Lattices and thick subcategories I

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Overview

Part 1: Background

- Lattices
- Thick subcategories
- Two motivating examples

Part 2: The many shapes of lattices of thick subcategories

Part 3: Approximating triangulated categories by spaces

※Extra slides

Marking of extra slides

Slides not explicitly discussed in the Lectures are marked by ※, and have a light blue background.

Lattices

Definition

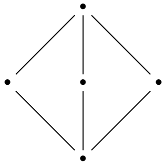
A lattice is a partially ordered set (poset) $(L, \leq)$ such that for all $a, b \in L$ there exists

- a join $a \vee b = \min\{c \mid a \leq c, b \leq c\}$ and
- a meet $a \wedge b = \max\{c \mid c \leq a, c \leq b\}$.

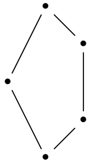
In particular, finite non-empty joins and meets exist.

Examples of lattices

- $(\mathbb{Z}, \leq)$
- the diamond



- the pentagon



Example: Topology

Lattice of opens

Let X be a topological space with set of open subsets $\mathcal{O}(X)$.
Then $\mathcal{O}(X)$ is a lattice under inclusion $\subseteq$.
For $U, V \in \mathcal{O}(X)$ we have

- $U \vee V = U \cup V$ and
- $U \wedge V = U \cap V$.

Example: Representation theory

Lattice of submodules

Let R be a ring, and let M be a right R-module.
The set of submodules $\mathrm{Sub}(M)$ forms a lattice under inclusion $\subseteq$.
For $N_1, N_2 \in \mathrm{Sub}(M)$ we have

- $N_1 \vee N_2 = N_1 + N_2$ and
- $N_1 \wedge N_2 = N_1 \cap N_2$.

Example: Combinatorics

Definition

Let $(\mathcal{L}, <)$ be a linearly ordered set. A non-crossing partition of $\mathcal{L}$ is a partition $\mathcal{P} = \{B_i \mid i \in I\}$ of $\mathcal{L}$

$$\mathcal{L} = \bigsqcup_{i \in I} B_i, \; B_i \neq \varnothing \text{ for all } i \in I,$$

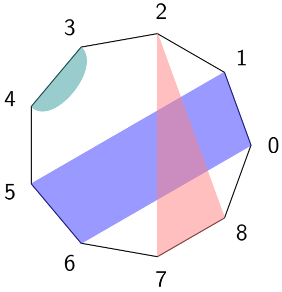
such that the following holds:
If we have $a, b \in B_i$ and $c, d \in B_j$ with

$$a < c < b < d$$

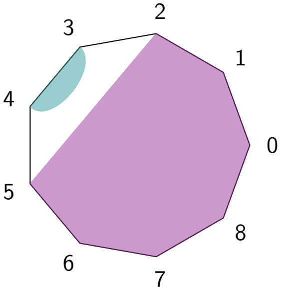
then we must have $i = j$.

Non-crossing partitions

Take $\mathcal{L} = \{0, 1, \ldots, 8\}$.



A crossing partition



A non-crossing partition

Lattice of non-crossing partitions

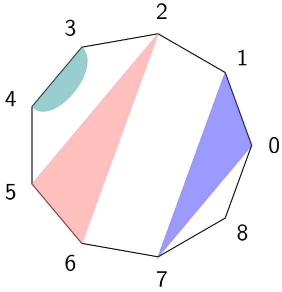
Denote by $\mathrm{NC}(\mathcal{L})$ the set of non-crossing partitions of a linearly ordered set $\mathcal{L}$.

Theorem (Kreweras)

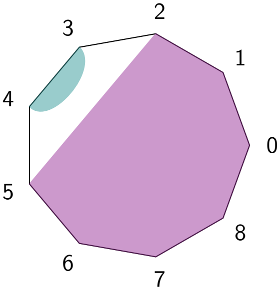
The set $\mathrm{NC}(\mathcal{L})$ forms a lattice under reverse refinement: For $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{NC}(\mathcal{L})$ we have $\mathcal{P}_1 \leq \mathcal{P}_2$ if for all $B_1 \in \mathcal{P}_1$ there exists a $B_2 \in \mathcal{P}_2$ such that $B_1 \subseteq B_2$.

