Deep points in cluster varieties

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Cluster algebras [Fomin-Zelevinsky]

Combinatorial input: a *quiver* (= an oriented graph), or (more generally) a *skew-symmetrizable* matrix.

The *cluster seed* $t = (\mathbf{x}, Q)$ consists of a quiver together with a set of algebraically independent cluster variables corresponding to its vertices.

Mutations change a quiver (by a certain explicit rule) and one of the cluster variables:

$$x_i x_i' = \prod_{i \to j} x_j + \prod_{k \to i} x_k$$

Some vertices of Q are declared to be *frozen*: we never mutate at them.

The corresponding variables are also called frozen.

Cluster algebras [Fomin-Zelevinsky]

The **cluster algebra** $A(Q)$ is the subalgebra in the ring of Laurent polynomials in the initial cluster variables generated by cluster variables and inverses of frozen variables in all seeds (obtained by iterated mutations).

Laurent phenomenon: each cluster variable is a Laurent polynomial in cluster variables in each seed.

The **upper cluster algebra** $U(Q)$ is the intersection over all seeds of the rings of Laurent polynomial in cluster variables in the seed.

The Laurent phenomenon ensures that $A(Q) \subseteq U(Q)$. The inequality is a priori strict, but in examples coming from Lie theory, we usually have $A(Q) = U(Q)$ (see Fan Qin's talk). This happens, in particular, if $A(Q)$ is *locally acyclic*. For simplicity, we will always assume this.

A_1 with 1 frozen

Consider the initial seed

$$(Q = 1 \to 2, \{x_1, x_2\})$$

where a blue variable means that it is frozen.

We can mutate at vertex 1 and obtain the new seed

$$\left(1 \leftarrow 2, \left\{x_1' = \frac{x_2 + 1}{x_1}, x_2\right\}\right).$$

Mutating at vertex 1 again, we obtain the initial seed back $(1 \to 2, \{x_1, x_2\})$.
Thus $A(Q) = \mathbb{C}[x_1, \frac{x_2+1}{x_1}, x_2^{\pm 1}]$. Note that $x_2 = x_1 x_1' - 1$, so $A \cong \mathbb{C}[x_1, x_1'][(x_1 x_1' - 1)^{-1}]$.

Cluster varieties and cluster manifolds

Geometrically, the non-vanishing locus of all cluster variables $\widetilde{x}_i$ in a seed $\widetilde{t} = (\widetilde{\mathbf{x}}, \widetilde{Q})$ gives an open torus, called *cluster chart* $\mathbb{T}_{\widetilde{\mathbf{t}}} \subseteq \operatorname{Spec} A(Q)$.

Definition

- The *cluster variety* associated with $A(Q)$ is

$$\mathcal{A}(Q) \ := \ \operatorname{Spec}(A(Q)).$$

- The *cluster manifold* is

$$\mathcal{M}(Q) \ := \bigcup_{\widetilde{\mathbf{t}}} \mathbb{T}_{\widetilde{\mathbf{t}}} \subseteq \mathcal{A}(Q).$$

- The *deep locus* is

$$\mathcal{D}(Q) \ := \ \mathcal{A}(Q) \backslash \mathcal{M}(Q).$$

Basic properties and examples

Cluster charts cover $\mathcal{A}(Q)$ up to codimension 2. In other words, $\mathcal{D}(Q)$ has codimension at least 2 in $\mathcal{A}(Q)$.

The singular locus $\mathrm{Sing}(\mathcal{A}(Q))$ is a subset of $\mathcal{D}(Q)$. In general, it is a proper subset. In particular, we can have smooth $\mathcal{A}(Q)$ with non-empty $\mathcal{D}(Q)$.

Example

The cluster variety for a quiver with n frozen vertices and no arrows is $(\mathbb{C}^\times)^n$. It has 1 cluster chart and empty deep locus.

A_1 with 1 frozen: geometry

Consider again the initial seed

$$(Q = 1 \to 2, \{x_1, x_2\})$$

Recall that $A(Q) \cong \mathbb{C}[x_1, x_1'][(x_1 x_1' - 1)^{-1}]$, so

$$\operatorname{Spec}(A(Q)) = \mathbb{C}^2 \setminus \{x_1 x_1' - 1 = 0\}.$$

We have two cluster tori, with coordinates given by

$$\mathbb{T}_1 = \{x_1 \neq 0, x_1 x_1' - 1 \neq 0\}$$

and

$$\mathbb{T}_2 = \{x_1' \neq 0, x_1 x_1' - 1 \neq 0\}.$$

Thus, the cluster manifold is

$$\mathcal{M}(Q) = \mathbb{T}_1 \cup \mathbb{T}_2 = \mathbb{C}^2 \setminus (\{x_1 x_1' - 1 = 0\} \cup \{(0,0)\}),$$

so $\mathcal{D}(Q) = \{(0,0)\}$.

Torus actions

Let **x** have n mutable and m frozen variables. There is a natural action of the $(n+m)$-dimensional torus $\widetilde{T}$ on the initial seed by rescaling the variables.

