (G.-Stevenson)

The lattice of thick subcategories $T(K)$ is algebraic.

Idea

One can show that the compact elements of $T(K)$ are precisely the thick subcategories $\langle k \rangle$ generated by an object $k \in K$. Now take any thick subcategory $T \in T(K)$. We have

$$T = \bigvee_{t \in T} \langle t \rangle,$$

hence it is a join of compact elements.

Corollary (G.-Stevenson)

A lattice of thick subcategories $T(K)$ is a spatial frame if and only if it is distributive.

Modular lattice

Given a ring R and a right R-module M, its lattice of submodules $\mathrm{Sub}(M)$ is a lattice under inclusion, with $\wedge = \cap$ and $\vee = +$.

Definition

A lattice is *modular* if for all a, b, c in L with $a \leq c$ we have

$$a \vee (b \wedge c) = (a \vee b) \wedge c.$$

Distributivity and modularity

Distributivity implies modularity: Indeed, if L is distributive and $a, b, c \in L$ with $a \leq c$ then

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c.$$

However, there are modular lattices which are not distributive.

※$\mathrm{Sub}(M)$

Let R be a ring and let M be a right R-module.

- The lattice $\mathrm{Sub}(M)$ is modular: For all submodules N_1, N_2, N_3 with $N_1 \subseteq N_3$ we have

$$N_1 \vee (N_2 \wedge N_3) = N_1 + (N_2 \cap N_3) = (N_1 + N_2) \cap N_3.$$

- It is not in general distributive: Take $R = \mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We have

$$\langle(1,1)\rangle \cap (\langle(1,0)\rangle + \langle(0,1)\rangle) = \langle 1,1 \rangle$$

but

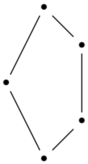
$$(\langle(1,1)\rangle \cap \langle(1,0)\rangle) + (\langle(1,1)\rangle \cap \langle(0,1)\rangle) = 0.$$

Minimal non-modular lattice

Question

Is $T(K)$ always modular?

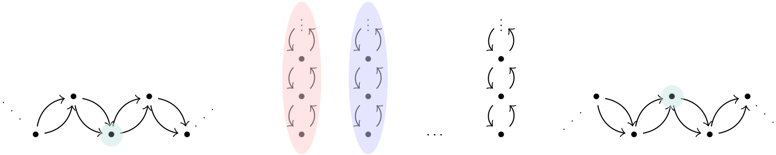
The pentagon



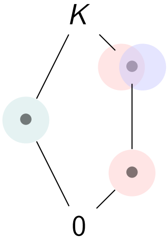
is the minimal non-modular lattice. A lattice is non-modular if and only if it contains the pentagon as a sublattice.

Non-modular $T(K)$

Consider $K = \mathrm{D}^{\mathrm{b}}(\mathbb{K} \bullet \rightrightarrows \bullet)$.



The lattice $T(K)$ contains the pentagon as a bounded sublattice:



Coherent frames

Definition

A frame F is *coherent* if there is exists a commutative ring R such that $F \cong \mathcal{O}(\mathrm{Spec}(R))$.

Theorem (Balmer, Buan-Krause-Solberg, Kock-Pitsch)

Let K be a tt-category. The lattice $T^{\sqrt{\otimes}}(K)$ of radical thick tensor ideals is a coherent frame.

Question

If $T(K)$ is distributive, is it also a coherent frame?

Spectral spaces

Definition

A space is *spectral* if

- it is sober;
- it is quasi-compact;
- it has a basis of quasi-compact open subsets whose finite intersections are also quasi-compact.

Theorem (Hochster)

A space is spectral if and only if it is isomorphic to $\mathrm{Spec}(R)$ *for some commutative ring* R*.*

Coherent frame

A space is spectral if

- it is sober;
- it is quasi-compact;
- its quasi-compact open subsets are closed under finite intersections and form a basis for the topology.

A frame F is coherent if

- $\mathrm{pt}(F)$ is sober (for free)
- its top 1 is compact.
- its compact elements are closed under finite meets and generate F under joins.

Coherent frame

A space is spectral if

- it is sober;
- it is quasi-compact;
- its quasi-compact open subsets are closed under finite intersections and form a basis for the topology.

A frame F is coherent if and only if the compact elements of F form a bounded sublattice and generate F under joins.

Non-coherent spatial frame $T(K)$

Consider $K = \mathrm{D}^{\mathrm{b}}_{\mathrm{tors}}(\mathrm{mod}\,\mathbb{K}[x])$, the full subcategory of $L = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\mathbb{K}[x])$ of objects with torsion cohomology. Assume $\mathbb{K} = \overline{\mathbb{K}}$.

- $K \subseteq L$ thick subcategory $\Rightarrow T(K) \hookrightarrow T(L)$ as lattices
- $T(L)$ distributive $\Rightarrow T(K)$ distributive. In particular, they are both spatial frames.
- K is not compact in $T(K)$: K has infinitely many tubes.

Consequence

The lattice $T(K)$ is a spatial frame which is not coherent.

※Most possibilities occur

For a complete lattice L we have the following implications:

$$\begin{array}{c}\text{coherent frame}\\ \Downarrow \\ \text{spatial frame}\\ \Downarrow \\ \text{frame}\\ \Downarrow \\ \text{distributivity}\\ \Downarrow \\ \text{modularity}\end{array}$$

Most possibilities occur

For the algebraic lattice $T(K)$ we have the following implications:

$$\begin{array}{c}\text{coherent frame}\\ \Downarrow \\ \text{spatial frame}\\ \Updownarrow \\ \text{frame}\\ \Updownarrow \\ \text{distributivity}\\ \Downarrow \\ \text{modularity}\end{array}$$

Open question

Question

Which algebraic lattices can be realised as lattices of thick subcategories?