Non-crossing partitions

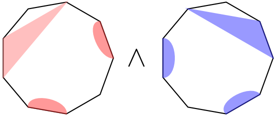
Take $\mathcal{L} = \{0, 1, \ldots, 8\}$.

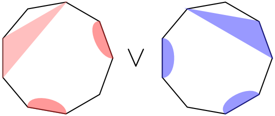


$\leq$

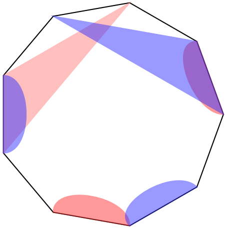


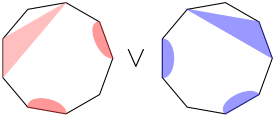
Meet and join of non-crossing partitions



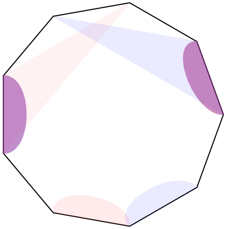


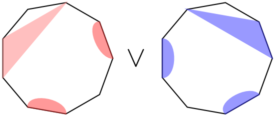
Meet and join of non-crossing partitions





Meet and join of non-crossing partitions





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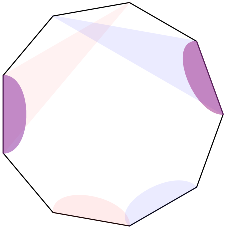
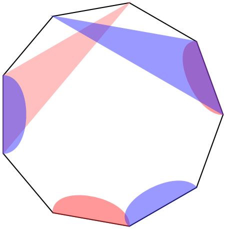
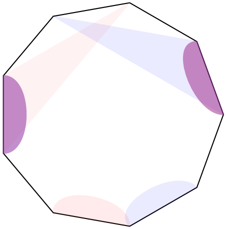
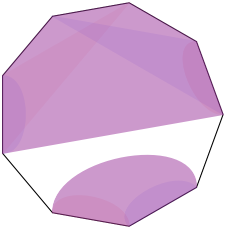




Meet and join of non-crossing partitions





Complete lattice

Definition

A lattice L is *complete* if arbitrary joins and meets exist: For every subset $S \subseteq L$ there exists

- a join: $\bigvee S = \min\{c \mid s \leq c \text{ for all } s \in S\}$;
- a meet: $\bigwedge S = \max\{c \mid c \leq s \text{ for all } s \in S\}$.

Existence of arbitrary meets suffices

A poset L has arbitrary meets if and only if it has arbitrary joins:

- $\bigwedge S = \bigvee\{l \in L \mid l \leq s \text{ for all } s \in S\}$;
- $\bigvee S = \bigwedge\{l \in L \mid s \leq l \text{ for all } s \in S\}$.

Examples

- The lattice $\mathbb{Z}$ under the standard order is not complete.
- The lattice of opens $\mathcal{O}(X)$ of a topological space X is complete: For a subset $S \subseteq \mathcal{O}(X)$ we have

$$\bigvee S = \bigcup S \text{ and } \bigwedge S = \text{int}\left(\bigcap S\right),$$

where $\text{int}(A)$ for a subset A of X denotes the *interior of A*, that is the largest open subset of X contained in A:

$$\text{int}(A) = \bigcup_{U \subseteq A} U.$$

Triangulated categories

"Definition"

A triangulated category is an additive category K with an autoequivalence $\Sigma\colon K \to K$ and a distinguished class of triangles

$$A \to B \to C \to \Sigma A,$$

satisfying certain axioms.

Thick subcategories

Let K be a triangulated category.

Definition

A *thick subcategory* of K is a full subcategory $T \subseteq K$ which is closed under:

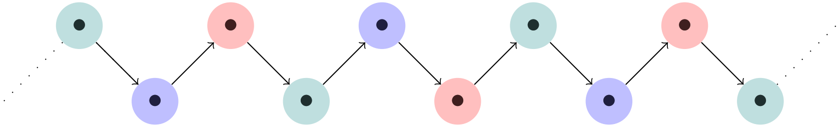
- the action of Σ: $\Sigma T = T = \Sigma^{-1} T$;
- "two out of three": If

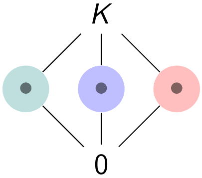
  $$A \to B \to C \to \Sigma A$$

  is a triangle and two out of A, B and C are in T, then so is the third;
- summands.