It does not extend to the action on the entire cluster algebra $A(Q)$, but there is a natural subtorus $T = \ker(f)$ whose action does extend. Here

$$f : \widetilde{T} \to (\mathbb{C}^\times)^n : (z_1, \ldots, z_{n+m}) \to (\prod_{i=1}^{n+m} z_i^{\widetilde{b}_{ij}})_{j=1}^n,$$

is defined using the exchange matrix of quiver Q with entries $\widetilde{b}_{ij} = \#\{i \to j\} - \#\{j \to i\}$.

Lemma (Gekhtman-Shapiro-Vainshtein)

The torus $T = \ker(f)$ acts naturally on $A(Q), \mathcal{A}(Q), \mathcal{M}(Q)$, and $\mathcal{D}(Q)$. Moreover, the action on $\mathcal{M}(Q)$ is free.

T is precisely the torus of *cluster automorphisms* (algebra automorphisms rescaling all cluster variables).

We see that the points with non-trivial stabilizers belong to $\mathcal{D}(Q)$.

Conjecture (CGSS, after Shende-Speyer for top positroid cells)

$\mathcal{D}(Q) = \{p \in \mathcal{A}(Q) \mid Stab_T(p) \neq \{1\}\} =: \mathcal{S}(Q)$ for locally acyclic skew-symmetric cluster algebras.

Theorem (CGSS)

Conjecture holds for a family of mutable quivers, with arbitrary choice of frozens, which includes:

- *Certain double Bott-Samelson varieties;*
- *Cluster algebras of types ADE;*
- *Unique top-dimensional open positroid cells in Grassmannians $Gr(2, n)$ and $Gr(3, n)$. In particular, $\mathcal{D}(\prod^{\circ}_{2,n}) = \emptyset$ if and only if n is odd, and $\mathcal{D}(\prod^{\circ}_{3,n}) = \emptyset$ if and only if $gcd(3, n) = 1$.*

We call the points in $\mathcal{D}(Q) \setminus \mathcal{S}(Q)$ mysterious.
[Beyer–Muller]: Compute deep points in cluster algebras of type A, of rank 2, of unpunctured surfaces, and in Markov cluster algebras.

Lemma

Let B be an $n \times n$ skew-symmetric matrix. Assume that there exists an $m \times n$ matrix C such that for the extended exchange matrix

$$\widetilde{B} = \begin{pmatrix} B \\ C \end{pmatrix}$$

we have:

1. $A(\widetilde{B})$ *is locally acyclic,*
2. $\widetilde{B}$ *has really full rank: $\mathbb{Z}$-span of the rows of $\widetilde{B}$ is $\mathbb{Z}^n$,*
3. $\mathcal{A}(\widetilde{B})$ *has no mysterious points.*

Then, for any $k \geq 0$ and any $k \times n$-matrix D, the cluster algebra $A\begin{pmatrix} B \\ D \end{pmatrix}$ has no mysterious points.

Beyond the locally acyclic case, the same **fails** for **upper** cluster algebras. E.g: Markov cluster algebra with principal coefficients.

Double Bott-Samelson varieties

For σ_i a generator of Br_n, we say that two full flags $F^\bullet, G^\bullet$ in $\mathbb{C}^n$ are in position σ_i if $F_j = G_j$ for $j \neq i$, and $F_i \neq G_i'$.
Let $\beta = \sigma_{i_1} \cdots \sigma_{i_l} \in \mathrm{Br}_W^+$ be a positive braid word.

Definition

The **half-decorated double Bott-Samelson variety** associated with β is

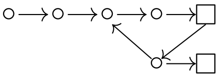
$$\mathrm{BS}(\beta) := \{(F_1^\bullet, \ldots, F_{l+1}^\bullet) \mid F_1 = F_{st}, F_k \xrightarrow{i_k} F_{k+1}, F_{l+1} \in B_- B_+ / B_+\}.$$

Theorem (Shen-Weng, Casals-E.Gorsky-MG-Le-Shen-Simental)

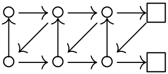
These are cluster varieties for locally acyclic cluster algebras of really full rank.

$\mathrm{BS}(\sigma) \cong \mathrm{C}^\times$;
$\mathrm{BS}(\sigma^2)$ is of type A_1 with 1 frozen.

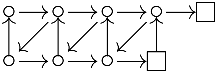
(a) For $\beta = \sigma_1^n$, the quiver Q_β is A_{n-1} quiver with a single frozen.
(b) For$\beta = \sigma_1^3\sigma_2\sigma_1^2\sigma_2$, Q_β is of type D_5 with two frozens:



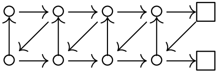
For $\beta = \sigma_1^{n-2}\sigma_2\sigma_1^2\sigma_2$, Q_β is a quiver of type D_n with two frozens.
(c) For $\beta = (\sigma_1\sigma_2)^4$, Q_β is this quiver of type E_6:



(d) For $\beta = (\sigma_1\sigma_2)^4\sigma_1$, Q_β is a quiver of type E_7:



(e) For the word $\beta = (\sigma_1\sigma_2)^5$, Q_β is a quiver of type E_8:



Deep loci in BS varieties

The maximal torus in PGL(n) acts diagonally on BS(β), and this coincides with the action of the cluster automorphisms torus T (if β contains each σ_i).