Example

Throughout, we fix a field $\mathbb{K}$. Let $K = \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \mathbb{K}A_2)$.





Lattice of thick subcategories

Lattice of thick subcategories

Fix an essentially small triangulated category K. The set

$$T(K) = \{\text{thick subcategories of } K\}$$

forms a complete lattice under inclusion $\subseteq$. For a subset $\mathcal{T} \subseteq T(K)$ we have

- $\bigwedge \mathcal{T} = \bigcap \mathcal{T}$;
- $\bigvee \mathcal{T} = \langle \bigcup \mathcal{T} \rangle$, the smallest thick subcategory containing $\bigcup \mathcal{T}$.

$\mathrm{D^b}(\mathsf{mod}\,\mathbb{K}[\mathrm{x}])$

Theorem (Hopkins, Neeman)

There is a lattice isomorphism

$$T(\mathrm{D^b}(\mathsf{mod}\,\mathbb{K}[x])) \cong \{\textit{unions of closed subsets of } \mathsf{Spec}(\mathbb{K}[x])\},$$

where the latter is a lattice under inclusion.

$\mathrm{D^b}(\mathrm{mod}\,\mathbb{K}[\mathrm{x}])$

Assume $\mathbb{K} = \overline{\mathbb{K}}$.

- indecomposables in $\mathrm{mod}\,\mathbb{K}[x]$: $\mathbb{K}[x]$ and $\mathbb{K}[x]/(x-\alpha)^n$, $\alpha \in \mathbb{K}$, $n \geq 1$
- $\mathrm{Spec}\,\mathbb{K}[x] = \{(x-\alpha) \mid \alpha \in \mathbb{K}\} \sqcup \{(0)\}$
- closed subsets of $\mathrm{Spec}\,\mathbb{K}[x]$: $\varnothing$, $\mathrm{Spec}\,\mathbb{K}[x]$, finite subsets of $\{(x-\alpha) \mid \alpha \in \mathbb{K}\}$

$\mathrm{D^b}(\mathrm{mod}\,\mathbb{K}[x])$	$\mathrm{Spec}\,\mathbb{K}[x]$
0	$\varnothing$
$\langle \mathbb{K}[x]/(x-\alpha) \rangle$	$\{(x-\alpha)\}$

Commutative ring R

Theorem (Hopkins, Neeman, Thomason)

There is a lattice isomorphism

$$T(\mathrm{Perf}(R)) \cong \{\textit{Thomason subsets of } \mathrm{Spec}(R)\},$$

where the latter is a lattice under inclusion.

A subset of $\mathrm{Spec}(R)$ is a *Thomason subset*, if it is a union of closed subsets with quasi-compact complements.

Controlled by a space

The *Hochster dual* of $X = \mathrm{Spec}(R)$ is the topological space $X^\vee$, whose underlying set is X and whose open subsets are the Thomason subsets of $\mathrm{Spec}(R)$.

Theorem (Hopkins, Neeman, Thomason)

There is a lattice isomorphism

$$T(\mathrm{Perf}\ R) \cong \mathcal{O}(\mathrm{Spec}(R)^\vee).$$

※tt-geometry

Let K be an essentially small symmetric monoidal triangulated category (*tt-category*).

Definition

A thick tensor ideal is a thick subcategory $I \subseteq K$ such that for all $k \in K$ and $t \in I$ we have $k \otimes t \in I$.
A thick tensor ideal $\mathcal{P}$ is *prime* if whenever $a \otimes b \in \mathcal{P}$ we have $a \in \mathcal{P}$ or $b \in \mathcal{P}$.
A thick tensor ideal I is *radical* if whenever $a \otimes \cdots \otimes a \in I$ then $a \in I$.

※tt-geometry

Let K be an essentially small symmetric monoidal triangulated category (*tt-category*).