Conjecture

TFAE:

1. $\mathcal{D}(Q)$ *is empty;*
2. T *acts freely on BS*(β)*;*
3. β *closes up to a knot.*

Theorem (CGSS)

(a) (2) $\Leftrightarrow$ (3) $\Rightarrow$ (1);

(b) *For braids of the form* $\sigma_1^a(\sigma_2\sigma_1)^b \in \mathrm{Br}_3$*, we also have* (1) $\Rightarrow$ (2).

Denote $X(a,b) := \mathrm{BS}(\sigma_1^a(\sigma_2\sigma_1)^b)$.

Theorem (CGSS)

(a) (2) $\Leftrightarrow$ (3) $\Rightarrow$ (1);

(b) *For braids of the form* $\sigma_1^a(\sigma_2\sigma_1)^b \in \mathrm{Br}_3$, *we also have* (1) $\Rightarrow$ (2).

The proof of (b) is given by constructing for each point $x \in \mathrm{BS}(\beta)$ with trivial stabilizer an explicit cluster chart covering x.

The cluster charts involved are obtained by using techniques from [Casals-E.Gorsky-MG-Le-Shen-Simental]: *Demazure weaves* and quasi-cluster transformations given by cyclic rotations of words at intermediate steps.

NB: For some non top-dimensional positroids in $\mathrm{Gr}(3, n)$, using only the charts corresponding to plabic graphs is not sufficient.
NB: For some more general *braid varieties*, using only the charts corresponding to Demazure weaves (without rotations) is not sufficient.

One motivation for the project comes from Homological Mirror Symmetry.

Conjecture

If the top Richardson stratum $V \subseteq G^\wedge/P^\wedge$ has empty deep locus, then for each $\lambda \in \mathbb{C}$, $DFuk_\lambda(G/P)$ is generated by objects supported on Lagrangian tori.

Theorem

$DFuk_\lambda(Gr(2, 2n+1))$ is generated by objects supported on Lagrangian tori and $DFuk_\lambda(Gr(2, 2n+1)) \simeq DSing(W^{-1}(\lambda))$ for all $\lambda \in \mathbb{C}$.

It's a quasi-equivalence of $\mathbb{Z}/2$-graded A_∞-categories, if one takes the category of matrix factorizations on the RHS:
Both sides have a generator with the same intrinsically formal endomorphism algebra (used works of Dyckerhoff, Orlov, Sheridan).

Geometry of the deep loci

More generally, $\mathcal{D}(Q)$ is not necessarily irreducible or equidimensional.

The irreducible components of $\mathcal{D}(Q_\beta)$ correspond, in a sense, to removing single components of the closure of β.

Intersections of irreducible components of $\mathcal{D}(Q_\beta)$ correspond to removing several components of the closure of β.

Theorem

Assume throughout that $b > 3$ and $a \geq 1$. Then,

(a) *If a is odd and $b \equiv 1 \bmod 3$, then $D(X(a,b))$ has three components C_1, C_2, C_3. Moreover,*
- *C_1 and C_2 are both cluster varieties of type $A_{\frac{2}{3}(b-1)-1}$ with 1 frozen.*
- *C_3 is a cluster variety of type $A_{a+\frac{1}{3}(2b-5)-1}$ with 1 frozen.*
- *$C_1 \cap C_2 = C_2 \cap C_3 = C_1 \cap C_3 = \{\mathrm{pt}\}$.*

In this case $D(X(a,b))$ is never smooth, and it is equidimensional if and only if $a = 1$.

(b) *If a is even and $b \equiv 0 \bmod 3$, then $D(X(a,b))$ is smooth, irreducible, and isomorphic to a cluster variety of type $A_{\frac{2}{3}(b-3)}$ with 1 frozen.*

(c) *If a is even and $b \equiv 1 \bmod 3$, then $D(X(a,b))$ is smooth, irreducible and isomorphic to a cluster variety of type $A_{a+\frac{2}{3}(b-4)}$ with 1 frozen.*

(d) *If a is even and $b \equiv 2 \bmod 3$, then $D(X(a,b))$ is smooth, irreducible, and isomorphic to a cluster variety of type $A_{\frac{2}{3}(b-2)}$ with 1 frozen.*

In all cases each irreducible component of the deep locus is again a cluster variety (with invertible frozens).

Theorem

(i) *Every irreducible component of the stabilizer locus $\mathcal{S}(BS(\beta))$ for an arbitrary double BS variety is again a double BS variety $BS(\beta)$. In particular, it is again a cluster variety.*

(ii) *All the intersections of the irreducible components are again double BS varieties and so in particular cluster varieties.*

Thus, the same is true for the deep locus as long as it coincides with the stabilizer locus.

The same holds for positroids, modulo the fact that some intersections might be empty.

Problem

Let $\mathcal{A}$ be a locally acyclic cluster variety of really full rank. Is every irreducible component of the deep locus of $\mathcal{A}$ again a cluster variety?

THANK YOU FOR YOUR ATTENTION!