Definition

The *Balmer spectrum*

$$\mathrm{Spc}(K) = \{\text{prime } \otimes\text{-ideals in } K\}$$

is a topological space under the Zariski topology.

Theorem (Balmer)

There is an isomorphism of lattices (under inclusion):

$$\{\textit{radical } \otimes\textit{-ideals of } K\} \cong \mathcal{O}(\mathrm{Spc}(K)^{\vee}).$$

※tt-geometry

Let K be an essentially small tt-category. The *support* of $x \in K$ is defined as $\operatorname{supp} x = \{\mathcal{P} \in \operatorname{Spc}(K) \mid x \notin \mathcal{P}\}$.

Theorem (Balmer)

There is an isomorphism of lattices (under inclusion):

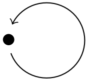
$$\{\textit{radical } \otimes\textit{-ideals of } K\} \xrightarrow{\cong} \{\textit{Thomason subsets of } \operatorname{Spc}(K)\}$$

$$I \longmapsto \bigcup_{x \in I} \operatorname{supp} x$$

$$\{x \in K \mid \operatorname{supp} x \subseteq V\} \longleftarrow\!\shortmid V$$

$\mathrm{D}^{\mathrm{b}}(\mathsf{gr}\,\mathbb{K}[\mathrm{x}])$

- $\mathsf{mod}\,\mathbb{K}[x]$



- $\mathsf{gr}\,\mathbb{K}[x]$

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

$D^b(\mathrm{gr}(\mathbb{K}[x]))$

Theorem (G.-Stevenson)

There is a lattice isomorphism

$$T(D^b(\mathrm{gr}(\mathbb{K}[x]))) \cong \mathrm{NC}(\mathbb{Z} \cup \{-\infty\}).$$

Idea: Identify the Σ-orbit of

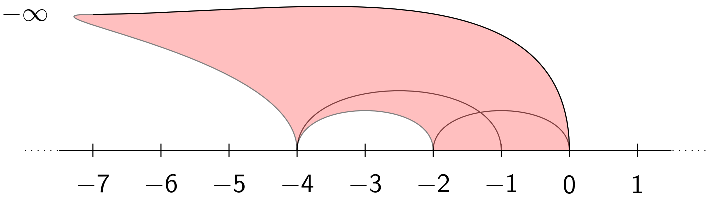
- $\mathbb{K}[x](j)$ with the pair $\{-\infty, j\}$;
- $\mathbb{K}[x]/(x^i)(j)$ for $i \in \mathbb{N}$, $j \in \mathbb{Z}$ with the pair $\{j-i, j\}$.

$\mathrm{D}^{\mathrm{b}}(\mathrm{gr}(\mathbb{K}[x]))$

Idea: Identify the Σ-orbit of

- $\mathbb{K}[x](j)$ with the pair $\{-\infty, j\}$;
- $\mathbb{K}[x]/(x^i)(j)$ for $i \in \mathbb{N}$, $j \in \mathbb{Z}$ with the pair $\{j-i, j\}$.

The thick subcategory $\langle \mathbb{K}[x]/(x^2), \mathbb{K}[x]/(x^3)(-1), \mathbb{K}[x] \rangle$ corresponds to the non-crossing partition



Not controlled by a space

Note

For every topological space X we have

$$T(\mathrm{D}^{\mathrm{b}}(\mathrm{gr}(\mathbb{K}[x]))) \ncong \mathcal{O}(X).$$

Problem: $T(\mathrm{D}^{\mathrm{b}}(\mathrm{gr}\,\mathbb{K}[x]))$ is not distributive.

Distributivity

Definition

A lattice L is *distributive* if for all $a, b, c \in L$ we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

A lattice is distributive if and only if for all $a, b, c \in L$ we have

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

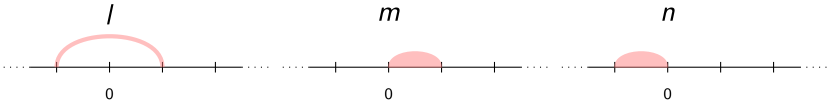
Examples

- Let X be a topological space. Then $\mathcal{O}(X)$ is distributive: For all U, V, W in $\mathcal{O}(X)$ we have

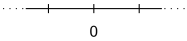
$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W).$$

- $T(\mathrm{D}^{\mathrm{b}}(\mathrm{gr}\,\mathbb{K}[x])) \cong \mathrm{NC}(\mathbb{Z} \cup \{-\infty\})$ is not distributive.

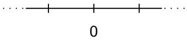
$T(\mathrm{D^b}(\mathbb{K}[\mathrm{x}]))$ is not distributive



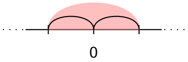
$l \wedge m = l \wedge n =$



$(l \wedge m) \vee (l \wedge n) =$



$m \vee n =$



$l \wedge (m \vee n) = l$

Non-crossing partitions beyond type A

Theorem (Ingalls-Thomas)

Let Q be an orientation of an ADE-diagram Δ. Then

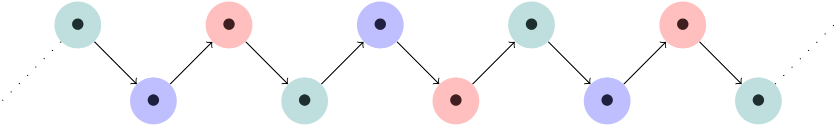
$$T(D^b(\operatorname{mod} kQ)) \cong \mathrm{NC}(\Delta),$$

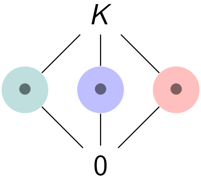
where $\mathrm{NC}(\Delta)$ denotes the non-crossing partitions of type Δ.

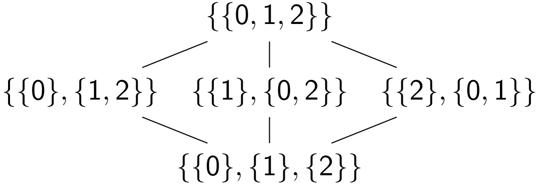
We have $\mathrm{NC}(A_n) \cong \mathrm{NC}(\{0, \ldots, n\})$.

Example

Let $K = \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \mathbb{K}A_2)$. We have $T(K) \cong NC(\{0,1,2\})$.







An obstruction to distributivity

Lemma

Let K be $\mathbb{K}$-linear and Hom*-finite with an exceptional pair (E_1, E_2) such that $\mathrm{Hom}_K(E_1, \Sigma^j E_2) \neq 0$ for some $j \in \mathbb{Z}$. Then $T(K)$ is* not *distributive.*

The pair (E_1, E_2) being exceptional means:

- For $i = 1, 2$ we have $\mathrm{Hom}_K(E_i, \Sigma^j E_i) = \begin{cases} \mathbb{K} & \text{if } j = 0 \\ 0 & \text{else;} \end{cases}$
- For all $j \in \mathbb{Z}$ we have $\mathrm{Hom}_K(E_2, \Sigma^j E_1) = 0$.

※An obstruction to distributivity

Proof.

Let $F = L_{E_1}E_2$ be the left mutation of E_2 along E_1.

$$F \to \mathrm{Hom}^\bullet(E_1, E_2) \otimes E_1 \to E_2 \to \Sigma F.$$

This is exceptional, and since $\mathrm{Hom}_K(E_1, \Sigma^i E_2) \neq 0$ we have $F \ncong \Sigma^j E_i$ for all $j \in \mathbb{Z}$, $i = 1, 2$. We obtain a sublattice of $T(K)$ of the form

$$\begin{array}{ccc} & \langle E_1, E_2 \rangle & \\ \diagup & | & \diagdown \\ \langle E_1 \rangle & \langle E_2 \rangle & \langle F \rangle \\ \diagdown & | & \diagup \\ & 0 & \end{array}$$

□

Finite dimensional algebras

Conclusion

Many examples of triangulated categories from finite dimensional algebras have non-distributive lattices of thick subcategories. They are not controlled by spaces.

Guiding questions

Questions

Part 2: What properties can $T(K)$ have?

Part 3: How can we *always* approximate $T(K)$ by a space?